PRACTICAL
ANE AND SOLID
GEOMETRY
ADVANCED STUDENTS

ISON AND BAXANDALL

PRACTICAL
PLANE AND SOLID GEOMETRY

PRACTICAL
PLANE AND SOLID
GEOMETRY
FOR ADVANCED STUDENTS
INCLUDING GRAPHIC STATICS

ADAPTED TO THE REQUIREMENTS OF THE
ADVANCED STAGE OF THE SOUTH
KENSINGTON SYLLABUS

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PREFACE

THE training in Practical Geometry suitable for the student of Art consists of working problems mainly in Plane Geometry and Perspective. The former because decorative designs are largely based on geometrical figures; and the latter because an artist constantly deals with appearances.

On the other hand, the student of Science finds in Practical Geometry a powerful engine of calculation, which may often with great advantage be employed in preference to analytical processes. The principal use to him of plane geometry is that it enables him to make graphical or semi-graphical computations. Descriptive geometry is of great importance since it is required in problems which involve three dimensions in space; and also because it shows how to exhibit the actual forms and dimensions of solid objects.

This book is written for Science students. The necessity of accurate draughtsmanship is insisted on throughout. We describe how the drawing instruments may be set and their efficiency maintained. And the numerical answers appended to many of the examples should help to prevent any relapse into slovenly and inaccurate work on the part of those students who are apt to become lax.

In Euclid's system of pure geometry the mechanical appliances and geometrical constructions allowed are severely restricted, the former being limited to a straight-edge, pencil, compasses, and a single plane surface on which to draw. A length may be transferred or a circle drawn by using the compasses, and a straight line may be drawn through two points. A line may not, however, be drawn to touch two circles without first finding the points of contact, although this adds nothing to the practical accuracy of the result.

These restrictions, which form an important part of Euclid's scheme of logic, may yet be quite unsuitable in actual drawing. And in fact, if strictly adhered to, they are found to hamper the student in his work, to be wasteful of time, to lead to inaccuracies which may nullify the result, and even to render problems impossible of solution, the difficulties of which may otherwise be successfully overcome. In illustration see Example 2, page 139, on the motion of a slide valve driven by a Gooch link.

It is true that in examinations in Practical Geometry the use of the tee- and set-squares has caused Euclid's restrictions to be somewhat relaxed, in that parallel and perpendicular lines are usually permitted to be drawn without construction. But does this go far enough? Is not our science still in leading-strings, tied to the systems of the pure geometrician, its humble function being to illustrate his principles?

No doubt a student already familiar with the proofs of Euclid may derive much benefit from illustrating the more important propositions by carefully drawing the figures to scale. For example he may learn how small need be the errors which are introduced into graphical work. At various parts of Section I. such examples have been inserted.

But in the authors' view a fuller conception of the scope and province of Practical Geometry is needed, one which, seeing in it a powerful instrument of mathematical investigation, recognises the limitations as to accuracy and the practical requirements of the draughtsman, and permits and teaches the use of any mechanical devices where these are found to be useful and beneficial—as, for example, the employment of transparent templates, to be adjusted by trial by hand, such as are described in connection with some of the problems and examples of this book.

Chapter VI., which deals with the use of squared paper, and the plotting of curves from co-ordinates, should be found useful in view of the increasing recognition of the importance of this branch of the subject.

In Chapter VII., where three planes of reference are used, the treatment is somewhat new. But the method has been tried and interests students, and is introduced with confidence. It is intended that a student shall read this chapter without much assistance from his teacher. Thus at the beginning of Descriptive Geometry he gets some very clear notions of projection and of the geometry of space, and is also trained to improvise models where these are helpful.

The chapter on metric projection goes rather beyond the requirements of the advanced stage. But to have omitted any part would have left the subject very incomplete.

The space at our disposal for the section on Graphics was very limited, but it is hoped that this part has not been unduly condensed.

It is no doubt convenient to have the figures and corresponding descriptions arranged so as to avoid the necessity of turning over a leaf when referring to a diagram, but occasionally this could only be done at the expense of crowding or condensation.

Examples bearing on the problems will generally be found following the latter or in close proximity thereto. The miscellaneous examples which close the chapters are mainly derived from the past examination papers of the Department of Science and Art, advanced stage, and by working these the student will be able from time to time to test his proficiency.

In order to ensure the attainment of the desirable degree of accuracy in graphical work generally, the figures should not be made too small. If in some cases it may appear that this condition has been ignored, the reason must be attributed to want of space. The student, with drawing-paper at his disposal, cannot urge this excuse.

July 1899.

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SECTION I

PRACTICAL PLANE GEOMETRY

CHAPTER I

GENERAL INTRODUCTION

1. Scope of subject.—Practical geometry is the science which is concerned with applications of the problems of pure geometry to cases in which the geometrical figures require to be drawn to scale, often with great accuracy.

In **pure geometry** we have a set of propositions, arranged in logical sequence, and deduced by strict reasoning based on a few fundamental axioms and certain mathematical abstractions or conceptions which are defined. Hand sketches serve the purpose of representing the ideal figures.

In **practical geometry** the conditions are different. The principal requirement is the careful drawing, to scale, of geometrical figures by means of mathematical instruments. More freedom is allowed in the use of instruments. Thus, by means of the edge of the *drawing-board*, by *tee-square* and *set-squares*, by *parallel rulers*, and by the *clinograph*, we draw parallel, perpendicular, and symmetrically disposed lines without any construction, such as would be required by Euclid.

2. **Limits of accuracy.**—In pure geometry a point is defined as having position, but not magnitude, and a line has no breadth. Now it is evident that, in order to be visible, a point must be of a definite size, and a line must have breadth. Moreover, it is impossible to actually draw a line which shall be perfectly straight, or a circle, curve, or, in fact, any figure with absolute accuracy. Consequently the results we obtain by graphical construction are subject to unavoidable errors, and in practical geometry these conditions are recognised at the outset.

In striving to approximate as nearly as may be to the ideal, and to obtain the greatest degree of accuracy warranted by the conditions of the particular problem under consideration, we fix working limits to the errors within which our results should be confined.

Thus a point, pricked in the paper with the point of a needle, need not be more than $\frac{1}{200}$ of an inch in diameter in order to be readily seen by a person with ordinary eyesight. A line drawn with a lead pencil of suitable lead and properly sharpened need not be more than $\frac{1}{200}$ of an inch wide; and with a good "straight-edge," no part of a line, of say one foot in length, need be out of the true alignment by so much as the $\frac{1}{500}$ of an inch. The lines on an engine-divided scale should be correct in position to within $\frac{1}{1000}$ of an inch, and with care we should be able to measure and set off distances to within $\frac{1}{200}$ of an inch. Set-squares are easily adjusted, so that the errors in the angles are kept within one minute or $\frac{1}{60}$ of a degree.

An expert draughtsman would be able to reduce these errors, if the circumstances were such as to warrant the increased care and expenditure of time that would be necessary. The limits given above are ordinary practical working limits, and should be expected to be attained by every student in his work. In order to maintain this standard of accuracy, the student should have access to the tools necessary to keep his instruments in good working order, such as the trying-plane, oil-stone, etc.

3. Instruments.—A list of the principal drawing instruments required by the student is now given, with a brief description of some of them.

The *drawing-board* should be of "imperial" size, 30″ × 22″, and the tee-square of corresponding length. A "half-imperial" board in addition is very convenient.

The 60° *set-square* may be 9″ or 10″ long, and the 45° *set-square* about 6″ long.

A *clinograph* is almost indispensable; see next article for illustration.

The *pencils* should be of pure quality, and of hard lead, say HH, or HHH.

The *compasses*, if the best, will have knee-joints and needle-points firmly secured, and the *dividers* will have a fine screw adjustment. The *inking-pens* need not have jointed nibs.

A *pricker* is requisite, and may be made by breaking a length of about 1″ from the sharp end of a stout needle, and inserting it in the wood of a penholder, leaving $\frac{1}{2}$″ or so of the point projecting.

A 6″ *semicircular protractor* is preferable to the common 6″ rectangular one, as an instrument for measuring and setting off angles with accuracy; but the best form is the *circular protractor*, and the diameter of this may be about 6″. It is desirable that angles should be capable of being read or marked off to within $\frac{1}{10}$ of a degree.

The *scale* is a very important instrument, and a special description of it is given in Art. 5.

French curves may be used as templates when lining in curves which have first been carefully drawn freehand.

Good *cartridge drawing-paper* with a smooth surface answers every purpose. *Drawing-pins* if used should have thin heads, but the latter always impede the movement of the squares. To overcome this, a good plan is to secure the corners of the paper with sealing-wax. The student should avoid inserting pins in his boards anywhere but near the edges.

4. Setting of instruments.—The *lead pencil* should be of hard quality, and finished to a fine chisel edge with glass paper as illustrated in the figure opposite. The leads of the compasses may be similarly sharpened.

The *edges* of the drawing-board and of the tee-square and set-squares should be trued up, and the angles of the latter made accurate by using a trying-plane.

The *clinograph* is useful for drawing series of lines which are respectively parallel, perpendicular, or symmetrically inclined. See Example 6.

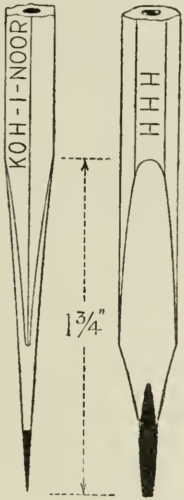
Examples.—**1.** Draw a fine line, and on it step off with the dividers ten divisions each $\frac{1}{5}''$ long; and from the last point step off one division in a direction at right angles to the others. Join the first and last points, and through the other points draw lines parallel to this line. The lines so drawn will be nearly $\frac{1}{50}''$ apart, and should be so fine as to be each distinct.

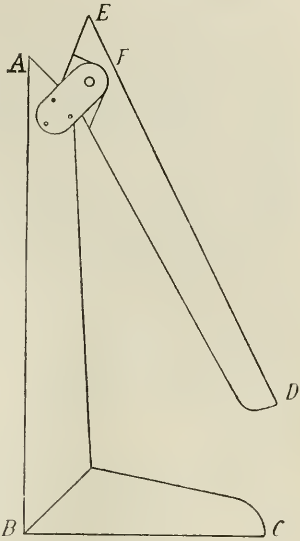
2. Repeat example 1, taking the divisions each $\frac{1}{10}''$ instead of $\frac{1}{5}''$. The lines will now be only $\frac{1}{100}''$ apart, and should still be distinct from one another.

3. By using the tee-square and set-square draw two lines crossing one another at right angles. Then starting from any point on one of the lines, and using the 45° set-square, draw four lines in succession across the angular spaces, meeting respectively on the lines first drawn. The last point should coincide with the first point, the four lines last drawn forming a square. If this figure be drawn big enough, it gives a very severe test of the accuracy of the 45° set-square.

4. Draw a circle of about 4″ radius, and through its centre draw the six possible lines with the tee- and 60° set-squares. Prick off the twelve points where these lines intersect the circle, and test whether the twelve chords joining these points are of equal lengths. Also test whether the directions of four of the chords agree with the 45° set-square, and whether the 60° and 45° set-squares can be arranged with two of their edges in contact, and one resting on the tee-square, so that the upper edge of the outer square shall coincide in succession with the eight remaining chords.

5. Describe a circle of say $1\frac{1}{2}''$ radius; draw the circumscribing hexagon by using the tee-square and 60° set-square. Prick off the corners of this hexagon, and test whether all the sides are of equal length. Also try whether the nine lines joining the alternate and opposite points agree in direction with one or other of the edges of the 60° set-square.





6. Let any line *XY* be drawn and a point *P* taken anywhere on the drawing-paper. It is required to draw through *P*, **(a)** a line parallel to *XY*; **(b)** a line perpendicular to *XY*; **(c)** a line symmetrically inclined with *XY*.

(a) Place the edge *AB* of the clinograph on the edge of the tee-square; move the tee-square along the edge of the board and the clinograph along the edge of the tee-square, setting the edge *DE* to coincide with *XY*. Again move tee-square (if necessary) and clinograph until *DE* passes through *P*, when the required line may be drawn.

(b) After setting the edge *DE* to coincide with *XY*, turn the clinograph over through a right angle so that *BC* rests on the edge of the tee-square; the required line through *P* is then readily drawn.

(c) Proceed as in **(a)**; then turn the clinograph on to its other face, still keeping *AB* on the edge of the tee-square; a line may then be drawn through *P* such that this line and *XY* are symmetrical with regard to horizontal and vertical lines.

5. The Scale.—The *most useful scales* for our work are those which are *decimally subdivided.* The subdivisions may extend along the whole length, or be confined to end main divisions, being technically known as "close" and "open" divided scales respectively. One close divided and two open divided scales can be got on one edge, so that if both faces be used, four different scales of the former kind and eight of the latter can be set off on one piece of boxwood, by using all the available space on it.

The figure shows portions at the ends of one face of a boxwood scale 12″ long, with four open divided scales of 1″, $\frac{1}{2}''$, $\frac{1}{4}''$, and $\frac{1}{8}''$ respectively, decimally subdivided. On the other face it would be very convenient to have four scales forming the set $\frac{3}{4}''$, $\frac{3}{8}''$, $\frac{3}{16}''$, $\frac{3}{32}''$, *or* the set $\frac{2}{3}''$, $\frac{1}{3}''$, $\frac{1}{6}''$, $\frac{1}{12}''$, also decimally subdivided. The student should be provided with one or other of the set of eight scales thus described.

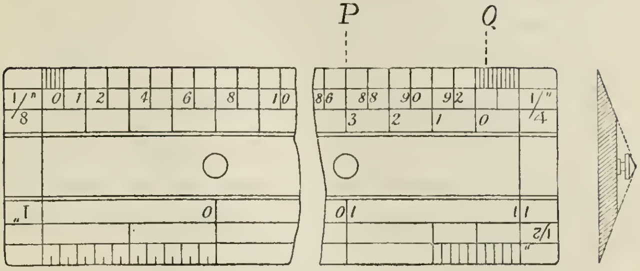
The divisions of the scale are marked 0, 1, 2, 3, . . ., but they may be read as 0, 10, 20, 30, . . ., or 0, 100, 200, 300, . . ., etc., or as 0, .1, .2, .3, . . ., or 0, .01, .02, .03, . . ., or 0, .001, .002, .003, . . ., and so on.

If the divisions are read, 0, 1, 2, 3, . . ., as marked, then the subdivisions represent figures in the first decimal place, and a fraction of a subdivision could be estimated by a figure in the second decimal place. Thus the distance *PQ* on the $\frac{1}{4}''$ scale would be read 3.24.

If the figures marked are read as tens, *i.e.* 10, 20, 30, . . ., the subdivisions become units, and the fraction of a subdivision is in the first place of decimals. Thus the distance *PQ* would now be 32.4.

If the readings of the open divisions are 100, 200, 300, . . ., *PQ* reads 327. If 1000, 2000, 3000, . . ., *PQ* is 3240, and so on.

While if the numbers at the open divisions stand for .1, .2, .3, . . ., then *PQ* represents .324. If the numbers are taken to mean .01, .02, .03, . . ., *PQ* is read equal to .0324, and so on.



Examples.—1. Measure the distance AB :—
(*a*) On the scale of 1″ to 1 unit. *Ans.* 2.37.
(*b*) On the scale of 1″ to 100 units. *Ans.* 237.
(*c*) On the scale of $\frac{1}{2}$″ to 10,000 units. *Ans.* 47500.
(*d*) On the scale of $\frac{1}{4}$″ to 0.1 unit. *Ans.* 0.95.
(*e*) On the scale of $\frac{1}{2}$″ to .001 unit. *Ans.* 0.00475.
(*f*) On the scale of $\frac{1}{3}$″ to 10 units. *Ans.* 71.3.
(*g*) On the scale of $\frac{3}{4}$″ to .01 unit. *Ans.* 0.0317.

Note.—The respective scales must be applied directly to the line, and the number of units read off, without the interposition of dividers.

2. Mark off lengths of—
(*a*) 3.78 units on the scale of 1″ to 1 unit.
(*b*) 697 units on the scale of $\frac{1}{2}$″ to 100 units.
(*c*) 0.913 unit on the scale of $\frac{1}{8}$″ to 0.1 unit.
(*d*) 0.001427 unit on the scale of $\frac{1}{4}$″ to .0001 unit.
(*e*) 0.092 unit on the scale of $\frac{3}{8}$″ to 0.1 unit.

Note.—The lengths must be marked off with a pencil or pricker direct from the scale, the latter being applied to the paper. Dividers should not be used.

3. Set off lengths representing :—
(*a*) 2.37 feet, to the scale of 1 inch to the foot.
(*b*) 64,500 lbs. to the scale of $\frac{1}{2}$ inch to 10,000 lbs.

6. Ratio and proportion.—When we speak of the *ratio* of one magnitude to another, say of A to B, written $A:B$ or A/B, we signify *how many times* the first contains the second.

Thus comparing a guinea with a crown, their respective money-values are in the ratio of 21 to 5, and this ratio $\frac{21}{5}$ equals 4.2. Thus a ratio may be expressed as a *pure number*, that is, one not associated with kind or quantity.

Or again the numbers on an ordinary scale give the ratios of the several lengths to the unit length.

Proportion implies the equality of two ratios. Let the ratio $A:B$ between two magnitudes be equal to the ratio $a:b$ between two other magnitudes, the four terms A, B, a, b, are said to form a proportion. This equality is expressed in symbols thus—

$$A : B : : a : b,$$

or

$$A : B = a : b,$$

or

$$\frac{A}{B} = \frac{a}{b};$$

and by algebra it is readily seen that

$$\frac{B}{A} = \frac{b}{a}, \text{ or } B : A : : b : a;$$

also

$$\frac{A}{a} = \frac{B}{b}, \text{ or } A : a : : B : b;$$

and

$$A \times b = a \times B.$$

Similar polygons afford typical illustrations of ratios and proportion. The student is acquainted with the fundamental property of such figures, viz. :—

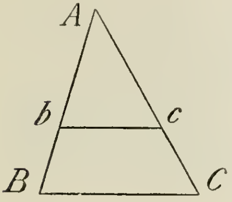
Theorem.—*In two similar rectilinear figures the ratios of pairs of corresponding lines are all equal.*

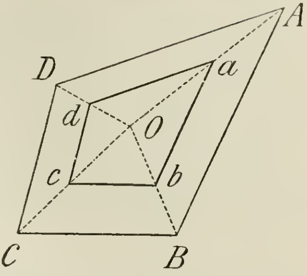
Thus in the similar triangles ABC, Abc we have

$$\frac{AB}{Ab} = \frac{BC}{bc} = \frac{CA}{cA};$$

or

$$AB : BC : CA = Ab : bc : cA.$$





And in the similar figures *OABCD*, *Oabcd* the ratios

$$\frac{AB}{ab}, \quad \frac{BC}{bc}, \quad \frac{CD}{cd}, \quad \frac{DA}{da}, \quad \frac{OA}{Oa}, \quad \frac{OB}{Ob}, \quad \frac{OC}{Oc}, \quad \frac{OD}{Od}$$

are all equal.

The following relations are important. The student should verify them by inserting simple numbers for the letters.

If $$\frac{M}{m}=\frac{N}{n},$$

then $$\frac{p\cdot M+q\cdot N}{p\cdot m+q\cdot n}=\frac{M}{m}\text{ or }\frac{N}{n},$$

and $$\frac{p\cdot M+q\cdot m}{r\cdot M+s\cdot m}=\frac{p\cdot N+q\cdot n}{r\cdot N+s\cdot n},$$

where p, q, r, and s are any numbers, positive or negative, whole or fractional. Thus in the figure,

$$\frac{Ab}{AB}=\frac{Ac}{AC}; \therefore \frac{AB-Ab}{AB}=\frac{AC-Ac}{AC}, \text{ or } \frac{bB}{AB}=\frac{cC}{AC}.$$

Examples.—**1.** Measure the sides of the triangles *ABC*, *Abc* to three significant figures on any suitable scale, say by applying the $\frac{1}{4}''$ decimal scale direct to the figure; and, by numerical calculation, verify the equality of the three ratios stated on p. 8.

2. Verify the equality of the given ratios for the quadrilaterals by measuring the lengths of all the lines and performing the divisions by arithmetic. Calculate to three significant figures; find the average of the eight values, and taking this as the correct value of the ratio, estimate the percentage error of the one most out of truth.

7. PROBLEM.—**To divide a given line AB into any number of equal parts, say seven.**

(*a*) *By construction* (no figure). Draw *AC* at any angle, say 45°, to *AB*, and, selecting any unit, step it off with the dividers seven times along *AC*, from *A* to *D*; *or* applying the scale to *AC*, mark off, by means of pencil or pricker, seven equal divisions. Join *DB*, and through the other points of division on *AD* draw lines parallel to *DB*; these intersect *AB* at the required points.

Note.—In choosing the unit, care should be taken that *AD* is neither too short nor too long, or the construction becomes "ill conditioned," and the points on *AB* are not determined with sufficient precision.

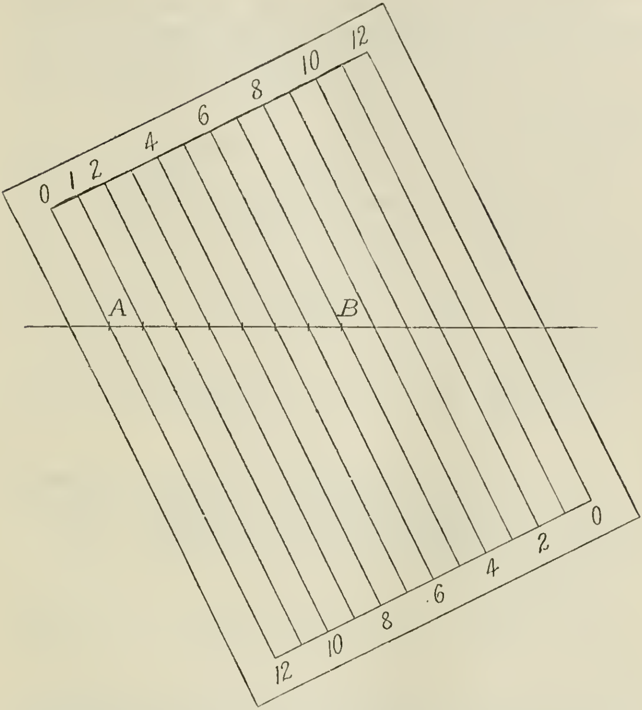
(*b*) *By trial with the dividers.* This is the method of trial and error and should be encouraged. It is a training in the estimation of lengths, and with practice the required division is quickly and accurately effected.

(*c*) *By the use of ruled (or squared) tracing-paper or ruled celluloid.* Place the ruled tracing-paper as shown over *AB*, with the line marked 0 passing through *A*. Put the point of a pricker through the tracing-paper at *A* and turn the paper round until the line 7 passes through *B*. Then carefully prick through at the places where the lines between 0 and 7 cross *AB*.

These ruled *templates* will be found very useful in graphical work, and it is worth while for the student to make them of celluloid, which may be done thus:—Paper accurately ruled (or squared paper) may be readily obtained, or failing this, carefully set out; the sheet celluloid is then placed over this and the lines scratched with a needle-point.

Two sizes of templates should be made. The smaller one may have the twelve main lines 4″ long and 0.2″ apart, each division being subdivided into five equal parts. Allowing a margin of $\frac{1}{2}''$ all round, the size of the template would be 5″ by $3\frac{1}{2}''$ outside. The larger template might conveniently have the twelve main lines 7″ long and $\frac{1}{3}''$ apart, each being subdivided into ten. The outside dimensions would be 8″ by 5″, leaving the same margin as before.

The lines should be scratched of different widths for the sake of clearness.



Examples.—1. Draw a straight line, and on it mark off a length *AB* of 3.67″. Bisect *AB*.

Note.—The bisection of a line is best done with the compasses, using the lead. Set the compasses to the half-length as nearly as can be guessed, and with *A* and *B* as centres describe short arcs nearly meeting on the line. A second guess may now be made which is very nearly, if not quite, accurate, the arcs drawn, and the mid point between them estimated.

2. Divide the line of Example 1 into 8 equal parts.

Note.—Here the method of continual bisection is useful.

3. On a straight line mark off a length of 3.14″, and divide this length into 5 equal parts, by each of the methods of this article.

8. PROBLEM.—**To divide a line AB into consecutive parts which have given ratios to each other,** say 2 : 3 : 5.

(*a*) Draw a line through A inclined at any angle to AB, and to a convenient scale mark off AC, CD, DE, respectively equal to 2, 3, 5 units.

Join EB and draw CF and DG parallel to EB.—Then $AF : FG : GB = AC : CD : DE = 2 : 3 : 5$ (Euclid, VI. 10).

(*b*) *By the use of ruled tracing-paper or celluloid.*—This may be effected in a manner similar to that already described in Art. 7. The sum of 2 and 3 is 5, and that of 2, 3, and 5 is 10. So by placing the line 0 over A, and the line 10 over B, the required points of division are obtained by pricking through where the lines 2 and 5 intersect AB.

If decimals occur in the given ratios, decimal subdivisions are required in the ruled template.

9. PROBLEM.—**The given line P represents a speed of 24·6 feet per second; construct the scale of speed, and measure the speed represented by the line Q.**

(*a*) On any line set off RS equal to P, and draw ST perpendicular to RS.

With centre R, radius 24.6 measured on a suitable scale, describe an arc intersecting ST in T, and join RT.

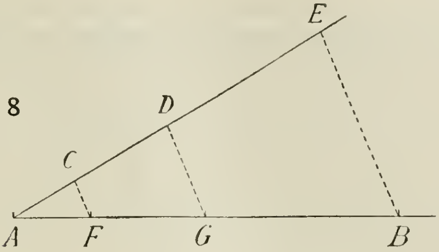
Set the scale used in measuring 24.6 with its edge along RT, and mark or prick off the decimal divisions and subdivisions shown.

Perpendiculars from these to RS will give the required divisions of the scale of speed. The scale is then numbered as shown.

Measuring Q on this scale, the speed represented by it is found to be 11.3 feet per second.

(*b*) *By the use of a ruled template.* Set the template with the zero line on R, and the 2.46 line over S, then prick through the decimal divisions and subdivisions of the scale.

8



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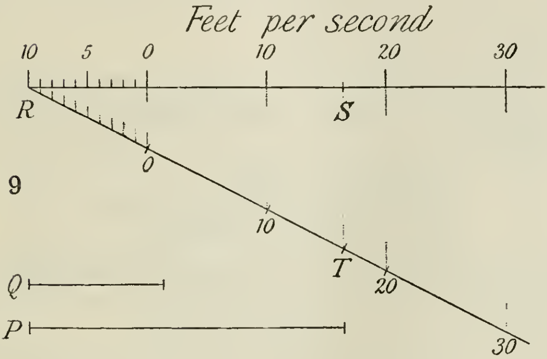
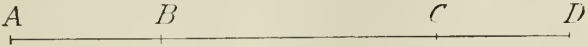




Examples.—**1.** Divide a line 2.87″ long into two parts which shall be in the ratio 4 : 7.

2. Divide a line 3.19″ long into two parts, so that the length of one part is to the length of the whole as 4 : 7.

3. On a straight line mark off a length of 2.71″, and divide the length into three consecutive segments, having the ratios to one another of 2.3, 3.6, 1.7.

4. Measure the ratio of AC to AB in the bottom figure. *Ans.* 2.80.

Place the zero line of the template on A, and rotate until one of the main division lines comes over B; read off the position of C, and divide by B.

5. Measure the ratios (*a*) $AB : AC : AD$. *Ans.* 1 : 2.80 : 3.70.
(*b*) $DC : DB : DA$. *Ans.* 1 : 3.06 : 4.20.
(*c*) $AC : BD$. *Ans.* 1.05 : 1.

10. PROBLEM.—**To construct a diagonal scale of $1\frac{1}{2}''$ to the foot, which shall measure feet, inches, and eighths of an inch.**

Set off eight equal divisions along a vertical line AB, and draw the nine horizontal lines through the points of division.

Draw a series of vertical lines, $1\frac{1}{2}''$ apart, for the main divisions of the scale, representing feet.

Apply the $\frac{1}{8}''$ scale to BC and AO, mark off twelve equal divisions on each of these lines, and draw the twelve diagonal lines between them.

Figure the scale as shown.

The length between the two dots represents $1'\ 2\frac{3}{8}''$.

11. PROBLEM.—**Linear measure in Western India being as follows:—**

1 tasu = 1·125 inches
1 hath = 16 tasu = 1 foot 6 inches
1 gaz = $1\frac{1}{2}$ hath = 2 feet 3 ,,

Draw a diagonal scale of $\frac{1}{30}$ showing gaz, hath, and tasu.

Mark off on it two lengths respectively equal to 2 gaz, 0 hath, 12 tasu; and 1 gaz, 1 hath, 4 tasu. (1898)

Mark off sixteen equal divisions along a vertical line oD, and draw the seventeen equally spaced horizontal lines.

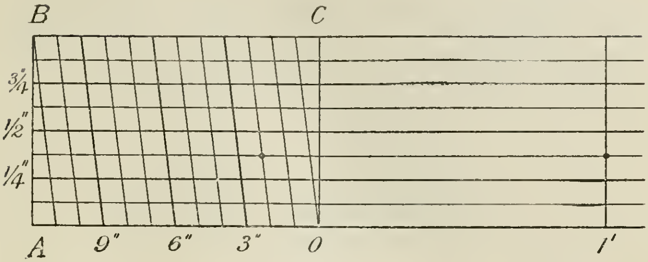
By calculation, $\frac{1}{30}$ of $2'\ 3'' = \frac{27}{30}'' = 0.9''$. Therefore draw the vertical lines of the scale at horizontal intervals of $0.9''$ to represent gaz.

Set off $DE = 1$ hath $= \frac{2}{3}$ gaz $= 0.6''$. Draw the diagonal line EO, and through 8 draw 8 1 parallel to EO.

Figure the scale as shown.

The two required distances are indicated on the scale by dots.

Note.—Diagonal scales have the disadvantage that the divisions do not extend to the edge, and they are not more accurate in use than ordinary scales. They are however useful where, as above, it is desired to represent quantities which are expressed in three denominations.



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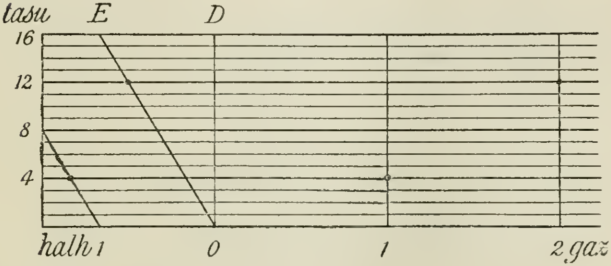

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Examples.—**1.** Construct a diagonal scale, the representative fraction of which is $\frac{1}{36}$, reading yards, feet, and inches.

2. Draw a scale showing hundredths of an inch by diagonal division.

3. Draw a scale of feet to measure all distances between 70 feet and 1 foot, where $5\frac{1}{2}$ feet is represented by .52 inch; and by diagonal division, make this scale available for reading inches.

4. Construct a scale of 120 feet to an inch, to measure 700 feet, from which single feet can be taken.

5. Construct a scale of 100 fathoms, with 18 fathoms represented by 1 inch, from which feet may be measured.

6. Construct a scale of 76 miles to 1.3 inches, to read to single miles, and to exhibit 500 miles.

7. A volume of 374 cubic inches is represented by a line 4.61 inches long. Construct a decimal scale of volume, and by its use measure the volume represented by a line 5.6 inches long. *Ans.* 455 cubic inches.

12. Solution of the right-angled triangle.—Many subsequent problems reduce to that of solving a right-angled triangle, having given two of its elements.

The *five elements* (exclusive of the right angle) are—

the height	a,
the base	b,
the hypothenuse	c,
the base angle	A,
the vertical angle	B.

There are nine possible cases. To solve the triangle—

1. Given the height and base a, b,
2. given the base and hypothenuse b, c,
3. given the hypothenuse and height c, a.

4. Given the height and base angle a, A,
5. given the base and base angle b, A,
6. given the hypothenuse and base angle c, A.

7. Given the height and vertical angle a, B,
8. given the base and vertical angle b, B,
9. given the hypothenuse and vertical angle c, B.

The student will readily solve all these cases himself. We have placed them here for convenience of reference, and to direct attention to them.

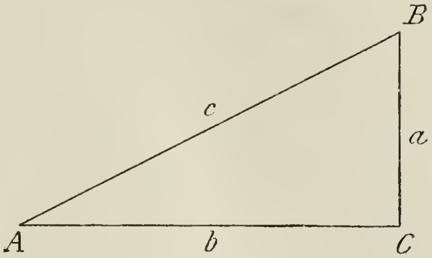
Examples.—Solve the following four right-angled triangles by accurately drawn figures, and measure the results :—

1. Given the hypothenuse 3.2″ and the height 1.9″.
Ans. $b=2.57''$, $A=36.4^\circ$, $B=53.6^\circ$.

2. Given the height 2.18″, base angle 36.3°.
Ans. $b=2.97''$, $c=3.68''$, $B=53.7^\circ$.

3. Given the base 3.06″, vertical angle 49.1°.
Ans. $a=2.66''$, $c=4.05''$, $A=40.9^\circ$.

4. Given the hypothenuse 3.92″, vertical angle 55.7°.
Ans. $a=2.21''$, $b=3.24''$, $A=34.3^\circ$.



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5. Measure all the sides and angles of the triangle ABC, the linear scale being $\frac{1}{4}$. Square the two sides a and b and add, and compare this with the square of c.

6. The shadow cast by a vertical post 57″ high on level ground was measured and found to be 87.6″; find the altitude of the sun above the horizon. *Ans.* 33.1°.

7. Wishing to know the height of an electric arc light, I placed a 5-feet rod vertically upwards on the floor, and found its shadow to measure 4.2 feet; on moving the rod 6.2 feet along the floor directly away from the light its shadow measured 7.5 feet. Required the height of the light above the floor. *Ans.* 14.4 feet.

8. Two knots on a plumb-line at heights of 7 feet and 2 feet above the floor cast shadows at distances of 11.4 feet and 1.65 feet respectively from the point where the line meets the floor. Find the height of the source of light above the floor. *Ans.* 12.1 feet.

9. A river AC, whose breadth is 217 feet, runs at the foot of a tower CB, which subtends an angle BAC of 27.4° at the edge of the opposite bank. Required the height of the tower. *Ans.* 112.4 feet.

10. A person on the top of a tower 68 feet high observes the angles of depression of two objects on the horizontal plane, which are in line with the tower, to be 32.4° and 48.6°. Find their distance from each other and from the observer. *Ans.* 47.1 feet, 90.7 feet, 127 feet.

11. The hypothenuse of a right-angled triangle is 3.45 feet, and one of the sides is double the other; determine the sides and angles. *Ans.* 1.54 feet, 3.08 feet, 26.5°, 63.5°.

12. The hypothenuse of a right-angled triangle is 43.5 feet, and one of the adjacent angles is double the other; find the sides and angles. *Ans.* 21.7 feet; 37.7 feet.

13. Ways of defining angles.—Let the student draw two straight lines at random, meeting at a point, and including some angle. By the use of any instrument except the protractor, let him obtain some information from the angle, which, being given to a neighbour, shall enable the latter to construct an angle of equal size. A number of different ways of doing this will now be given, but the student should think one or two out for himself before reading further.

Definitions.—Let AOB be any angle. With centre O, and any radius, describe the arc RS, intersecting OB, OA, in R and S, and draw the chord RS. Then it will be quite clear that if the lengths of the radius OR and chord RS were given, this would be sufficient information to enable one to construct an angle equal to AOB.

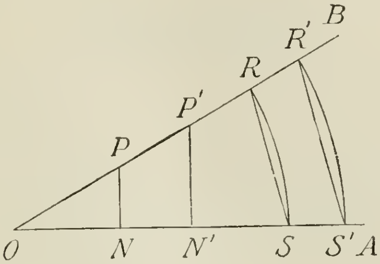
Again, in OB take any point P, and draw PN perpendicular to OA, it is evident that a knowledge of the lengths of ON and NP would enable the angle to be reproduced in size.

We might, instead, give the lengths of NP and PO, or of ON and OP. Whatever pair we take, it will be a simple matter to construct an angle equal to AOB, remembering always that the angle ONP must be a right angle.

But observe carefully that it is not the *actual lengths* of, say, ON and NP which are *necessary*, for the lengths of ON' and $N'P$ would do equally well. What, then, is it *sufficient* to give? The answer is that we may give the *ratio* of ON to NP, or NP to OP, or ON to OP, or RS to OS, or $R'S'$ to $O'S'$; for then the student may take any convenient lengths for the pair of lines, so long as they are in the given ratio.

For example, if ON were taken *three* inches and NP *one* inch, the same angle would be determined as by taking ON' 6 inches and $N'P'$ 2 inches, 3 to 1 being the same ratio as 6 to 2.

It will be convenient to give names to these ratios. The following are in constant use:—



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$\frac{\mathbf{RS}}{\mathbf{OR}}$ is called the **chord of the angle** *AOB*
$\frac{\mathbf{NP}}{\mathbf{OP}}$ ” **sine** ” ”
$\frac{\mathbf{ON}}{\mathbf{OP}}$ ” **cosine** ” ”
$\frac{\mathbf{NP}}{\mathbf{ON}}$ ” **tangent** ” ”

These are abbreviated to **cho AOB, sin AOB, cos AOB, and tan AOB** respectively. Thus—

$$\text{cho AOB} = \frac{\text{chord}}{\text{radius}} = \frac{\text{RS}}{\text{OR}}$$
$$\text{sin AOB} = \frac{\text{height}}{\text{hypothenuse}} = \frac{\text{PN}}{\text{OP}}$$
$$\text{cos AOB} = \frac{\text{base}}{\text{hypothenuse}} = \frac{\text{ON}}{\text{OP}}$$
$$\text{tan AOB} = \frac{\text{height}}{\text{base}} = \frac{\text{PN}}{\text{ON}}$$

Three-figure tables of the values of these ratios are given in the next article for angles between 0° and 90°, at intervals of one-tenth of a degree.

14. Trigonometrical Tables.

SINES OF ANGLES

											Difference to be added.								
	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	·1°	·2°	·3°	·4°	·5°	·6°	·7°	·8°	·9°
0°	.000	.017	.035	.052	.070	.087	.105	.122	.139	.156	2	3	5	7	9	10	12	14	16
10°	.174	.191	.208	.225	.242	.259	.276	.292	309	.326	2	3	5	7	8	10	12	13	15
20°	.342	.358	.375	.391	.407	.423	.438	.454	.469	.485	2	3	5	6	8	9	11	12	14
30°	.500	.515	.530	.545	.559	.574	.588	.602	.616	.629	1	3	4	6	7	9	10	11	13
40°	.643	.656	.669	.682	.695	.707	.719	.731	.743	.755	1	2	4	5	6	7	9	10	11
50°	.766	.777	.788	.799	.809	.819	.829	.839	.848	.857	1	2	3	4	5	6	7	8	9
60°	.866	.875	.883	.891	.899	.906	.913	.920	.927	.934	1	1	2	3	4	4	5	6	7
70°	.940	.946	.951	.956	.961	.966	.970	.974	.978	.982	0	1	1	2	2	3	3	4	4
80°	.985	.988	.990	.993	.995	.996	.998	.999	.999	1.000	0	0	0	1	1	1	1	1	1

COSINES OF ANGLES

											Difference to be subtracted.								
	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	·1°	·2°	·3°	·4°	·5°	·6°	·7°	·8°	·9°
0°	1.000	1.000	.999	.999	.998	.996	.995	.993	.990	.988	0	0	0	1	1	1	1	1	1
10°	.985	.982	.978	.974	.970	.966	.961	.956	.951	.946	0	1	1	2	2	3	3	4	4
20°	.940	.934	.927	.920	.913	.906	.899	.891	.883	.875	1	1	2	3	4	4	5	6	7
30°	.866	.857	.848	.839	.829	.819	.809	.799	.788	.777	1	2	3	4	5	6	7	8	9
40°	.766	.755	.743	.731	.719	.707	.695	.682	.669	.656	1	2	4	5	6	7	9	10	11
50°	.643	.629	.616	.602	.588	.574	.559	.545	.530	.515	1	3	4	6	7	9	10	11	13
60°	.500	.485	.469	.454	.438	.423	.407	.391	.375	.358	2	3	5	6	8	9	11	12	14
70°	.342	.326	.309	.292	.276	.259	.242	.225	.208	.191	2	3	5	7	8	10	12	13	15
80°	.174	.156	.139	.122	.105	.087	.070	.052	.035	.017	2	3	5	7	9	10	12	14	16

TANGENTS OF ANGLES

											Difference to be added.								
	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	·1°	·2°	·3°	·4°	·5°	·6°	·7°	·8°	·9°
0°	.000	.017	.035	.052	.070	.087	.105	.123	.141	.158	2	4	5	7	9	10	12	14	16
10°	.176	.194	.213	.231	.249	.268	.287	.306	.325	.344	2	4	6	7	9	11	13	15	17
20°	.364	.384	.404	.424	.445	.466	.488	.510	.532	.554	2	4	6	8	11	13	15	17	19
30°	.577	.601	.625	.649	.675	.700	.727	.754	.781	.810	3	5	8	10	13	16	18	21	23
40°	.839	.869	.900	.933	.966	1.000	1.036	1.072	1.111	1.150	4	7	11	14	18	21	25	28	32
50°	1.19	1.23	1.28	1.33	1.38	1.43	1.48	1.54	1.60	1.66	1	1	2	2	3	3	4	4	5
60°	1.73	1.80	1.88	1.96	2.05	2.14	2.25	2.36	2.48	2.61									
70°	2.75	2.90	3.08	3.27	3.49	3.73	4.01	4.33	4.70	5.14									
80°	5.67	6.31	7.12	8.14	9.51	11.43	14.30	19.08	28.64	57.29									

CHORDS OF ANGLES

	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	Difference to be added.								
											·1°	·2°	·3°	·4°	·5°	·6°	·7°	·8°	·9°
0°	.000	.017	.035	.052	.070	.087	.105	.122	.140	.157	2	3	5	7	9	10	12	14	16
10°	.174	.191	.209	.226	.243	.261	.278	.296	.313	.330	2	3	5	7	9	10	12	14	16
20°	.347	.364	.382	.399	.416	.433	.450	.467	.484	.501	2	3	5	7	9	10	12	14	15
30°	.518	.534	.551	.568	.585	.601	.618	.635	.651	.667	2	3	5	7	8	10	12	13	15
40°	.684	.700	.717	.733	.749	.765	.781	.797	.813	.829	2	3	5	6	8	10	11	13	14
50°	.845	.861	.867	.892	.908	.923	.939	.954	.970	.985	2	3	5	6	8	9	11	12	14
60°	1.000	1.015	1.030	1.045	1.060	1.075	1.089	1.104	1.118	1.133	1	3	4	6	7	9	10	12	13
70°	1.147	1.161	1.175	1.190	1.203	1.218	1.231	1.245	1.259	1.272	1	3	4	6	7	8	10	11	12
80°	1.286	1.299	1.312	1.325	1.338	1.351	1.364	1.377	1.389	1.402	1	3	4	5	6	8	9	10	12

SINES, TANGENTS, CHORDS, AND RADIANS OF SMALL ANGLES

	0°	·1°	·2°	·3°	·4°	·5°	·6°	·7°	·8°	·9°	Diff. to be added.
	0	6′	12′	18′	24′	30′	36′	42′	48′	54′	
0°	.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	1′ .0003
1°	.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	2′ .0006
2°	.0349	.0367	.0384	.0401	.0419	.0436	.0454	.0471	.0489	.0506	3′ .0009
3°	.052	.054	.056	.058	.059	.061	.063	.065	.066	.068	4′ .0012
4°	.070	.072	.073	.075	.077	.079	.080	.082	.084	.085	5′ .0015

Examples.—1. Required the sine of 34·5°. Referring to the table of sines, we find

Sine of 34° .	·559
Add for .5° diff.	7
Sine of 34.5° .	.566

2. Required the angle whose chord is ·824.

From the table, the chord of 48°	is	·813
For the difference 11 add	.7°	11
The required angle is	48.7°	

3. Required the cosine of 41·4°.

Cosine of 41° . .	·755
For excess .4° subtract diff.	5
Cosine of 41.4° . .	.750

15. Problem.—**To set off any given angle, say 37·6°.**

(*a*) *By the protractor.* If a circular protractor be used, then to ensure great accuracy, mark off *two* points for 37·6°, at opposite ends of a diameter.

(*b*) *By reference to a table of sines.* From the table the sine of 37.2° is seen to be .611.

Now construct the right-angled triangle *NPO*, making *NP* = .611, *PO* = 1. Thus set off *NP* = 6.11 and *PO* = 10 on the $\frac{1}{4}''$ or $\frac{1}{2}''$ scale.

The angle *PON* equals 37.6°.

(*c*) *By reference to a table of cosines.* From the table we find cos 37.6° to be .793.

To any convenient scale set off *ON* = .793 (or 7.93), and make *OP* = 1 (or 10).

The angle *PON* will be 37.6°.

(*d*) *By reference to a table of tangents.* We find from the table that tan 37.6° = .770.

Mark off *ON* = 1 (or 10), and *NP* = .770 (or 7.70); join *OP*.

Then the angle *PON* = 37.6°.

(*e*) *By reference to a table of chords.* The chord of 37.6° is found to be .645.

With centre *O*, radius unity (or 10), describe the arc *EP*.

With centre *E*, radius .645 (or 6.45), cut this arc in *P*; join *OP*.

Then angle *PON* = 37.6°.

(*f*) *By means of a scale of chords.* A scale of chords marked *CHO* is generally given on a rectangular protractor.

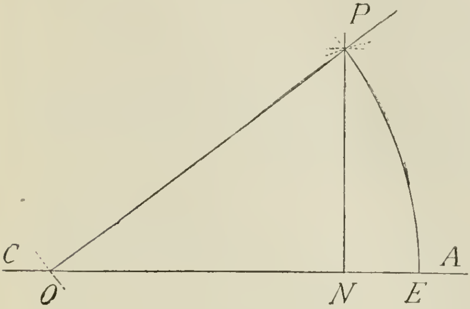
With centre *O*, radius 0 to 60, describe the arc *EP*.

With centre *E*, radius 0 to 37.6, cut this arc in *P*.

Join *OP*, then the angle *PON* = 37.6°.

16. Problem.—**An angle AOB being given, to measure it in degrees.**

With centre *O*, radius unity (or 10 read as 1) on any convenient scale, describe the arc *PE*. Draw the chord *PE* and the perpendiculars *PN*, *EQ*.



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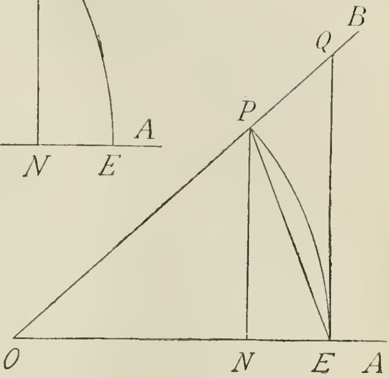

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(*a*) Measure the angle directly with the protractor;
(*b*) or measure PN and refer to the table of sines;
(*c*) or measure ON and refer to the table of cosines;
(*d*) or measure QE and refer to the table of tangents;
(*e*) or measure PE and refer to the table of chords;
(*f*) or measure PE on the scale of chords, the arc PE having been struck with the radius 0 to 60.

Examples.—1. Draw an isosceles triangle with sides of 5″, 5″, and 2.2″. Determine and measure the vertical angle by each of the six methods of this article. Take the mean of these, and observe by how much each separate result differs from the mean. *Ans.* The angle is 25.4°.

2. The sine of an angle is .820, what is the tangent? *Ans.* 1.44.

Note.—This example should be worked in two ways: first, by construction; secondly, by an inspection of the tables.

17. Miscellaneous Examples.

The student, from his previous study, should be able to work such examples as the following :—

1. Determine by construction a perpendicular through a given point to a given line, for various positions of the point and restrictions as to the length of the line.

2. Through a given point draw a line to meet a given line at a given angle.

3. Draw a triangle, having given (*a*) the three sides ; (*b*) two sides and the included angle ; (*c*) two sides and an angle opposite to one ; (*d*) two angles and the intermediate side ; (*e*) two angles and a side opposite to one of them.

4. Construct an irregular quadrilateral, having given (*a*) the four sides and one diagonal ; (*b*) the four sides and one angle ; (*c*) three sides and the two diagonals ; (*d*) three sides, one diagonal and one angle ; (*e*) three sides and two angles, etc., etc.

5. Construct an irregular pentagon, having given the five sides and two diagonals ; the five sides and two angles, etc.

6. Construct any irregular polygon, having given adequate data as to the sides, diagonals, and angles.

7. $ABCDE$ are the five consecutive corners of an irregular pentagon. Construct the figure, having given
Sides—$AB = 2.91''$; $BC = 2.31''$; $CD = 3.56''$; $DE = 1.34''$.
Diagonals—$AC = 4.63''$; $BD = 3.98''$. Angle $DEA = 133.2°$.

The polygon is not to have any internal angles greater than 180° (called re-entrant angles).

Measure the remaining side and diagonals.

Ans. $EA = 3.04''$; $CE = 4.57$; $DA = 3.66''$.

8. Set out all the pentagons which comply with the data of Ex. 7, re-entrant angles being allowed.

Note.—There are four different polygons with re-entrant angles.

9. Copy a given triangle so that a specified side shall have an assigned position.

10. Copy a given quadrilateral so that a specified side or diagonal shall occupy an assigned position.

11. Copy any given rectilinear figure so that a specified line shall have a given position on the paper.

12. Draw a triangle similar to a given triangle, the length and position of one side of the former being given.

13. Copy the figures on page 9, (*a*) the same size ; (*b*) double size.

14. Copy the figures on page 27, (*a*) the same size ; (*b*) double size.

15. Reduce any given quadrilateral to a triangle of equal area ; and reduce the triangle to an equivalent rectangle.

16. Find the centre of any given circle or circular arc.

17. Describe a circle $4\frac{1}{2}''$ diameter. Suppose this circle to be given to you on the paper, and that you have only a pencil and a ruler with two parallel edges $1\frac{1}{4}''$ apart; show how the centre of the circle can be obtained. (1888)

18. Determine a circle to pass through three given points, *or* draw the circle which circumscribes a given triangle.

19. Draw a circle to touch three given lines, *or* determine the inscribed and escribed circles for a given triangle.

20. Draw a tangent to a given circle from a given point on the circumference.

21. Find the point of contact of a tangent to a circle drawn from an external point.

22. Draw a circle of given radius to touch (*a*) a given line at a given point; (*b*) a given circle at a given point.

23. Draw a circle of given radius (*a*) to pass through two given points; (*b*) to touch two given lines; (*c*) to pass through a given point and to touch a given line.

24. Draw a circle of given radius (*a*) to touch two given circles; (*b*) to touch a given line and circle; (*c*) to pass through a given point and touch a given circle.

25. Draw a circle of given radius which shall have its centre on a given line, and touch (*a*) a second given line; (*b*) a circle.

26. Illustrate the theorem of Note 1, Prob. 21, in the following way: Draw on tracing-paper two lines AP, AQ meeting at any angle, say make $A = 30°$, and AP, AQ, $3''$ and $4''$. Between AP and AQ draw a series of lines P_1Q_1, P_2Q_2, P_3Q_3, . . . parallel to PQ. These may be about $\frac{1}{8}''$ apart. Glue a small piece of paper on the tracing at A for strength.

Next draw in ink on paper any figure, say a semicircle $3''$ diameter. Mark a point O inside the semicircle, say near the middle; place the tracing with A at O, and insert a pin at this point.

Then rotate the tracing, and as points on one of the lines, say on AP, come in succession on the boundary of the figure, prick through the corresponding points on the other line AQ. The locus of the latter will be a semicircle, *larger* in the linear ratio 3 : 4, and *turned through an angle A* about O.

Again pin the vertex A of the tracing to a point outside the semicircle, say near one end. Rotate, and as the points Q come on the boundary, prick through the corresponding points P. A copy of the semicircle will again be obtained; this time *reduced* in the ratio 4 : 3, and turned through an angle A.

CHAPTER II

SIMILAR RECTILINEAL FIGURES—AREAS

18. Similar polygons. *Definition.*—Two equiangular polygons which have the sides about their equal angles proportional are said to be *similar*.

Thus the pentagons $ABCDE$ and $abcde$ are similar, being equiangular and having

$$AB : BC : CD : DE : EA = ab : bc : cd : de : ea\ ;$$

or

$$\frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \frac{DE}{de} = \frac{EA}{ea}.$$

Except in the case of the triangle, polygons may be equiangular without being similar, for example, $ABCDE$ and $A'B'C'D'E'$. And evidently, with the same reservation, the sides may be proportional without the corresponding angles being equal.

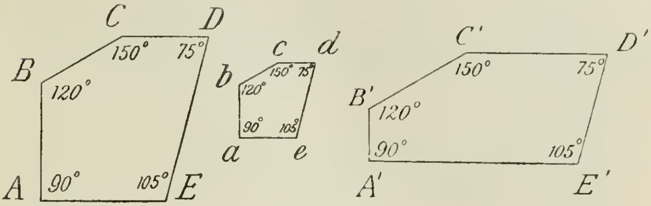
We now give some of the more important properties of similar figures, on which the constructions of many problems are based.

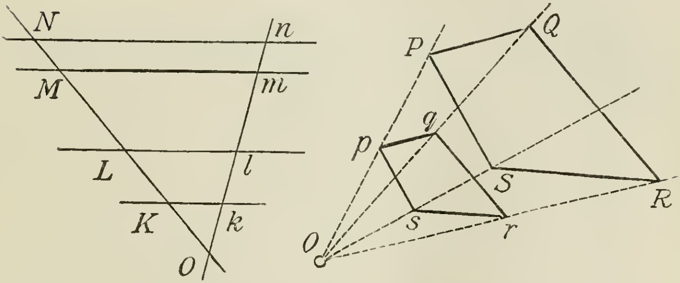
An instrument called a *pantograph*, for the tracing of similar figures, is sometimes used.

Theorem 1.—*If two lines be cut by a series of parallel lines, the ratios of corresponding segments are all equal.*

$$\text{Thus } \frac{KL}{kl} = \frac{LM}{lm} = \frac{MN}{mn} = \frac{KN}{kn} = \frac{LN}{ln} = \frac{OK}{ok} = \frac{OL}{ol} = \ . \ . \ . \text{ etc.}$$

Theorem 2.—*If two polygons have their sides respectively parallel, and the lines joining their corresponding corners all converging to a point, the figures are similar ; and conversely.*





Thus the polygons *PQRS* and *pqrs* which satisfy this condition are similar, and the following relations hold :—

$$\frac{PQ}{pq}=\frac{QR}{qr}=\frac{RS}{rs}=\frac{SP}{sp}=\frac{OP}{Op}=\frac{OQ}{Oq}=\frac{OR}{Or}=\frac{OS}{Os}.$$

The point O is called the pole.

The student will find many illustrations of this theorem. Thus in the figure of Art. 6 the pole is situated *within* the polygon. In Prob. 36 the pole coincides with one corner A. In Prob. 24 the pole is inaccessible ; it might be at an infinite distance away, when the lines through it would be parallel.

Theorem 3.—*The areas of similar figures are proportional to the squares on any two corresponding sides.*

Thus $$\frac{\text{area } PQRS}{\text{area } pqrs}=\frac{(PQ)^2}{(pq)^2}=\frac{(QR)^2}{(qr)^2}=\frac{(OP)^2}{(Op)^2}=\text{ etc.}$$

or $$\frac{\text{area } ONn}{\text{area } OKk}=\frac{(Nn)^2}{(Kk)^2}=\frac{(ON)^2}{(OK)^2}=\text{ etc.}$$

19. Problem.—**Two triangles ABC, DEF are given. It is required to draw a triangle def with its vertices d, e, f in BC, CA, AB, and its sides de, ef, fd parallel to DE, EF, FD.**

Between AB and AC draw any line $F'E'$ parallel to FE; draw $F'D'$, $E'D'$ parallel to FD, ED.

Join AD', and let this line (produced) meet BC in d.

Through d draw de, df parallel to DE, EF, and join ef.

Then def is the triangle required.

20. Problem.—**A triangle ABC and a quadrilateral DEFG are given. It is required to draw a quadrilateral defg similar to DEFG, with its side de in AB, and its vertices f, g in BC, CA respectively.**

Copy the figure $DEFG$ in the position $D'E'F'G'$; that is, with $D'E'$ in AB, and G' in AC.

Draw AF' to intersect BC in f.

Then draw fg, gd, fe parallel to $F'G'$, $G'D'$, $F'E'$.

We thus obtain the inscribed quadrilateral as required.

21. Problem.—**Two triangles ABC, DEF are given. It is required to draw a triangle def similar to DEF, with f at a given point in AB, and d, e in BC, CA respectively.**

From the given point f draw fE' to meet BC at an angle $fE'B = FED$. From E' draw $E'e$, making the angle $CE'e = DFE$.

Join ef. From e draw ed, where the angle $fed = FED$. Join df. Then def is the triangle required.

Note 1.—This solution is based on the following theorem.

Theorem.—If a triangle of varying size but constant shape rotate or swing about one vertex A, while a second vertex P traces any figure, then the third vertex Q traces a similar figure turned through an angle A, the linear dimensions of the two figures being in the ratio $AP : AQ$.

The figure shows two positions $fD'E'$, fde of such a triangle. As one vertex moves from D' to d, the other moves along $E'e$, inclined at β to $D'd$. See also Ex. 26, p. 25.

Note 2.—With modified data some of the points would be situated in the sides AB, BC, CA produced. The student will now be able to work examples relating to triangles, squares, parallelograms, etc., inscribed in and circumscribed about triangles.

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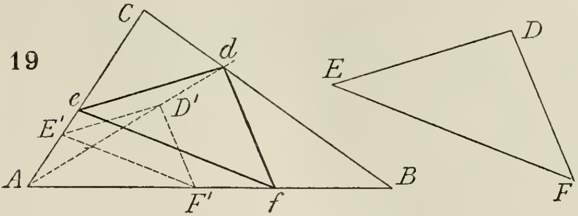

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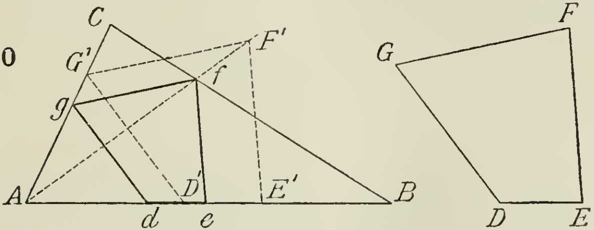

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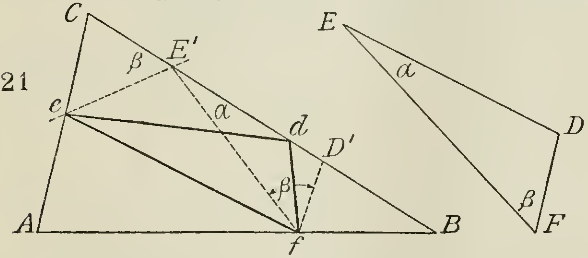


Examples.—1. Draw a triangle *ABC*, making *AB* = 4″, *BC* = 3.5″, *CA* = 2.5″. In *ABC* inscribe an equilateral triangle with one side inclined at 45° to *AB*.

2. In the triangle of Ex. 1 inscribe a square with one side in *AB*.

3. In *ABC* inscribe an equilateral triangle with one vertex bisecting *AB*.

4. In *ABC* of Ex. 1 inscribe a similar triangle *abc*, with *c* at the middle point of *BC*, *a* in *AB*, and *b* in *AC*.

5. In *ABC* inscribe a parallelogram with sides in the ratio 2 : 3, and included angle 60°, a longer side lying in *BC*.

6. Draw a circular arc, 3″ radius, and two radii including 60°. In this sector inscribe a square with a side along a radius.

22. PROBLEM.—**Two lines AB and CD and a point P are given. It is required to draw through P a line terminated by the given lines, and divided at P into two segments which are in a given ratio, say 2 : 3.**

Through *P* draw any line meeting one of the given lines, say *AB*, at *E*.

Bisect *PE* at *G*, and set off $PF = 3\ PG$. Draw *FS* parallel to *AB*, meeting *CD* in *S*.

Then the line through *S* and *P* meeting *AB* in *R* is the one required.

For, since the triangles *PFS*, *PER* are similar,

$$\frac{PR}{PS} = \frac{PE}{PF} = \frac{2PG}{3PG} = \frac{2}{3}.$$

Note.—If *PF* were made equal to *PE*, then *SR* would be bisected at *P*.

23. PROBLEM.—**To draw the straight line bisecting the angle between two given straight lines AB and CD, the intersection of the latter being inaccessible.**

Draw any line *EF* between the given lines, and bisect the angles *AEF* and *CFE* by lines *EO* and *FO* meeting at *O*.

Also bisect the angles *BEF* and *DFE* by lines *EO′* and *FO′* meeting at *O′*.

Then the line *OO′* will be the line required.

For it is obvious that *O* is equidistant from *AB*, *EF*, and *CD*; so also is the point *O′*.

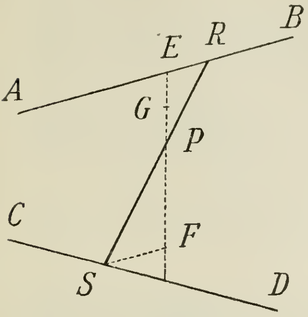
Hence *OO′* bisects the angle between *AB* and *CD*.

24. PROBLEM.—**Having given two straight lines AB and CD and a point P, it is required to draw the straight line through P, such that the three lines converge to the same point, the point being inaccessible.**

Take any convenient points *E* and *F* in *AB* and *CD* respectively, and join *EF*, *FP*, and *PE*.

Parallel to *EF* draw any corresponding line *GH*; draw *GQ*, *HQ* parallel to *EP*, *FP*. Join *PQ*.

Then the lines *AB*, *CD*, and *PQ* if produced would all meet at the same point.



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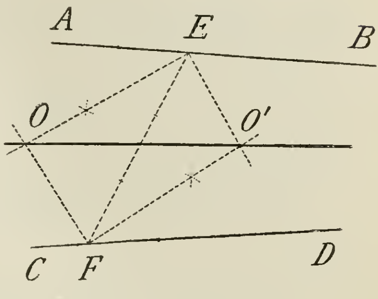

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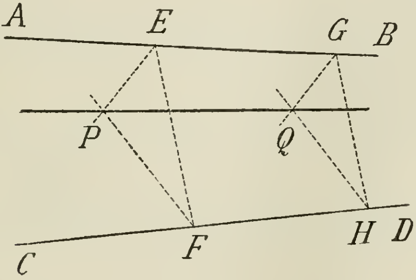

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Examples.—**1.** Draw a quadrilateral $ABCD$ of the following dimensions :

Sides—$AB = 3.7''$; $BC = 3.0''$; $CD = 3.40''$; $DA = 3.30''$. Diagonal—$BD = 4.4''$. Draw the two bisectors of the angles between AB, DC, and AD, BC.

2. Let the diagonals of $ABCD$, Ex. 1, intersect in P. Through P draw the two lines which converge respectively to the same points as the pairs of opposite sides.

3. Through the point P, Ex. 2, draw the two lines between the pairs of opposite sides of the quadrilateral which are bisected at P.

4. Draw an equilateral triangle ABC of $3''$ side, and take points D, E in AB, AC distant $1.9''$ and $1.8''$ from A. Determine the line through A which converges to the same point as BC and DE.

25. PROBLEM.—**To find a fourth proportional D, to three given lines A, B, and C; that is, to find a line D such that A : B : : C : D.**

Take any two lines intersecting at a point O; along one set off OA, OC equal to A and C respectively; and along the other set off OB equal to B. Join AB, and draw CD parallel to AB.

Then OD is the required length of D.

If AE be drawn parallel to CB, then OE is the length of a line E such that $C : B :: A : E$.

26. PROBLEM.—**To find a third proportional C, to two given lines A and B; that is, to find a line C such that A : B : : B : C.**

Take any two lines intersecting at O; along one set off OA, OB' equal to A and B respectively; and along the other set off OB equal to B.

Join AB and draw $B'C$ parallel to AB. Then OC is the required length of C, such that $A : B :: B : C$.

27. PROBLEM.—**To find a mean proportional C between two given lines A and B; that is, to find a line C such that A : C : : C : B.**

Take a point O in any line, and set off in opposite directions OA equal to A, and OB equal to B.

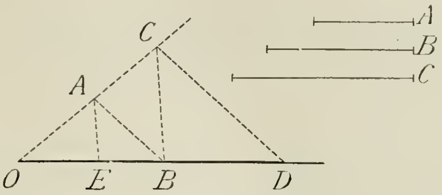
Describe a semicircle on AB. Then OC, drawn perpendicular to BA, is the required length of C (Euc. VI. 13).

C is also called the *geometrical mean* between A and B.

28. PROBLEM.—**To divide a line AB in extreme and mean ratio; that is, to find a point C in AB such that AB : AC : : AC : CB.**

Bisect AB at D; draw BE perpendicular to AB, and equal to BD, and join EA. With centre E describe arc BF; and with centre A describe arc FC.

This gives C, the required point of division (Euc. VI. 30).



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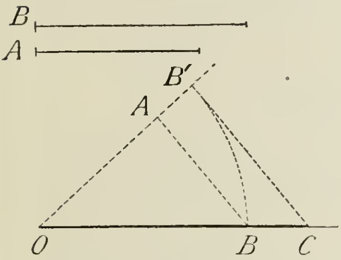

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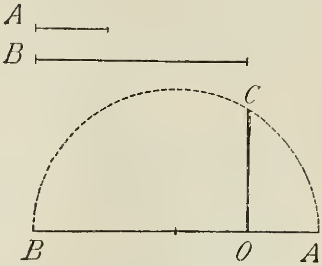

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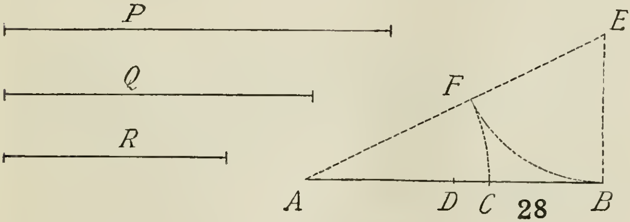

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Examples.—1. Find the fourth proportionals to P, Q, R; to R, Q, P; and to Q, P, R. *Ans.* .92″; 2.78″; 1.45″.

2. Find the third proportionals to P and Q; to Q and P; and to R and P. *Ans.* 1.28″; 2.53″; 3.48″.

3. Find the mean proportionals between P and Q; Q and R; and R and P. *Ans.* 1.79″; 1.36″; 1.52″.

4. Divide P in extreme and mean ratio, and give the value of the ratio. *Ans.* 1.62.

29. PROBLEM.—**To find the harmonic mean between two given lengths a and b; that is, to determine a length k such that b − k : k − a : : b : a.**

From any point H in a straight line mark off HA, HB equal to a and b.

Through A and B draw any two lines AO, BO perpendicular to one another. Join OH and set off the angle AOK equal to the angle AOH.

Then HK is the required length of k, the harmonic mean between a and b; and HA, HK, HB are in harmonic progression.

Notes.—The relation $b - k : k - a = b : a$ is equivalent to

$$\frac{KB}{AK} = \frac{HB}{HA}; \text{ or } \frac{AH}{KA} = \frac{BH}{BK};$$

and comparing these equations we see that BK, BA, BH are also in harmonic progression.

The line AB is said to be *harmonically divided* in H and K, and the line HK in A and B. And four such points are spoken of as a *harmonic range*.

The series of lines or *pencil of rays* formed by joining H, A, K, B with any point O' is called a *harmonic* pencil, because any transverse line or *transversal* can be shown to be cut harmonically by the pencil.

Thus H', A', K', B' and B_1, H_1, A_1, K_1 are harmonic ranges. Stated formally—

Theorem 1.—*A harmonic pencil divides all transversals harmonically.*

The alternate points or rays are said to be *conjugate* to one another. Thus A and B, or OA and OB are conjugate. Likewise OH and OK.

Theorem 2.—*A transversal which is parallel to a ray is bisected by the conjugate.*

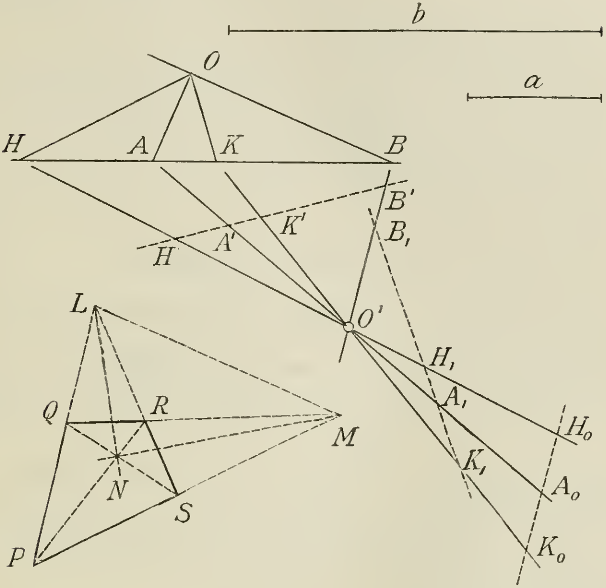
Thus the transversal K_0H_0 is parallel to $O'B$, and is bisected in A_0. We also have

Theorem 3.—*If a transversal parallel to any one of the four rays of a pencil is bisected, the pencil is harmonic.*

Thus if OC bisect any angle AOB, and OD be taken perpendicular to OC, the pencil is harmonic because any transversal parallel to OD or OC is bisected.

A *complete quadrilateral* affords an example illustrating harmonic pencils. See the figure.

Let the diagonals of any quadrilateral $PQRS$ intersect in N, and the pairs of opposite sides produced in L, M. Join L, M, N. Then the pencils radiating from L, M, N are all harmonic.



Examples.—**1.** Take a and b each double the length given in the figure above, and find the harmonic mean k.

2. Find the arithmetical mean r, and the geometrical mean g, between a and b, and vertify the theorem that r, g, and k are in geometrical progression.

3. Draw any complete quadrilateral $PQRSLMN$, and test whether the pencils through L, M, N are harmonic, by applying Theorem 3 of this article.

4. Taking the lines a and b in the figure above, find a line c such that a, b, c are in harmonic progression. *Ans.* $-2.46''$.

5. Draw OA, OH, OB, making the consecutive angles AOH, HOB equal to 25° and 55° respectively. Find the fourth ray of the harmonic pencil, and measure the angle it makes with OB. *Ans.* 70.8°.

6. Through the point b, Fig. 6, p. 47, draw a straight line cutting oa, oc in d and e, such that the triangles odb, obe are equal in area.

7. Draw a line AB 2″ long, and divide it internally and externally at K and H into segments which have the same ratio, say 3 : 4.

30. Problem.—**To reduce a given polygon ABCDEF to a triangle of equal area.**

Join D to B, A, and F. Draw CG parallel to DB to meet AB produced in G, and join DG. Draw EH parallel to DF to meet AF produced in H, and draw HK parallel to DA to meet BA produced in K. Join DK.

Then DKG is a triangle having an area equal to the polygon.

This solution is based on the theorem that triangles on the same base and between the same parallels are equal in area.

31. Problem.—**To construct a square equal in area to a given rectangle.**

Determine a mean proportional between the sides of the rectangle (Prob. 27); this is equal to the side of the required square.

32. Problem.—**To construct a square equal in area to a given triangle.**

Let a rectangle be drawn, one side of which is equal to the base of the given triangle, and an adjacent side equal to half the altitude.

The rectangle so drawn is equal in area to the triangle.

Hence the problem reduces to the last one.

33. Problem.—**To construct a square equal in area to a given polygon.**

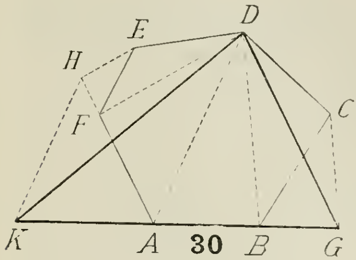
Reduce the given polygon to an equivalent triangle (Prob. 30). Then apply Problem 32.

34. Problem.—**To construct a rectangle equal in area to a given rectangle, and having one side equal to a given line.**

Let $ABCD$ be the given rectangle, and AE the given side of the required rectangle.

Draw EF equal and parallel to BC; join FA cutting BC at H; through H draw MK parallel to AE.

Then ME is the required rectangle.



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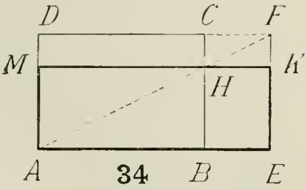

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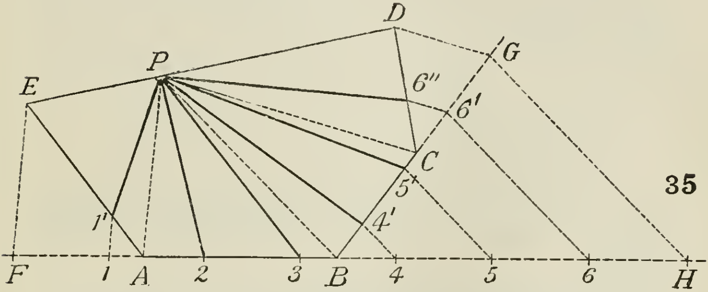

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35. Problem.—**To divide a given polygon ABCDE into a number of equal areas by lines drawn from a given point P in one side. Say into seven parts.**

As in Prob. 30, draw EF parallel to PA; and DG, GH parallel to PC, PB, thus reducing the polygon to an equivalent triangle with vertex P and base FH.

Divide FH into the required number of equal parts.

Then, reversing the construction, draw $1 1'$ parallel to FE; $4 4'$, $5 5'$, $6 6'$ parallel to HG; and $6' 6''$ parallel to GD.

Join P to $1'$, 2, 3, $4'$, $5'$, $6''$, and the problem is solved.

Examples.—1. Draw a pentagon $ABCDE$ as follows:—
Sides $AB = 1\frac{3}{4}''$; $BC = 2''$; $CD = 2\frac{1}{4}''$; $DE = 2\frac{1}{2}''$.
Angles $ABC = 120°$; $BCD = 80°$; $CDE = 125°$.

Reduce the figure to an equal triangle with vertex D and base along AB. Then reduce the triangle to an equal square.

2. Divide the pentagon of Ex. 1 into eight equal parts by lines drawn through the middle point of DC.

36. Problem.—**It is required to draw a polygon similar to a given polygon, and having an area which bears to the area of the latter a given ratio, say 3 : 5.**

Let $ABCDE$ be the given polygon. In *any* side BA, produced in either direction, say beyond A, set off Ap and AP equal to 3 and 5 on any suitable scale.

On Bp, BP draw semicircles as shown; and from A draw a line perpendicular to AB, meeting these semicircles in q and Q. Join QB, and draw qb parallel to QB.

Then Ab is one side of the required figure, and the latter $AbcdeA$ is completed by drawing the diagonals AC, AD, and then the sides bc, cd, de, parallel to BC, CD, DE.

For $(Aq)^2 = AB \,.\, Ap$, (Euc. III. 35, or Prob. 27),
and $(AQ)^2 = AB \,.\, AP$;

$$\text{then } \frac{\text{area } AbcdeA}{\text{area } ABCDEA} = \frac{(Ab)^2}{(AB)^2} = \frac{(Aq)^2}{(AQ)^2} = \frac{AB \cdot Ap}{AB \cdot AP} = \frac{Ap}{AP} = \frac{3}{5}.$$

37. Problem.—**To divide a given triangle ABC into two equal areas, by a line drawn through any given point P.**

Bisect the sides of the triangle in D, E, F.

Join AD, BE, CF and bisect these lines in G, H, K.

Join HK, intersecting OD in L, and bisect OL in M. And through H, M, K draw a *fair* curve *touching* BH and CK at H and K.

In like manner draw the curves GH, GK.

Through P draw PQ so as to touch one or other of the three curves, according to the situation of P.

Then PQ divides the triangle into two equal areas.

Note.—The true curves should be hyperbolas, to which the sides of the triangle are asymptotic (see Chapter IV.). The bisection of OL in M makes them parabolas, from which the true curves for such small arcs do not visibly differ.

Alternative Solution.—Along AB, AC set off AN, AN each equal to the geometrical mean between AB and AE. Bisect NN in T. Join AT and produce to S, making $AS = AN$. Then T is the vertex, S a focus, and A the centre of the hyperbola KH, and the tangent through P may be found by a construction analogous to that of Prob. 94.

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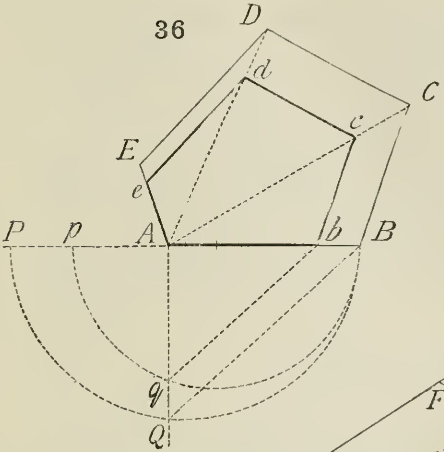

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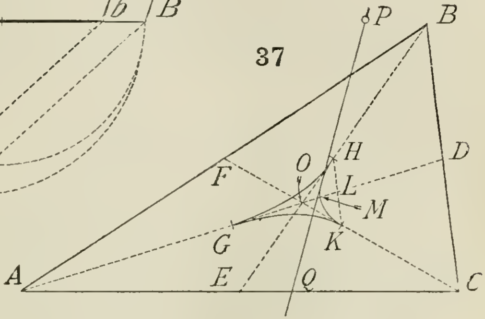


Examples.—1. Draw a triangle ABC having given $AB = 3.0''$, $BC = 3.2''$, $CA = 1.8''$. Divide the triangle into two equal areas by a line parallel to AB.

2. Divide the triangle of Ex. 1 into five equal areas by lines parallel to BC.

Note.—Divide AB into five equal parts, and from the points of division draw perpendiculars to meet a semicircle on AB in 1, 2, 3, 4. With centre A draw arcs through 1, 2, 3, 4 to meet AB in 1′, 2′, 3′, 4′. Draw lines through the latter points parallel to BC.

3. Draw a pentagon $ABCDE$ as follows:—

Sides—$AB = 1.7''$; $BC = 1.35''$; $CD = 1.45''$; $DE = 1.65''$; $EA = .8''$. Diagonals—$AC = 2.5''$, $AD = 2.2''$.

Draw two similar pentagons, one $\frac{2}{3}$ and the other $\frac{3}{2}$ the area of $ABCDE$.

4. Divide the pentagon of Ex. 3 into four equal areas by lines parallel to the sides.

5. Divide the triangle of Ex. 1 into two parts of equal area, by a line passing through a point P outside the triangle, the position of P being given by $AP = 1''$; $BP = 2.8''$.

38. Problem.—**To find the number of square units of area in any given polygon.**

Let *ABCD* be the square equal in area to the polygon, obtained as in Prob. 33.

Produce one side *DA* to *S*, making *AS* one unit of length. Join *SB*, and draw *BT* perpendicular to *BS*.

Then *AT*, measured on the scale having *AS* as unity, gives the number of square units of area required.

Thus if *AS* be set off 1″, the area in square inches is obtained by measuring *AT* on the 1″ scale. Or if *AS* be 1 centimetre long, then *AT* measured on the centimetre scale gives the area in square centimetres.

Note that *AT* is a third proportional to *AS* and *AB*; and that *AB* is a mean proportional between *AS* and *AT*.

To construct a square of given area, say 3 square inches, draw a rectangle 3″ long and 1″ broad, and reduce to an equivalent square. Or set off $AT = 3''$, $AS = 1''$, and find the mean proportional *AB*.

39. Problem.—**To construct a polygon similar to a given polygon ABCD, and having an area equal to that of a given polygon Q.**

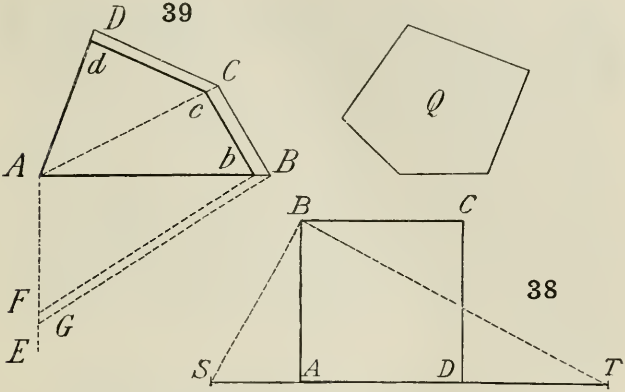
As in Prob. 40, determine *AF*, the side of a square equal in area to *Q*; and also *AG*, the side of a square equal in area to *ABCD*. (Construction not here shown.)

Set off *AF* and *AG* along a line *AE*, making any angle with *AB*; join *GB*, and draw *Fb* parallel to *GB*.

Join *AC*, and draw *bc* and *cd* respectively parallel to *BC* and *CD*. Then *Abcd* is the required rectilinear figure.

For $$\frac{\text{area of } Abcd}{\text{area of } ABCD} = \frac{Ab^2}{AB^2} = \frac{AF^2}{AG^2} = \frac{\text{area of } Q}{\text{area of } ABCD}.$$

It is important that the reader should observe that this problem is the *general case* of a distinct *type* or class of problems, and that the method of solution is of a *general* character. Two figures, which may vary in size and shape, are given, and a third figure has to be determined, such that it is similar to one of the given figures and equal in area to the other. This is the type of problem. The *data* of the problem may be varied considerably, in so far that two triangles, rectangles, squares, etc., or any combination of these, may be substituted for the pair of figures given above; but the problem remains of the same character, and the same *method* of solution is employed.



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Such observations should render it unnecessary to explain in detail the solutions of further problems of the same kind, such as the examples below.

It may here be remarked that the student should acquire the habit of closely observing the type of any problem which comes under his consideration, and constantly aim at classifying problems, with the object of detecting general methods where such may exist, rather than to regard each problem as having no relation or resemblance to previous problems. In this manner the power of successfully attacking examples may often be greatly increased.

Examples.—1. Construct an isosceles right-angled triangle equal in area to a given regular pentagon $1\frac{1}{4}''$ side.

2. Construct an isosceles triangle, vertical angle 40°, equal in area to a triangle whose sides are $2\frac{1}{2}''$, $1\frac{3}{8}''$, and $2.1''$.

3. Construct a rhombus, one angle of which is 60°, equal in area to a given rectilineal figure *ABCD*, of which *AD* is 2″, angle *BAD* 105°, *AB* 2″, *BC* $3\frac{1}{2}''$, *DC* 1.7″.

4. Determine an equilateral triangle having an area of 4 square inches. Measure the side. *Ans.* 3.04″.

5. Construct a regular heptagon having an area of 5 square inches.

6. Construct a rectangle equal in area to a hexagon of $1\frac{1}{4}''$ side, the ratio of the sides of the rectangle being 2 : 3.

7. Determine graphically the area of the quadrilateral *ABCD* of Ex. 3, in square inches and also in square centimetres. *Ans.* 4.62 ; 29.8.

40. Problem.—**Having given two similar rectilinear figures, it is required to construct a similar figure, the area of which shall be equal to the sum or difference of the areas of the given figures.**

Let $ABCDE$, $Abcde$ be the two given similar figures.

For the sum—draw BN at right angles to AB, and make $BN = Ab$.

With A as centre, describe the arc NB' to meet AB produced in B'; draw $B'C'$, $C'D'$, $D'E'$ respectively parallel to BC, CD, and DE, to meet AC, AD, AE produced in C', D', and E'. Then $A'B'C'D'E'$ is the required figure.

For the difference—describe a semicircle on AB as diameter.

With centre B, radius Ab, cut this semicircle in M;

With centre A, radius AM, draw the arc MB''.

Then a polygon with AB'' as one side, and similar to the others, is the figure required.

Note.—These constructions are based on the following theorem, of which Euc. I. 47 is a particular case.

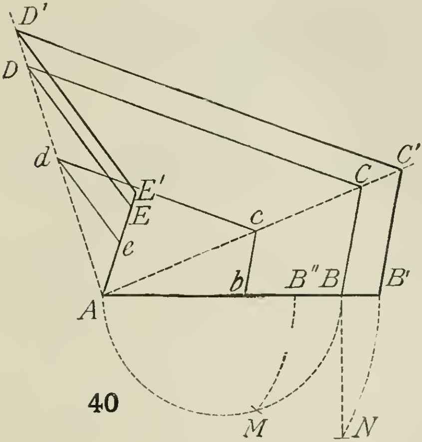
Theorem.—If similar polygons be described on the three sides of a right-angled triangle ABC, so that AB, BC, CA are corresponding sides of the three figures respectively, then the areas of the polygons described on the sides AB and BC are together equal to the area of that described on the hypothenuse AC.

41. Problem. — **Having given any three rectilinear figures, which may be denoted by A, B, and C, to determine a fourth figure which shall be similar to C, and have an area equal to the sum or difference of the areas of A and B.**

Determine, by means of Prob. 39, a figure E similar to A, and equal in area to B.

By Prob. 40 determine a figure F similar to A (or E), and having an area equal to the sum or difference of the areas of A and E.

Finally, by means of Prob. 39, determine a figure G similar to C, and having an area equal to that of F; then G is the required figure.



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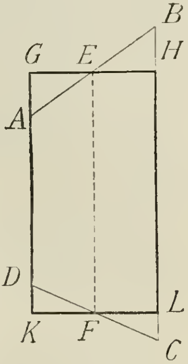

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42. Problem.—**To find a rectangle equal in area to a given trapezoid ABCD.**

Let AD, BC be the parallel sides of the trapezoid.

Bisect AB, CD in E, F. Then EF is parallel to AD and BC, and is midway between them. Through E and F draw GH and KL perpendicular to EF.

Then the rectangle GL is equal in area to the trapezoid, and the area is equal to $EF \times KL$, or $\frac{AD + BC}{2} \times KL$.

Examples.—1. Draw two equilateral triangles respectively equal to the sum and difference of two equilateral triangles of 2″ and 3″ sides.

2. Draw two circles respectively equal to the sum and difference of two circles of $1\frac{1}{2}''$ and $2\frac{3}{4}''$ diameters.

3. Draw a square equal to the sum of the areas of two equilateral triangles of 1.7″ and 2.8″ sides.

4. Draw an equilateral triangle equal to the difference of two squares of 1.2″ and 2.7″ sides.

5. Draw a square equal to the sum of an equilateral triangle and a regular pentagon, each of $1\frac{1}{2}''$ side.

6. Draw an equilateral triangle equal to the difference of a square and a regular hexagon, each of 2″ side.

43. Problem.—**To determine approximately the area enclosed by a given irregular curve AB, a base line CD, and two perpendicular ordinates CA, DB.**

A perpendicular erected at any point on the base *CD* of the figure is called an *ordinate.*

Draw a series of equidistant ordinates between *CA* and *DB*, dividing the area into a number of strips of equal width; in this case six. One of these strips is shown with shade lines.

Draw a second series of intermediate ordinates, midway between those first drawn, and let $y_1, y_2, \ldots y_6$ denote their lengths.

Determine the mean ordinate $y_m = \frac{y_1 + y_2 + \ldots + y_6}{6}$.

Take *CK* equal to y_m and through *K* draw *KL* parallel to *CD.* Then the area of the rectangle *KD* is equal to area required.

The result is only approximate, because the upper boundary lines of the strips are curved instead of straight. By increasing the number of strips, and so making the width of each less, the error can be reduced to a very small proportion of the total area.

We now state some other rules, available for calculating approximately the mean ordinate y_m; that is, the height *CK* of the equivalent rectangle.

In employing these we require the ordinates which *bound* the strips (not the middle ordinates of the spaces). In the figure there are seven such ordinates to the six divisions, *CA* being the first, and *DB* the last. These may be denoted by $h_0, h_1, h_2, h_3, h_4, h_5, h_6$.

If there were n equal divisions, or strips, there would be $n+1$ ordinates

$$h_0, h_1, h_2, \ldots \ldots h_{n-1}, h_n.$$

With this notation, the rules in question may be written as follows:

Ordinary rule.—Any number of divisions.

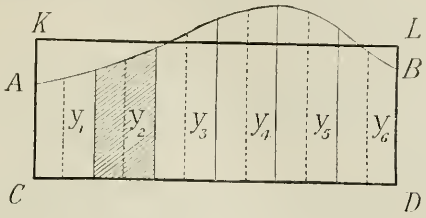
$$y_m = \frac{1}{n}\left\{\frac{1}{2}(h_0 + h_n) + h_1 + h_2 + h_3 + \ldots + h_{n-1}\right\}.$$

Simpson's first rule.—Number of divisions even.

When $n = 2$. $y_m = \frac{1}{6}(h_0 + 4h_1 + h_2)$.

When $n = 2$, or 4, or 6, etc.

$$y_m = \frac{1}{3n}\left\{h_0 + h_n + 2(h_2 + h_4 + \ldots + h_{n-2}) + 4(h_1 + h_3 + \ldots + h_{n-1})\right\}.$$



Simpson's second rule.—Number of divisions a multiple of 3.
When $n=3$. $y_m=\frac{1}{8}(h_0+3h_1+3h_2+h_3)$.
When $n=3$, or 6, or 9, or 12, etc.

$$y_m=\frac{3}{8n}\{(h_0+h_n)+2(h_3+h_6+\ .\ .\ .\ +h_{n-3})+3(h_1+h_2+h_4+h_5+.\ .\ .+h_{n-2}+h_{n-1})\}.$$

Weddle's rule.—Number of divisions a multiple of 6.
When $n=6$. $y_m=\frac{1}{20}\{h_0+h_2+h_4+h_6+5(h_1+h_3+h_5)+h_3\}$.
When $n=6$, or 12, or 18, etc.

$$y_m=\frac{3}{10n}\{h_0+h_2+.\ .\ .+h_n+5(h_1+h_3+.\ .\ .+h_{n-1})+h_3+h_6+.\ .\ .+h_{n-3}\}.$$

Examples.—1. Draw a quadrant of a circle 3″ radius. Divide one of the radii into six equal parts, and draw the ordinates, which will be parallel to the other radius. Determine the mean ordinate by each of the rules given, and compare the results. Set out the equivalent rectangle. Reduce to an equivalent square. And find the area of the quadrant in square inches. *Ans.* Mean ordinate = 2.36″. Side of square = 2.66″. Area = 7.07 sq. inches.

2. Plot the curve for which thirteen equidistant ordinates, spaced $\frac{1}{2}$″ apart, have the following successive values :—

. 4.1″, 4.0″, 4.0″, 3.9″ ; 3.55″, 3.16″, 2.73″, 2.34″, 2.05″, 1.77″, 1.60″, 1.45″, 1.0″.

Obtain the mean ordinate by each of the rules given. Draw the equivalent rectangle, and determine the area under the curve in square inches. *Ans.* Mean ordinate = 2.76″. Area = 16.6 sq. ins.

3. In Ex. 2, Art. 156, find the average school attendance for the six years 1892-97. *Ans.* 119,100.

44. Miscellaneous Examples.

Note.—The figures are to be copied twice size.

***1.** Divide the given triangle ABC into four equal parts by lines drawn parallel to AB. (1896)

***2.** Draw a triangle similar to the given triangle ABC, but with a perimeter equal to the harmonic mean between the lines LM and PQ. (1896)

***3.** Divide the given line A into two parts, such that the sum of the squares on them shall be equal to the square on the given line B. State what limitations are necessary as to the length of B in order that the problem may be possible. (1888)

Hint.—The solution of a problem is often suggested by an algebraical investigation.

Thus let x be one part, and therefore $A - x$ the other. Then by the conditions $x^2 + (A - x)^2 = B^2$. Solving which, we find

$$x = \tfrac{1}{2}\left\{A \pm \sqrt{(\sqrt{2}B)^2 - A^2}\right\}.$$

Now $\sqrt{2}B$ is the diagonal of a square on the side B. If this be made the hypothenuse of a right-angled triangle, and A one of the sides, the other side is $\sqrt{(\sqrt{2}B)^2 - A^2}$. The rest of the construction is simple.

***4.** Between the lines ab, cd produced if necessary, place a line parallel to ef in such a position as to form a four-sided figure with an area of 2 square inches. (1891)

Hint.—Produce ab, cd to meet in o. Find the area of the triangle ofe. Call this m square inches. Then find the triangle oFE similar to ofe, the areas of the two having the ratio $2 + m : m$.

***5.** Draw a line ab parallel to the base AC of the given triangle ABC, cutting off segments Aa on AB, Cb on CB, such that their difference $Aa - Cb$ shall be equal to one inch. (1895)

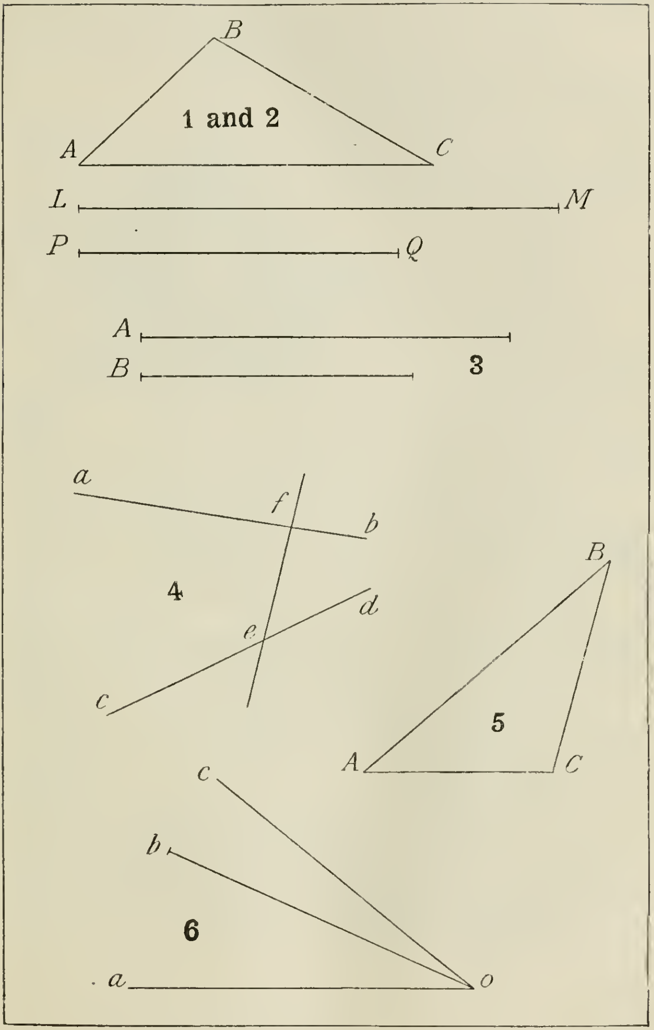
Hint.—Aa may be found as the fourth term in the following proportion—

$$AB - BC : AB :: Aa - Cb : Aa..$$

***6.** Construct an equilateral triangle which has one angular point at b on the line ob, and the remaining angular points on oa and oc respectively. (H. 1886)

7. Draw a triangle ABC with the following dimensions. $AB = 4''$, $BC = 2\frac{3}{4}''$, $AC = 2''$. Inscribe in the triangle a rhombus, one side lying in AC, and the adjoining sides inclined to AC at 45°. (1894)

8. Given two lengths $P = 2.25''$, $Q = 2.70''$. Find a line whose length x is such that $P^2 = (x - Q)\,x$. Write down the value of x. (1894)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Hint.—P is a mean proportional between $(x-Q)$ and x. Therefore in Fig. 27, page 33, make $OB=P$, $OC=Q$. Draw CA perpendicular to CB. Then $AB=x$.

9. In Ex. 8 find x if $P^2=(Q-x)x$.

***10.** Reduce the given hatched figure to a triangle. (1883)

Hint.—Reduce the whole figure and the hole to triangles. Then determine a triangle equal to the difference of the two.

***11.** Through the given point p draw two lines cutting the two given lines in such a way that the included area is $3\frac{1}{4}$ square inches. (1887)

Hint.—Through p draw any two lines intersecting the given lines in a, A; b, B. Reduce $AabB$ to a square, and find its area in square inches by Prob. 38. The length of AB required for an area of $3\frac{1}{4}$ square inches may now be found as a fourth proportional to area $AabB$, $3\frac{1}{4}$, AB.

***12.** In what sense can areas be represented by straight lines? If the area of the given triangle ABC is represented by a line 1″ long, draw a line representing the area $DEFG$. (1885)

***13.** Reduce the given figure to a rectangle on the base ab. (1886)

Hint.—Apply a construction similar to that of Prob. 30. Thus draw fl, lm, mn respectively parallel to ge, gd, gc, to meet the lines ed, dc, ca. Join gn, and through its middle point draw a line parallel to ab. We thus obtain the required rectangle.

***14.** Draw a straight line from A and terminating on FG, such that the sum of the areas included between it and the given zigzag line on the one side is equal to the sum of those on the other. (1885)

15. Draw a quadrilateral figure having an area of 3 square inches, and made up of two isosceles triangles having a common base which is a mean proportional between the sides of the triangles. The vertical angle of the smaller triangle is 90°. (1881)

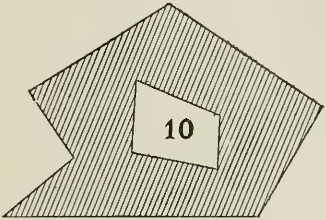
16. A line $1\frac{1}{2}''$ long represents a square of 2″ side. Determine a length of line which would represent on the same scale a hexagon of $1\frac{3}{4}''$ side. (1883)

17. Draw a right-angled triangle with hypothenuse 2″ and one side $1\frac{3}{4}''$. Prove by construction that the square on the hypothenuse is equal to the sum of the squares on the sides. (1893)

Hint.—One method is to apply the construction of Prob. 38.

18. Draw an equilateral triangle of 2″ side, and circumscribe this by a right-angled isosceles triangle, so that one of the equal sides of the latter is bisected.

Copy the figures double size



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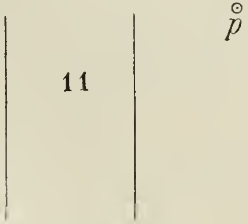

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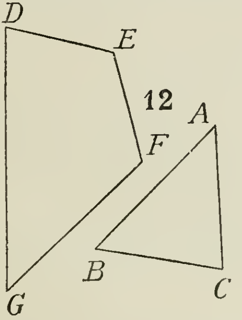

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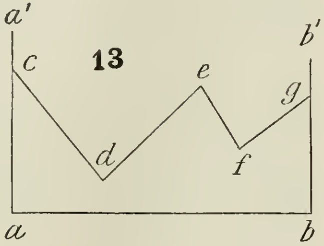

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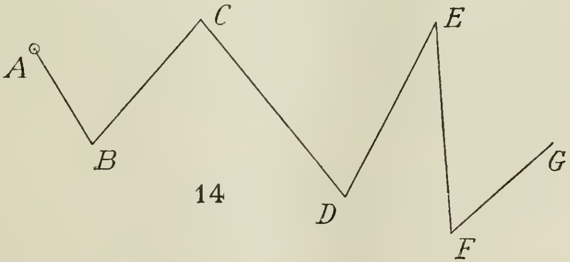

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CHAPTER III

TRIANGLES, CIRCLES AND LINES IN CONTACT

45. **Introduction.**—The problems in this chapter relate mainly to the contact of lines and circles, and the construction of triangles from adequate data.

Euclid permits a line to be drawn between two definite points, but does not allow a common tangent to be drawn to two given circles without having previously determined the points of contact by a special construction.

This construction may be necessary in a strict system of deductive reasoning like Euclidean geometry, but does not add to the accuracy in a scale drawing, as the student may easily test for himself.

So in "practical" geometry a tangent may be drawn to a circle from an external point, or a common tangent to two circles, by simply adjusting the straight-edge and drawing the line without any previous construction. Then if the point of contact be required, it is necessary to draw the perpendicular radius.

If a tangent is required at a point on the circumference, it must be drawn perpendicular to the radius to the point.

With care, a circle may be drawn with a given centre to touch a given line, without first finding the point of contact. But in this case it is generally preferable to draw the perpendicular from the point to the line before drawing the circle.

Before working the problems of this chapter, the student should illustrate the truth of the following theorems, by making accurate drawings to scale, measuring the results and making calculations where necessary. For several reasons this is a most valuable form of exercise; by comparing his results with the true ones he may observe how small need the errors be which he introduces into his graphical work. It will not be necessary to add that he must see that his pencil is in proper condition.

Theorem 1.—*Angles in the same segment of a circle are equal to each other* (Euc. III. 21).

Theorem 2.—*The two tangents drawn from a point to a circle are equal to each other* (Euc. III. 17).

Theorem 3.—*If through any point in the circumference of a circle a chord and tangent be drawn, the angles between them are equal to the angles in the alternate segments of the circle* (Euc. III. 32).

Theorem 4.—*If any point P be taken inside or outside a circle and two lines be drawn through P, one cutting the circle at A and B, and the other at C and D, the product of PA and PB is equal to the product of PC and PD* (Euc. III. 35).

Also if P be outside the circle and a line PT be drawn to touch the circle at T, the square of PT is equal to the product of PC and PD, or of PA and PB (Euc. III. 36).

Theorem 5.—*If a line which bisects the vertical angle A of any triangle ABC, cut the base BC in D, the ratio of BD to DC is the same as the ratio of BA to AC* (Euc. VI. 3).

Also if a line bisecting the exterior angle at A cut the base BC produced in D', the ratio of $D'C$ to $D'B$ is the same as that of AC to AB (Euc. VI. 3).

D and D' are said to divide BC *internally* and *externally* in the ratio of the sides of the triangle. (Or AB, AD, AC, AD' form a harmonic pencil. See Prob. 29.)

Theorem 6.—*If A and B are two fixed points, and a point P move so that the ratio $PA : PB$ is constant, then the* locus *of P is a circle.*

46. PROBLEM.—**On a given line AB, to describe a regular polygon; say a heptagon.**

First Method.—Produce *AB*, and with centre *B*, radius *BA*, describe the semicircle *AC*.

By *trial*, divide the semicircle into as many equal parts as the polygon has sides, in this case seven. Join *B* to the *second* point of division from *C*.

Then *B*2 is a second side of the required heptagon.

Draw a circle through *A*, *B*, 2. This is the circumscribing circle of the required polygon, and the length *AB* should step exactly seven times round the circumference.

Second Method.—Produce *AB*, and set off the angle *CB*2 equal to $360^\circ \div 7$, that is to 51.4°.

Make *B*2 equal to *BA*. Then proceed as before.

Note.—To inscribe a regular polygon of n sides in a circle, divide the circumference by trial with the dividers into *n* equal parts; *or*, find one side by setting off from the centre an angle equal to $360^\circ \div n$.

47. PROBLEM.—**On a given line AB, to construct a segment of a circle which shall contain an angle equal to a given angle α.**

Draw *CD* bisecting *AB* at right angles, and make the angle *DCE* equal to α. Draw *AO* parallel to *CE*.

Describe a circle with *O* as centre, radius *OA*; then that segment of the circle, on the side of *AB* on which *D* lies, is the one required.

If the given angle is a right angle, the point *O* coincides with *C*. If the angle is obtuse, *O* is on *DC* produced.

48. PROBLEM.—**From a given circle to cut off a segment which shall contain an angle equal to a given angle.**

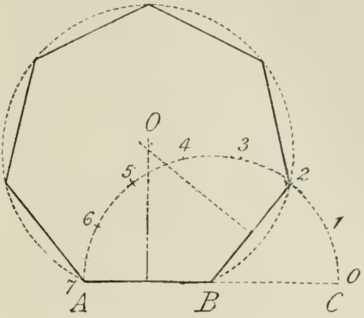
Take a point *A* on the circumference and draw the tangent *AD*.

Make the angle *DAB* equal to the given angle.

Then the segment *ACB* is the one required.

49. PROBLEM.—**In a given circle to inscribe a triangle the sides of which shall be in a given ratio.**

Draw any triangle having its sides in the given ratio.



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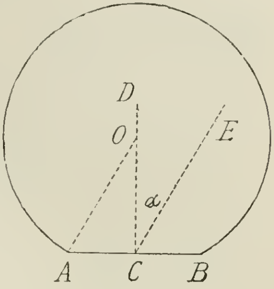

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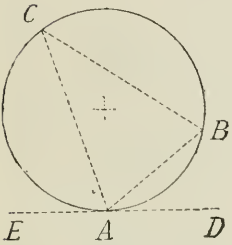

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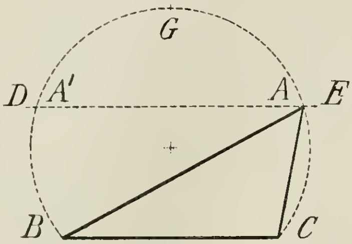

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At any point *A* in the circumference of the given circle draw the tangent *DE*. Make the angles *DAB*, *EAC* respectively equal to two of the angles of the triangle. Join *BC*. Then the triangle *ABC* is the one required.

50. Problem.—**To construct a triangle, having given the base, vertical angle, and altitude.**

Let *BC* be the given base; describe on it the segment of a circle containing an angle equal to the given vertical angle. Draw *DE* parallel to *BC*, the distance between these lines being equal to the given altitude. If *A*, *A′* are the points in which *DE* intersects the segment, either of the triangles *ABC*, *A′BC* will satisfy the given conditions.

51. PROBLEM.—**The perimeter of a triangle is 5″; the vertical angle is 70°; and one of the sides is half the base. Construct the triangle.**

Take any line *AB*; on *AB* draw a segment of a circle containing an angle of 70° (Prob. 47).

With centre *A*, radius half *AB*, describe an arc cutting the circle in *C*. Join *CA*, *CB*.

Then *ABC* is a triangle similar to the one required.

Produce *AB* both ways; make *AE* equal to *AC*, and *BD* equal to *BC*.

Divide a line 5″ long into segments having the ratio *EA* : *AB* : *BD* (Prob. 8). Then these segments are equal to the required sides of the triangle.

52. PROBLEM.—**To construct a triangle, having given the base, one of the base angles, and the perimeter.**

Let *BC* be the given base. Draw *BE* such that the angle *CBE* is equal to the given base angle. Along *BE* set off *BD* equal to the sum of the remaining sides.

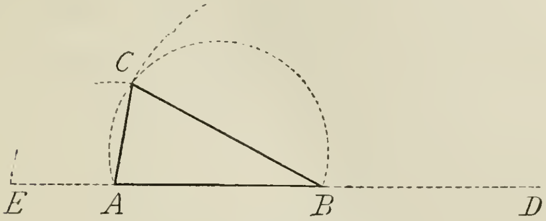
Join *DC*; draw *FA* bisecting *DC* at right angles, and join *AC*. Then the triangle *ABC* is the one required.

53. PROBLEM.—**To construct a triangle, having given the base 2″, perimeter 5″, and area 0·85 square inch.**

In the method here given we first determine the side of a square 0.85 square inch in area, and then obtain a rectangle equal in area to this square, one side of the rectangle being equal to the given base, 2″. The other side of this rectangle will be equal to half the altitude of the required triangle.

Make $OS = 1''$ and $ON = 0.85''$. On *SN* describe a semicircle, and draw *OX* perpendicular to *SN*; then *OX* is the length of the side of the square in question.

Now make *OR* = given base = 2″. Bisect *RX* at right angles by the line *YO′*, and with *O′* as centre and radius *O′X* describe the arc *XT*; then the area of the rectangle $RO.OT = (OX)^2 = 0.85$ square inch, and therefore *OT* = half altitude of required triangle. The problem may now be completed as in Prob. 55.



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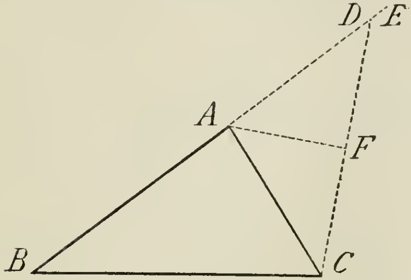

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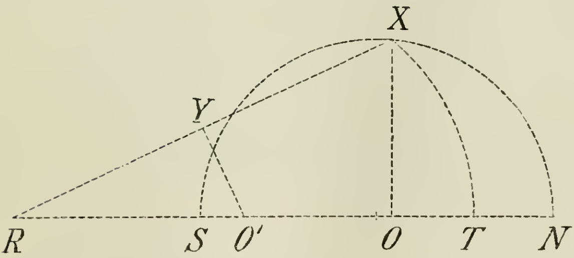

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Examples.—1. Describe a regular pentagon on a line $1\frac{3}{4}''$ long.

2. In a circle $2\frac{1}{2}''$ diameter inscribe a triangle two angles of which are 45° and 65°.

3. Construct a triangle having the base $1\frac{3}{4}''$, perimeter $4\frac{3}{4}''$, and area .95 of a square inch.

54. Problem.—**To construct a triangle, having given the vertical angle, altitude, and perimeter.**

Draw two lines AD, AE containing an angle equal to the given vertical angle. With centre A, and radius equal to the given altitude, describe the arc MN.

Along AD and AE set off AF and AG each equal to half the given perimeter.

Draw FO and GO perpendicular to FA and GA, and with O as centre, describe the arc GF. If now a common tangent be drawn to this arc and the arc MN, so as to meet AD and AE in B and C respectively, then ABC is the required triangle.

55. Problem.—**To construct a triangle, having given the base, altitude, and the perimeter.**

Let $F'F$ be the given base; bisect $F'F$ in C, and make $CA = CA' =$ half the sum of remaining sides, *i.e.* $= \frac{1}{2}$ (perimeter – base). Draw CE perpendicular to AA', and with F as centre and CA as radius, describe an arc intersecting CE in B. Make $CQ =$ given altitude, and $CG = CB$. Join GQ, and draw $A'K$ parallel to GQ. On AA' describe a semicircle, and draw KP' parallel to $A'A$. Draw $P'P$ parallel to BC, and QP parallel to KP', then PFF' is the required triangle.

Note.—The point P is on an ellipse, major axis AA', foci F, F'. The above construction is based on the properties of an ellipse.

56. Problem.—**To construct a triangle, having given H, K, L, the lengths of the three medians.**

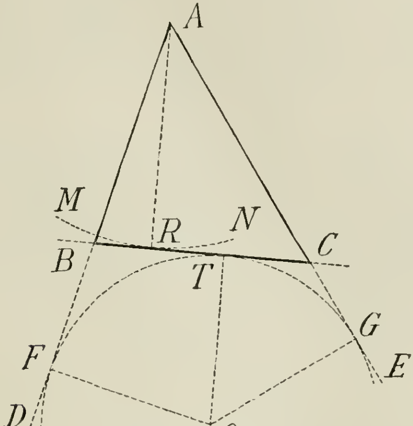
A *median* is a line drawn from any vertex to the middle point of the opposite side.

Make $AD = H$, and produce it to G such that $DG = DO = \frac{1}{3}AD$.

With O and G as centres, and $\frac{2}{3}$ K and $\frac{2}{3}$ L as radii respectively, describe two arcs intersecting at B.

Join BD and produce it to C, making $DC = DB$, then ABC will be the required triangle.

Note.—For explanation, compare the construction with the triangle *abc* in which the medians have been drawn.



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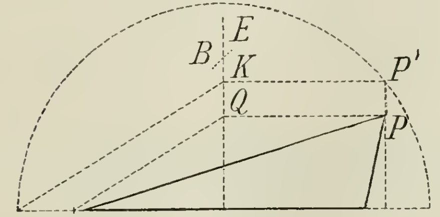

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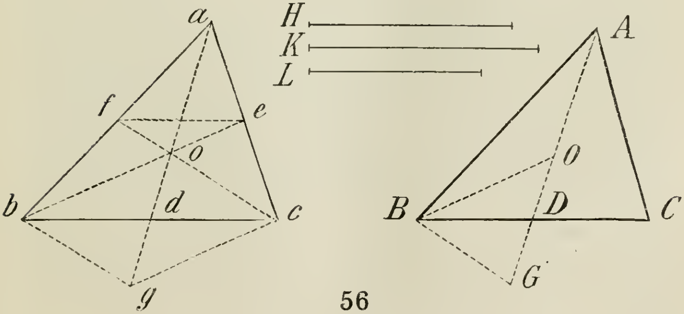

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57. Problem.—**To construct a triangle, having given (a) the base, vertical angle, and the ratio of the sides, say 3 : 5; (b) the base, altitude, and the ratio of the sides, 3 : 5.**

(a) Let AB be the given base. Select any convenient unit, and construct the triangle $AP'B$, making $BP' = 3$ units, and $AP' = 5$ units. Bisect the angle $AP'B$ by the line $P'D$, and draw $P'D'$ perpendicular to $P'D$.

On DD' as diameter describe a circle; this circle must contain the vertex of the required triangle. On AB describe the segment ABE of a circle, containing an angle equal to the given vertical angle (Prob. 47). Then the point P in which this segment intersects the circle on DD' is the required vertex, and APB the required triangle.

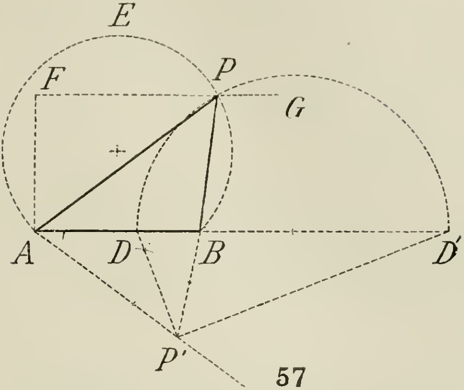
(b) Proceed as in (*a*) so far as obtaining the circle on DD'. Draw AF perpendicular to AB, making $AF =$ the given altitude; draw FG parallel to AB, intersecting, in P, the circle on DD', then APB is the required triangle.

The construction for this problem is based on Theorems 5 and 6, Art. 45.

58. Problem.—**To construct a triangle equal in area to a given triangle ABC, and having two of its sides respectively equal to two given lines D and E.**

Draw AF perpendicular to BC, bisect AF in G, and complete the rectangle $BHKC$. Draw LM equal to one of the given lines, say D, and mark off LN equal to BC. Draw MO perpendicular to ML, making MO equal to FG; join LO, and draw NP parallel to MO, then the rectangles $QRML$ and $BHKC$ are equal in area (Prob. 34). Produce NP to S, making PS equal to PN, and draw ST parallel to LM. With L as centre, and the other given line E as radius, describe an arc intersecting TS in V, then the triangle VLM satisfies the required conditions.

Note.—The arc described with L as centre would intersect ST produced in V', hence there are *two* solutions.



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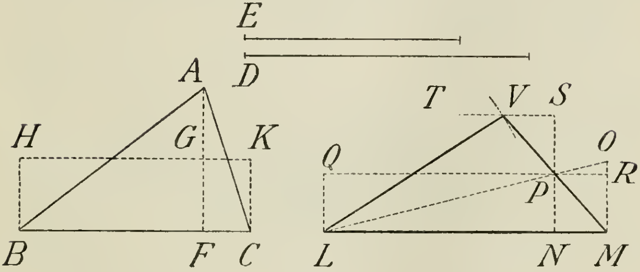

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Examples.—1. Construct a triangle having the vertical angle 55°, altitude 2″, and perimeter 5 inches.

2. Construct a triangle having the base 2″, altitude $1\frac{1}{2}''$, and perimeter 5 inches.

3. The three medians of a triangle are $1\frac{3}{4}''$, $2\frac{1}{4}''$, and 3″; draw the triangle.

4. Construct a triangle having its base 3″, vertical angle 120°, and the ratio of sides 3 : 5.

5. Construct a triangle having its base 2″, altitude $2\frac{1}{4}''$, and ratio of sides 4 : 7.

6. Draw a triangle with sides of 2″, $2\frac{1}{4}''$, $2\frac{3}{4}''$, and construct another triangle having the same area, two of its sides being 2″ and 3″.

59. PROBLEM.—**To describe a circle to pass through two given points P and Q, and touch a given line AB.**

Join *P* and *Q*, and produce *PQ* to cut *AB* in *R*.

Along *AB* set off *RT* (either to the right or left) equal to the mean proportional between *RQ* and *RP*.

Draw *TD* at right angles to *AB*, and bisect *PQ* at right angles by *NO*, meeting *TD* in *O*.

Then *O* is the centre of *one* circle satisfying the required conditions.

60. PROBLEM.—**To describe a circle which shall have its centre on a given line CD, pass through a given point P, and touch a given line AB.**

Let fall a perpendicular *PN* from *P* on to *CD*; and produce *PN* to *Q*, making *NQ* equal *PN*. Then, by Prob. 59, determine a circle to pass through *P* and *Q* and touch *AB*.

61. PROBLEM.—**To describe a circle to pass through a given point P, and touch two given lines BA, AC.**

Draw *AD* bisecting the angle *CAB*.

With any point *O′* in *AD* as centre, describe the circle touching *AC* and *AB*.

Draw *AP*, and produce it to cut this circle in *P′* (and P_1 not shown). Join *P′O′*, and draw *PO* parallel to *P′O′*.

Then *O* is the centre of *one* circle satisfying the conditions.

62. PROBLEM.—**Between two given lines AB, AC, to inscribe a succession of circles in contact.**

Draw *AD* to bisect the angle *BAC*. Take any point *O* on *AD* as centre, and describe a circle touching *AB* and *AC*, and cutting *AD* in *E*, *F*.

Draw *OG* perpendicular to *AB*, and join *EG*, *FG*. Draw *EH*, *HP* respectively parallel to *FG*, *GO*; and *FK*, *KQ* to *EG*, *GO*.

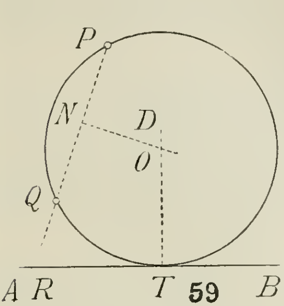
With centre *P* and *Q*, describe circles through *E* and *F*.

The three circles will touch the two lines and each other as required. The circles may be extended by repeating the construction.

63. PROBLEM.—**Two straight lines AB and CD are given, their point of intersection being inaccessible. It is required to describe a circle which shall touch these lines and pass through a given point P.**

By Prob. 22, draw *EF* to bisect the angle between *AB* and *CD*. With any centre *O* in *EF*, draw the circle which touches *AB*.

By Prob. 24, draw *PQ* to converge to the same point as *AB*, *CD*, cutting the circle in *R* and *P′*. Draw *PO′* parallel to *P′O*. Then the circle through *P*, with centre *O′*, is *one* solution.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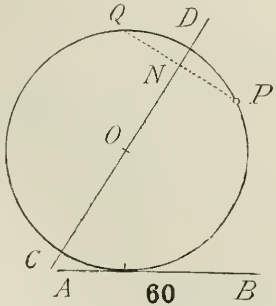
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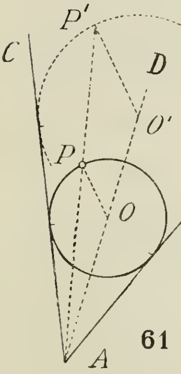
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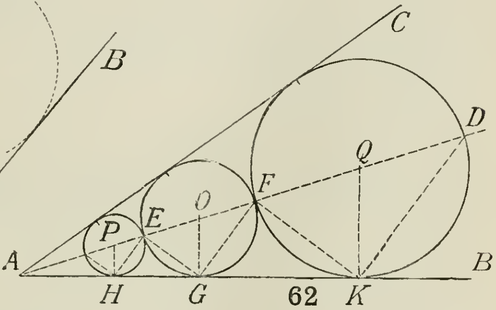
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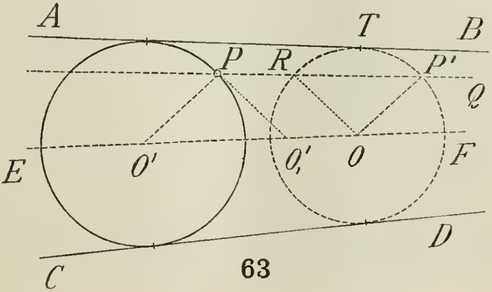
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64. Problem.—**To describe a circle which shall pass through two given points P and Q, and touch a given circle, centre G.**

Join *PQ*, and draw *NK* bisecting *PQ* at right angles.

Select any point *C* on *NK* such that the circle, with *C* as centre and *CQ* as radius, will intersect the given circle in *E* and *F* say.

Join *FE*, and produce it to meet *PQ* produced in *R*. Through *R* draw the tangent *RT* (or *RT′*), and produce *GT* (or *T′G*) to meet *NK* in *O* (or *O′*); then *O* (or *O′*) is the centre of a circle satisfying the given conditions.

For $RT^2 = RE \,.\, RF = RQ \,.\, RP$ (Euc. III. 36).

Hence the circle through *P*, *Q*, *T* touches *RT* at *T* (Euc. III. 37), and therefore also the given circle at *T*. Consequently the centre of the required circle lies on *GT* produced (converse of Euc. III. 12); but it must lie on *NK* (Euc. III. 1); hence it must be at *O* (or *O′*).

65. Problem.—**To describe a circle which shall have its centre on a given line CD, pass through a given point P, and touch a given circle, centre G.**

Let fall a perpendicular *PN* from *P* on to *CD*, and produce *PN* to *Q*, making *NQ* equal *PN*.

Then, by Prob. 64, determine a circle to pass through *P* and *Q*, and touch the given circle, centre *G*.

66. Problem.—**To describe a circle to touch two given circles, centres A, B, and pass through a given point P.**

Draw a common tangent *Tt* to meet *AB* produced in *S*; join *PS*. Through *G*, *F*, *P* describe a circle cutting *PS* in *Q*.

By Prob. 64 describe a circle to pass through *P*, *Q*, and to touch the circle with centre *B*; *D* being the point of contact. (There are **two** such circles.) This circle will also touch the circle with centre *A*.

Also, by drawing the internal tangent to the given circles, meeting *AB* in *S′*, two others may be found by a similar construction.

67. Problem.—**To describe a circle which shall touch a given circle, centre G, pass through a given point P, and touch a given straight line AB.**

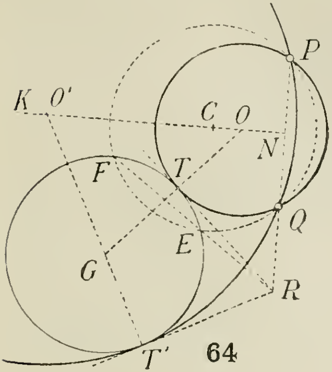
Draw *GE* perpendicular to *AB*, meeting the given circle in *C* and *D*; join *CP*.

Describe a circle to pass through *P*, *D*, *E*, cutting *PC* in *F*.

By Prob. 59 describe a circle passing through *P*, *F*, and touching *AB*; this will be the required circle.

Note.—There are *two* circles passing through *P*, *F* and touching *AB*.

Also, by reading *D* for *C*, and *C* for *D*, in the above instructions, two other circles will be found to be possible.



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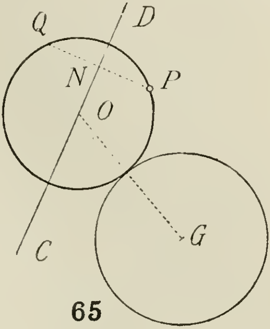

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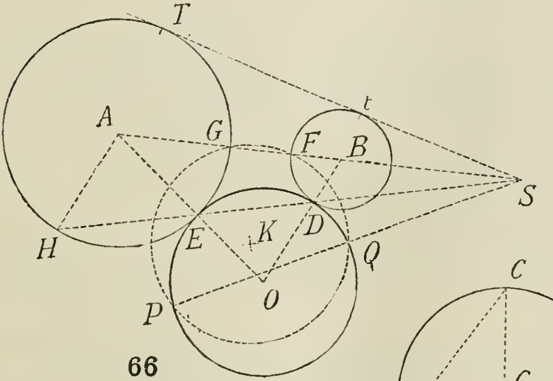

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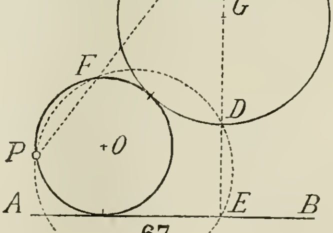

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68. PROBLEM.—**To describe a circle which shall touch three given circles, centres A, B, C.**

Let r_1, r_2, r_3 denote the lengths of the radii of the circles, centres A, B, C respectively, the first being the least.

With centre B and radius $r_2 - r_1$, describe a circle, and with centre C and radius $r_3 - r_1$, describe another circle.

By means of Prob. 66 determine O, the centre of a circle which passes through A and touches the two circles just described. Join OA, OB, OC, meeting the circles at the points indicated. This construction ensures that $AD = EG = FH$; from which it follows that the circle described with O as centre and OD as radius will touch three given circles *externally*.

In general there will be eight circles, satisfying the given conditions; three being such that for each one there is internal contact with two of the given circles, and external contact with the third; three others which have external contact with two of the given circles and internal contact with the third; one circle which has internal contact with the given circles; finally, the one in the figure.

69. PROBLEM.—**To describe a circle which shall touch two given lines AB, AD, and a given circle, centre C.**

Draw EF parallel to AD, and GH parallel to AB, the distance between the parallel lines in each case being equal to the radius of the given circle.

Now draw a circle which shall pass through C and touch EF and GH, Prob. 61 (there are two such circles); let O be its centre. Join OC, cutting the given circle in T.

Then the circle with centre O and radius OT is one solution.

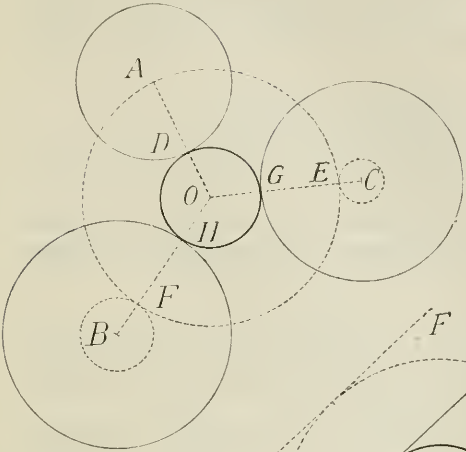
Note.—If EF and GH be drawn on the other side of AD and AB respectively, a second solution will be obtained, the circles having *internal* contact.

70. PROBLEM.—**To describe a circle which shall touch a given line AB and two given circles, centres C, D.**

Draw EF parallel to AB, at a distance equal to the radius of one of the other circles, say the one with centre C. With centre D describe a circle whose radius is equal to the difference of the radii of the given circles.

By Prob. 67 describe a circle to touch the latter circle, pass through C and touch EF; let O be its centre. Join OC, cutting the circle, centre C, at G; then the circle with O as centre and OG as radius will satisfy the required conditions.

Note.—There will, in general, be *eight* solutions. In four cases EF will lie on one side of AB, and in the other four cases on the opposite side.



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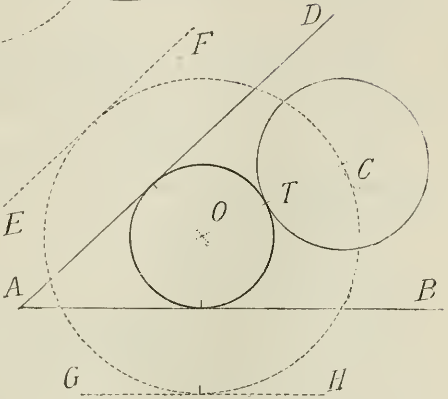

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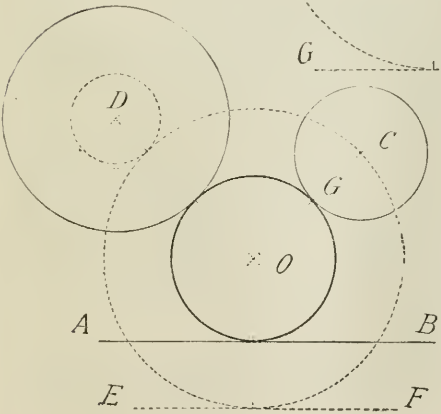

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71. Problem.—**To describe a circle which shall touch a given circle, centre G, and a given line AB at a given point T.**

1st. Externally. Draw *TF* and *GD* perpendicular to *AB*.

Join *DT*, intersecting the given circle at *E*.

Produce *GE* to *O*; then *O* is the centre of the required circle.

2nd. So as to include the given circle. Proceed as above, reading *D′*, *E′*, and *O′* for *D*, *E*, and *O*.

72. Problem.—**To describe a circle which shall touch a given line AB, and a given circle, centre G, at a given point E.**

Join *GE*, and produce it both ways.

Draw the tangent *ER*, and set off *RT* and *RT′*, each equal to *RE*.

Draw *TO* and *T′O′*, each perpendicular to *AB*; then *O* and *O′* are the centres of the two required circles, having respectively *external* and *internal* contact with the given circle.

73. Problem.—**To describe a circle which shall have its centre on a given line CD, and shall touch a given line AB and circle, centre G.**

By a method illustrating the use of a locus.

Draw *any* line *MN* at right angles to *AB*, and *any* line *GH*, cutting the given circle in *K*.

Along *MN* set off *any* lengths *M*1, *M*2, *M*3, etc., and along *KH* set off *K*1, *K*2, and *K*3 respectively equal to these.

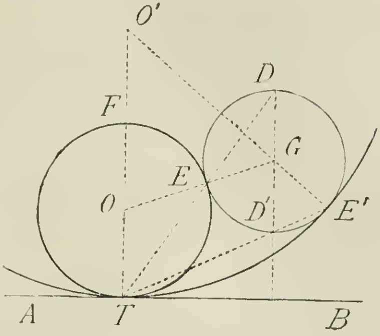
With *G* as centre describe arcs through the points 1, 2, 3, etc., on *KH*, to meet lines drawn parallel to *AB* through the corresponding points 1, 2, 3, etc., on *MN*.

These arcs and corresponding lines intersect at points 1, 2, 3, etc., through which points draw a fair curve.

This curve is the locus of the centre of a circle which moves so as always to touch the given line and circle, and the point *O* in which the curve intersects *CD* will be such that the circle described with *O* as centre, to touch *AB*, will also touch the circle, centre *G*.

Examples.—Take two points *A* and *B* 2″ apart. Draw a line *CD* distant 1″ from *A* and $1\frac{1}{2}''$ from *B*. On *CD* take *P*, $1\frac{1}{4}''$ from *A*. Describe a circle which shall pass through—

1. *A* and *B* and touch *CD*.
2. *A*, touch *CD*, and have its centre on *AB*.
3. *A* and *B*, and touch a circle, centre *P*, radius $\frac{1}{2}''$.
4. *A*, have its centre on *AB*, and touch circle centre *P*, radius $\frac{1}{2}''$.
5. *B*, touch *CD*, and a circle, centre *A*, radius $\frac{3}{4}''$.



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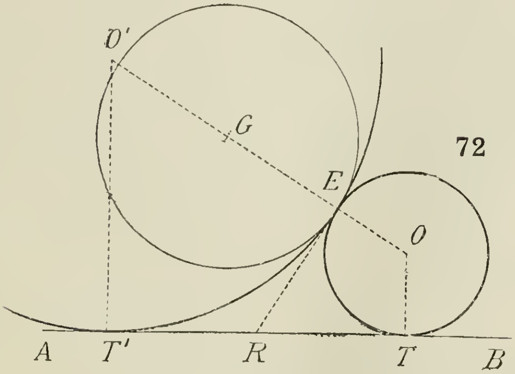

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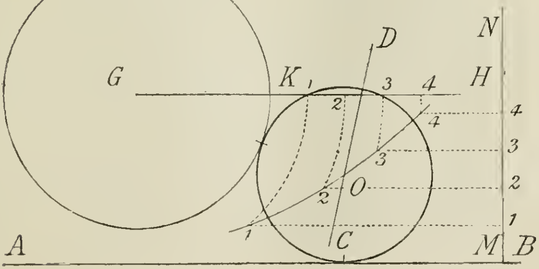

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74. Miscellaneous Examples.

***1.** In the given circle place a chord 1.8″ long and parallel to the line joining the given points *p*, *q*. Draw a second chord of the same length which, if produced, will pass through *p*. (1884)

Hint.—If two circles are concentric, all chords of the outer circle which touch the inner are equal.

2. Draw two lines including 50°. Describe a circle $2\frac{7}{8}''$ diameter cutting these lines in chords of $1\frac{1}{2}''$ and $2\frac{1}{2}''$. (1893)

3. Two points *a*, *b* are $3\frac{3}{8}$ miles apart. Determine the position of a point *p*, such that *pa* is $1\frac{5}{8}$ miles, and the angle *apb* 73°. Scale $1'' = 1\frac{1}{4}$ miles. (1886)

***4.** From *a* draw a line cutting the two given circles in equal chords.

N.B.—An auxiliary curve may be employed. (1893)

Hint.—Draw any line through *a* cutting the circles in chords *ab*, *cd*; along *cd* mark off $cx = ab$. Repeat this construction for other lines through *a*, and draw a fair curve through the points *x*. Let the locus of *x* cut the circle in *X*; then *aX* is one solution.

***5.** Through *A* draw two lines containing an angle of 30°, and intercepting a length of 2″ on the given line *BC*. (1889)

***6.** The figure represents a window of the decorated English style. The construction is sufficiently shown and dimensions are given. Draw the window to scale of 2″ to 1′ 0″. (1879)

7. Describe a circle enclosing and touching three other circles of 0.6″, 0.8″, and 1″ radius respectively, each of which touches the other two. (1882)

***8.** Draw the riband pattern to the given dimensions. (1880)

9. Three posts, *B*, *C*, *D*, are in a straight line at intervals of 100 yards. An observer at *A* finds that the angle *BAC* is 20°, and *CAD* 30°. Obtain the position of *A* (Scale $\frac{1}{2400}$). (1882)

10. Construct a regular pentagon having a diagonal of 3″. (1881)

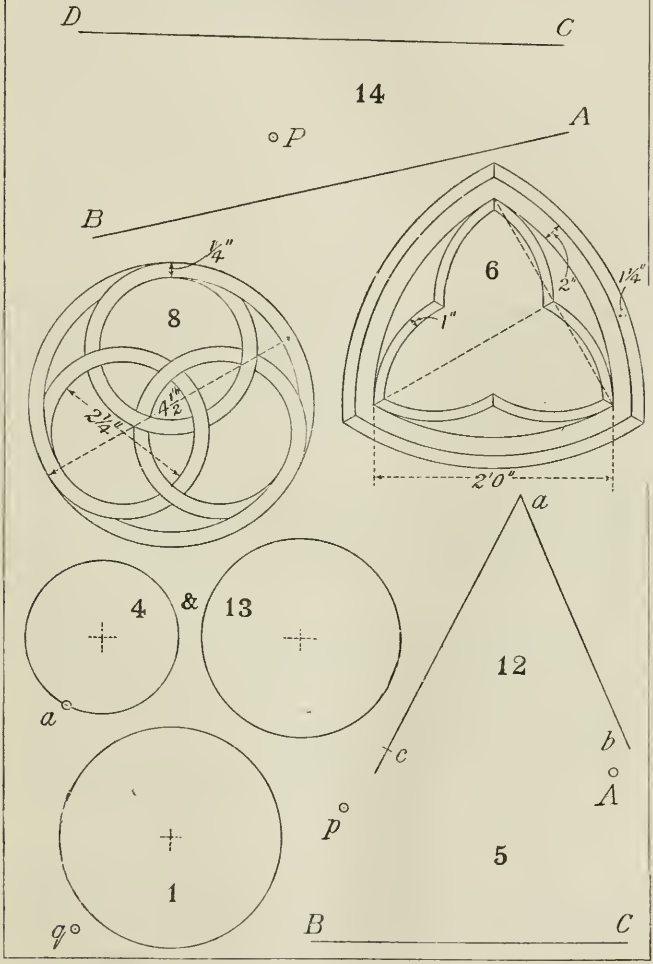
11. Draw a triangle from the following data :—
  (1) Base $2\frac{1}{4}''$, perimeter $7\frac{1}{2}''$, one base angle 55°. (1880)
  (2) Perimeter 6″, vertical angle 60°, altitude 1.25″. (1890)
  (3) Altitude 2.5″, perimeter $8\frac{1}{2}''$, one base angle 50°. (1877)

***12.** A boy starting from *a* runs in the direction *ab*. A man starts at the same moment from *c*. Supposing the man to run also in a straight line, but $1\frac{1}{2}$ times as fast as the boy, what direction must the former take to catch the latter? (1880)

***13.** Describe a circle of $2\frac{1}{2}''$ radius touching the two given circles. The given circles are to be *within* the required circle. (1877)

***14.** Describe three circles touching the two given lines *AB*, *CD*, and each other successively—
  (1) Such that the middle circle passes through *P*. (1878)
  (2) Such that the largest has a radius of 1 inch. (1881)

Copy the figures _double_ *size.*



CHAPTER IV

CONIC SECTIONS

75. Definitions.—Let one of two intersecting straight lines of indefinite length revolve about the other as a fixed axis, the inclination to each other and the point of intersection of the lines remaining fixed, then the two equal and opposite acute angles generate two equal and opposite conical solid angles of revolution, or a double cone of unlimited extent. The revolving line generates the surface of the cone, and in any position is called a generator of the surface, or simply a *generator*. The fixed line is called the *axis*, and the point of intersection the *vertex*, of the cone.

Any plane section of this cone is called a *conic section*.

The object of the present chapter is to consider the nature of these conic sections together with some of their more useful geometrical properties, and to give convenient methods of constructing the curves.

Classification of conic sections. — If the section plane be at right angles to the axis and do not contain the vertex, the conic section is called a **circle**.

If the section plane be *not* at right angles to the axis, nor parallel to a tangent plane, and be such that (supposed indefinitely extended) it cuts only one of the halves of the double cone, the section is called an **ellipse**.

If the section plane be parallel to one generator, and one only, that is, be parallel to a plane which touches the cone, the conic section is called a **parabola.**

If the section plane cut both halves of the double cone, and do not contain the vertex, the section is called an **hyperbola.**

If the section plane pass through the vertex, the conic section will be either a *point*, a *line*, or two *plane angles* opposite to each other. These may be considered as limiting cases of the ellipse, parabola, and hyperbola respectively.

From the manner in which the above conic sections are formed, it will be obvious that both the circle and ellipse are completely bounded figures, while the figure of the parabola extends indefinitely in one direction, and that of the hyberbola consists of two portions, which extend indefinitely and in opposite directions. These remarks refer merely to the general appearance of the sections; we have now to consider other relations which distinguish these figures.

Double meaning of the words "cone" and "conic section."—A circle, as defined by Euclid, is the plane figure enclosed within the circumference, not the circumference itself, though the word "circle" is frequently used as synonymous with the latter. The terms "ellipse," "parabola," and "hyperbola" are also used in two senses, denoting in one case *a plane figure*, and in the other the *bounding curve* or outline of the figure. In like manner the word "cone" is commonly employed with a double meaning, indicating either the *solid figure* or its *bounding surface.* So the sections of a cone may mean either the plane figures or their curved outlines.

In this book we shall have occasion to use the terms in both senses; the reader must infer the meaning from the context in any particular case.

In the next article the conic sections are examined from another standpoint, being regarded as curves traced by a moving point; and the nature of the motion is defined.

76. Properties of conic sections.—Let V be the vertex, and VO the axis of the cone. DP represents a *section plane* which determines the conic section PAQ. Let a sphere, centre I, be inscribed in the cone so as to touch the plane at F. This sphere touches the cone in a circle, centre O, the plane of which (HK) is perpendicular to the axis; let this plane intersect the section plane in the line DN.

Select any point P on the curve PAQ; take PM perpendicular to the plane HK, meeting it in M; draw PN perpendicular to DN, and join PV, PF, ML, MN. In this way two right-angled triangles PML, PMN are formed, since PM, being perpendicular to the plane HK, is therefore perpendicular to the lines ML, MN in the plane.

Now the angle PLM is equal to the complement of the semi-vertical angle of the cone, and is therefore constant for *any* position of P, hence the ratio $PL : PM$ is constant.

Again, the angle MNP, or α, is equal to angle between the planes HK, DP and is therefore constant, hence

the ratio $PN : PM$ is constant,

therefore the ratio $PL : PN$ is constant.

But PL and PF are tangents from P to the same sphere, therefore $PL = PF$, hence

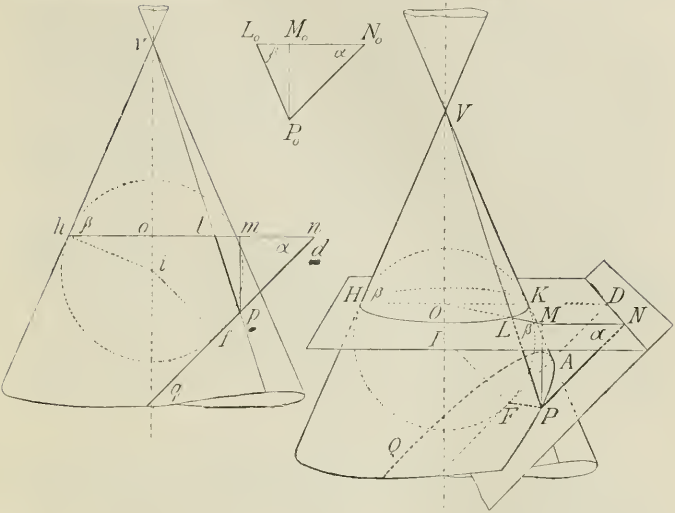
the ratio $PF : PN$ is constant.

That is to say, wherever P may be on the curve PAQ, its distance from F is always in a constant ratio to its distance from DN, and this is true for the parabola, ellipse, and hyperbola.

The point F is called a *focus*, the inscribed sphere a *focal sphere*, the line DN is a *directrix*, and the ratio $PF : PN$ is called the *eccentricity* of the conic section.

Thus we obtain the following definition:—

Definition.—A conic section is the curve described by a point which moves in a plane in such a manner that its distance from a fixed point in the plane (a focus) is in a constant ratio to its distance from a fixed line (a directrix) in the plane.



Referring again to the figure, the triangle PLM may be supposed to rotate about PM until its plane coincides with that of the triangle PMN, when LMN becomes one straight line. The triangles will then appear as shown detached at $P_0L_0M_0N$. The left-hand diagram of the figure is an elevation on a plane containing the axis of the cone and perpendicular to the section plane. In these diagrams corresponding points and angles are denoted by corresponding letters.

From the manner in which the curves were defined in Art. 75, it is obvious that for an *ellipse* the section plane must be so chosen that α is less than β, hence P_0L_0 is less than P_0N_0, or *PF is less than PN*. For a *parabola* α is equal to β, hence *PF is equal to PN*. And for a *hyperbola* α is greater than β, and consequently *PF greater than PN*.

Hence it follows that a conic section is an *ellipse*, a *parabola*, or a *hyperbola*, according as its eccentricity is *less than*, *equal to*, or *greater than* unity.

77. The ellipse.—Let the cone be cut by the plane DD' so that the section is an ellipse. A focal sphere is shown touching the section plane in a focus F, and determining the directrix DN, as explained in Art. 76. A second inscribed sphere is shown on the other side of the cutting plane touching the latter in F'. $D'N'$ is the line of intersection of the cutting plane with the plane containing the circle of contact $H'L'K'$.

Take any generator meeting the ellipse in P, and the circles of contact in L, L'. Let MPM' be a line parallel to the axis of the cone. Draw NPN' perpendicular to DN or $D'N'$. Join ML, MN, $M'L'$, $M'N'$, PF, PF'.

Suppose the triangles LMP, $L'M'P$ are turned about MM' into one plane as shown detached to a reduced scale, then it will be readily seen that

$$PF : PN = PF' : PN' = PL : PN, \text{ or } PL' : PN'.$$

Therefore P is similarly related to F' and $D'N'$ as to F and DN; thus *the ellipse has two foci*, F, F', *and two directrices*, DN, $D'N'$.

Let the line through FF' meet the curve in A, A'. Bisect FF' in C and draw BB' perpendicular to AA'.

Since the curve may be defined with reference to either the focus F and directrix DN, or the focus F' and directrix $D'N'$, it follows that BB' is an axis of symmetry; so also is AA'. Therefore *the ellipse has a centre which is* C.

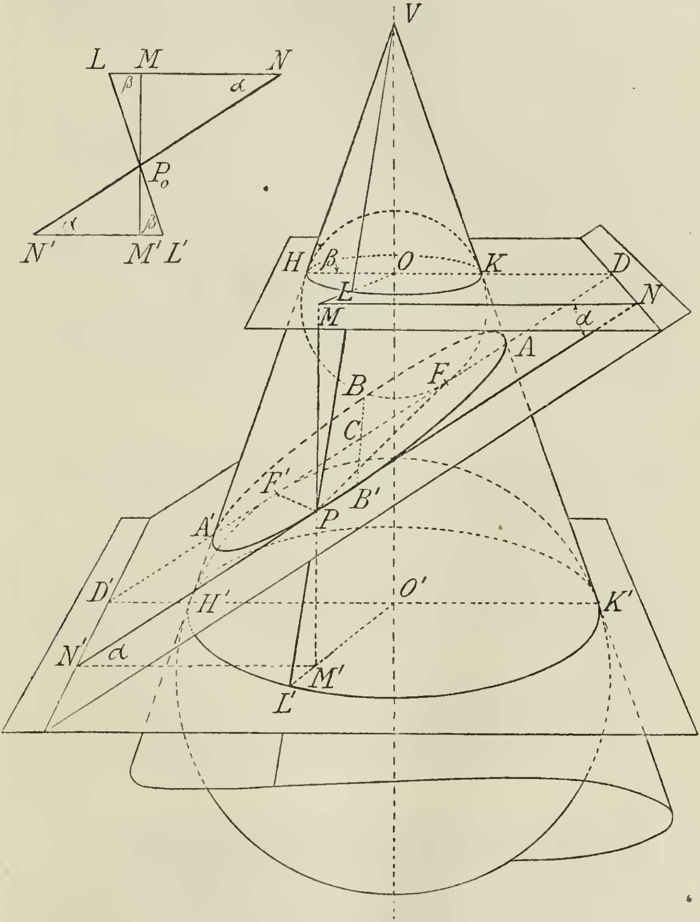
Any line through C and terminated by the curve is called a *diameter;* and *all diameters are bisected at* C.

AA' is called the *major axis*, and BB' the *minor axis;* they are respectively the longest and shortest diameters, and are sometimes referred to as the *principal axes.*

It appears from the figure that

$$PF + PF' = PL + PL' = KK' = AF + AF' = AA'.$$

Theorem 1.—*In any ellipse the sum of the focal distances is constant and equal to the major axis*, or *to the length of the portion of a generating line of the cone intercepted by the circles of contact of the focal spheres.*



Theorem 2.—It may be shown that CF, CA, CD *are in geometrical progression; also that the eccentricity* $FA : AD = CF : CA = CA : CD$.

78. PROBLEM.—**Having given the lengths of the major and minor axes of an ellipse, to determine the foci.**

Let AA' be the major axis.

Bisect AA' in C; draw CB, CB' at right angles to AA', making each equal to half the minor axis.

With B or B' as centre and CA or CA' as radius, describe arcs cutting AA' in F and F'.

Then F and F' are the required foci.

79. PROBLEM.—**Having given the foci and the major axis of an ellipse, to determine the minor axis.**

Let F, F' be the two foci and AA' the major axis.

Bisect AA' in C. With F and F' as centres and radius CA describe arcs to cut each other in B and B'.

Then BB' is the minor axis.

80. PROBLEM.—**To construct an ellipse, having given the major and minor axes AA′, BB′.** (First method.)

Determine F and F', the foci, as in Prob. 78.

In CF' take any convenient points 1, 2, 3 . . .

With $A1$ as radius describe arcs with centres F and F' respectively; similarly with $A'1$ as radius describe arcs with centres F' and F. These arcs intersect in the four points marked 1_1.

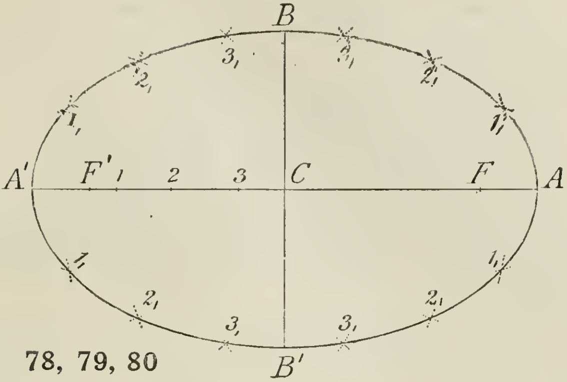
Repeat this construction for the points 2, 3, thus obtaining the points marked 2_1, 3_1.

Then the points 1_1, 2_1, 3_1 are on the required ellipse.

A fair curve should now be drawn through the points 1_1, 2_1, 3_1.

81. **Mechanical method of describing an ellipse.**

Insert pins at F, F', B. Take a piece of fine string, pass it round the pins so as to form a triangle $F'BF$, and tie the ends tightly together. Replace the pin B by a sharply-pointed pencil, with which trace out the curve, allowing the pencil to be guided by the string, but not pressing it against the string too tightly.



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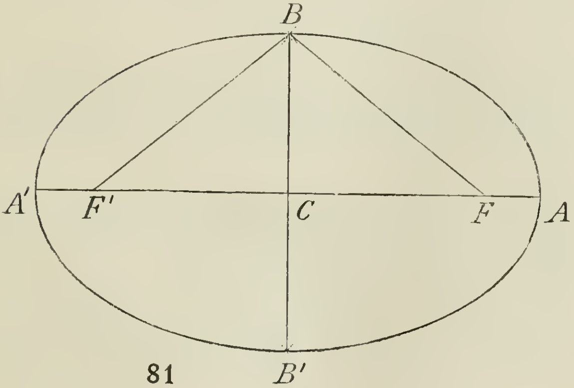

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Examples.—1. The lengths of major and minor axes of an ellipse are 4″ and 3″ respectively; determine and measure the distance between the foci. *Ans.* 2.64″.

2. The major axis of an ellipse is 3.6″ long, and the foci are 2.3″ apart; find the length of the minor axis. *Ans.* 2.78″.

3. The minor axis of an ellipse is 2.8″ long, and the foci are 2″ apart; find the length of the major axis. *Ans.* 3.44″.

4. Construct the ellipse of Ex. 1 by the method of Prob. 80.

5. Trace the ellipse of Ex. 3 by the method of Art. 81.

82. Projective properties of the ellipse.—It may be proved that *any orthogonal or radial projection of a conic section is a conic section.* It will be sufficient here to state this theorem in the less general form :—

The orthogonal projection of an ellipse is an ellipse.

In this theorem the ellipse is to be considered as including the circle and the straight line as extreme cases.

By considering an ellipse as the projection of a circle, and a circle as the projection of an ellipse, we shall be able to prove in a simple manner some useful properties of the curve. In this connection two additional theorems of projection will be required.

A system of parallel straight lines project orthogonally into a system of parallel straight lines, the lengths of the projections bearing the same ratios to each other as the lengths of the corresponding lines bear to each other.

If a line be tangential to a curve at any point, then any projection of the line will be tangential to the projection of the curve at the projection of the point.

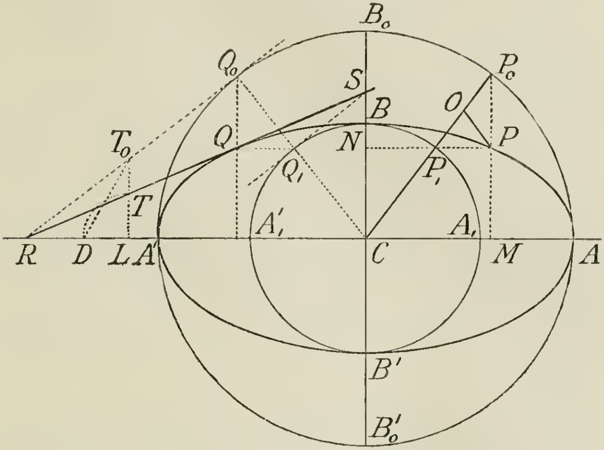
Let AA', B_0B_0' be two diameters of a circle perpendicular to each other.

Suppose the circle to be turned about AA' as axis until the projection of B_0B_0' on the plane of the paper is BB'; then the projection of the circle will be the ellipse shown, of which AA', BB' are the principal axes.

Next conceive the ellipse, the plane of which is that of the paper, to be turned about the minor axis until the major axis projects into the line A_1A_1' of length equal to BB'; then the projection of the ellipse will be the circle with BB' as diameter.

Definitions.—The circle described on the major axis of an ellipse as diameter is called the **major auxiliary circle**, or simply the *auxiliary circle* or the *major circle.* The circle with the minor axis as diameter is called the **minor auxiliary circle** or the *minor circle.*

If P_0 be any point on the major auxiliary circle, then when this circle is turned about AA' and projected into



the ellipse, the projection of P_0 will be P, where P_0P is perpendicular to CA. And from the principles of projection

$$P_0M : PM = B_0C : BC = CA : CB = \text{constant}$$

for all points on the curve.

Again, if the ellipse be turned about the minor axis and projected into the minor auxiliary circle, the point P projects into P_1, where PP_1 is perpendicular to CB and

$$PN : P_1N = AC : A_1C = CA : CB = P_0M : PM.$$

From this relation it is easy to prove that the line P_0P_1 if produced will pass through C.

These theorems may be stated as follows :—

Theorem 1.—*The ratio of the ordinate of any point of an ellipse to the corresponding ordinate of the major auxiliary circle, is constant and equal to the ratio of the minor to the major axis. And this is equal to the inverse ratio of the abscissa of the point to the abscissa of the corresponding point on the minor auxiliary circle.*

See Art. 144 for definitions of ordinate and abscissa.

Theorem 2.—If P be any point on an ellipse, the corresponding points P_0 and P_1 on the auxiliary circles lie on the same radius.

Next, let the tangent at any point Q on the ellipse intersect the major and minor axes in R and S. If the ellipse, with the tangent, be turned about the minor axis so as to project into the minor auxiliary circle, then Q will come to Q_1, S will not move, and the tangent SQ to the ellipse will project into the tangent SQ_1 to the minor auxiliary circle. Similarly RQ_0 will be the tangent to the major auxiliary circle corresponding to tangent RQ to the ellipse. And RQ_0, SQ_1 are parallel to each other since Q_0, Q_1 lie on the same radius; we thus have the following theorem :—

Theorem 3.—A tangent to an ellipse and the corresponding tangent to the auxiliary circle intersects the major axis at the same point. Also the tangent to the ellipse and the corresponding tangent to the minor auxiliary circle intersects the minor axis at the same point. And the two auxiliary tangents are parallel to each other.

Bisect P_1P_0 in O and join OP. Then it is clear that for all positions of the point P

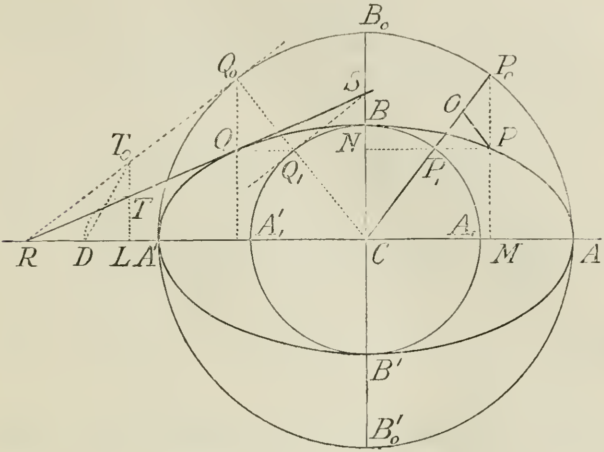
$$\mathbf{CO} = \frac{CP_1 + CP_0}{2} = \frac{\mathbf{CA} + \mathbf{CB}}{2} = \text{constant},$$

and also that

$$\mathbf{OP} = OP_1 = OP_0 = \frac{\mathbf{CA} - \mathbf{CB}}{2} = \text{constant}.$$

Further, since both the triangles P_1OP, P_0OP are isosceles, it is evident that *CO and OP are always equally and symmetrically inclined to CA*; they are also equally and symmetrically inclined to CB. We are thus led to the following theorem :—

Theorem 4.—If two lines CO and OP of constant length move round a fixed point C so as to be always symmetrically inclined to a given fixed line, the locus of P will be an ellipse



of which the principal axes are lines through C parallel and perpendicular to the fixed line, the lengths of the semi-axes being respectively equal to the sum and difference of CO and OP.

A *mechanism for drawing ellipses* has been constructed on this principle. CO is a crank free to rotate on a pin fixed at C to a frame. OP is a second crank pinned to the first one at O, and constrained by mechanism to turn (*i.e.* alter its angular position) at the same rate that CO turns, but in the opposite direction. In order to be able to trace an ellipse of any given size, it is necessary that the lengths of both cranks be capable of adjustment, one being made equal to half the sum, and the other to half the difference of the semi-axes. The figure shows OP as the shorter crank, but the *same ellipse* would be traced if the lengths of CO and OP were interchanged so that OP were the longer.

Example.—Construct the above figure when the lengths of the axes AA', BB' are $4''$ and $2\frac{1}{4}''$.

83. Problem.—**To construct an ellipse, having given the major and minor axes.** (Second method.)

Let AA' and BB' be the major and minor axes.

With centre C describe the two auxiliary circles on AA', BB' as diameters.

Divide the arc AB_0 into a number of equal parts, say six; draw the radii to the points of division, meeting the inner circle in corresponding points. Draw lines through the outer set of points perpendicular to the major axes to meet parallels thereto through the inner set.

A fair curve must be drawn through the points thus found. The remaining portions of the ellipse may be similarly determined, or obtained by constructions depending on symmetry.

84. Problem.—**Having given the principal axes of an ellipse, to find points in the curve mechanically by means of a paper trammel.**

Let AA', BB' be the two axes. Take a strip of paper with one edge straight, and from a point P on this edge mark off Pa, Pb, equal to CA, CB, the semi-axes.

Place the strip in successive positions with a and b always on the minor and major axes respectively, and mark corresponding positions of P; these will be points on the ellipse.

For, bisect ab in O and join CO. Then

$$CO = Ob = \tfrac{1}{2}ab = \text{constant}$$

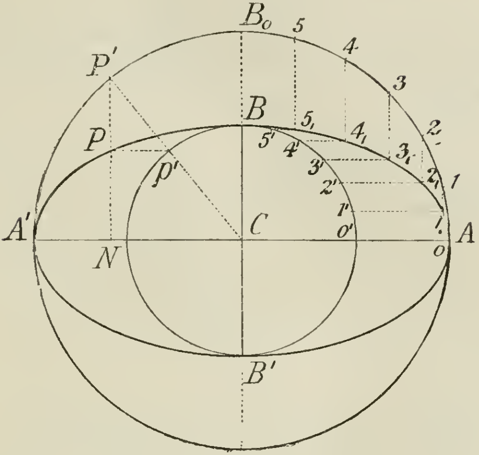
and $$\text{angle } OCb = \text{angle } ObC.$$

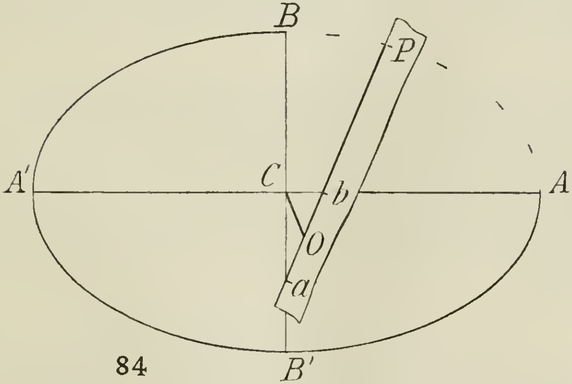
Hence CO, OP, two lines of constant length, move round C, and in all positions are symmetrically inclined to a fixed line CA; therefore by Theorem 4, Art. 82, the locus of P is an ellipse of which

$$\text{the semi-major axis} = PO + CO = PO + Oa$$
$$= Pa = CA$$

$$\text{and the semi-minor axis} = PO - CO = PO - Ob$$
$$= Pb = CB.$$

An instrument called a *trammel* for drawing ellipses is constructed on this principle.





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If *Pa* and *Pb* had been set off opposite ways along the edge of the trammel, the same ellipse would have been described. *CO* would then have been the longer of the two cranks.

Note.—Tracing-paper may with advantage be used for the trammel, in which case the points *P*, *a*, *b* need not be on its edge.

85. PROBLEM.—**A triangle moves in such a manner that two of its angular points always lie respectively in two fixed lines, to determine the locus of the third angular point.**

Let PQR be one position of the triangle, the points Q and R being on the fixed lines CY and CX respectively; the locus of P is required.

Describe a circle to pass through C, Q, R, and let O be its centre; join and produce PO, cutting the circle in b and a. Draw the lines CaB, CbA, and make $CA = Pa$ and $CB = Pb$. The locus of P will be an ellipse of which CA and CB are the major and minor semi-axes.

For, join OR, and suppose the circle and the lines PO, OR to be drawn on the plane of the triangle and to move with it.

The radius of the circle, the chord QR, and the angle QCR remain constant in magnitude, therefore as the triangle and circle move together the latter always passes through C (Euc. III. 21). Hence

CO moves about C as centre . . . (i.)

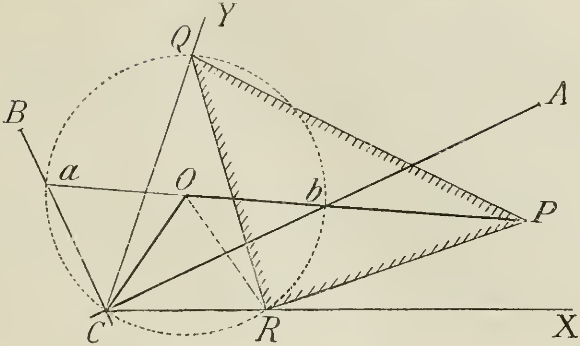
Now suppose that the triangle turns through any angle θ, say *clockwise;* then OR will do the same (since it is attached to the triangle), and the angle ORC will be increased by an amount θ. But since OCR is isosceles the angle OCR will also be increased by an amount θ, and therefore OC will turn through an angle θ *anti-clockwise.* Hence

CO and OP turn about C and O respectively
at the same rate but in opposite directions . (ii.)

But in the position shown in the figure, CO and OP are symmetrically inclined to the fixed line CA, since OCb is isosceles, therefore from (ii.)

CO and OP turn about C and are *always*
symmetrically inclined to the fixed line CA (iii.)

Hence from (iii.), as stated in Theorem 4, Art. 82, it follows that the locus of P is an ellipse of which



the semi-major axis $= PO + OC = Pa = CA$

and the semi-minor axis $= PO - OC = Pb = CB$,

the directions of which are CA and CB.

It will be noticed that Pba is the trammel for the double crank CO, OP.

The *triangular* trammel PQR, the double crank CO, OP, and the principal straight line trammel Pba are thus seen to be equivalent to one another in determining the path of P.

An indefinite number of equivalent trammels could be found, either triangular or straight; for the former by taking any two fixed lines CX', CY', intersecting the circle in R', Q', and joining PQ', PR'; or for the latter by drawing any line through P intersecting the circle in a', b', and then taking Ca', Cb' as the fixed lines.

Examples.—**1.** The lengths of the major and minor axes of an ellipse are respectively 4″ and 3″. Set out the curve (*a*) by the method of Prob. 83; (*b*) by the method of Prob. 84, using tracing-paper for the trammel.

2. Work Prob. 85, having given—
Angle $XCY = 70°$, $QR = 2.5''$, $QP = 2.75''$, $RP = 1.0''$.
Measure the lengths of the principal axes of the ellipse described by P, and the angle which the major axis makes with CX.
Ans. 6.55″, 1.27″, 13.7°.

86. Conjugate diameters of an ellipse.

Let tangents be drawn to a circle at the ends of two diameters perpendicular to each other, thus forming a circumscribing square as shown in the figure; and let PQ, PR be any two chords of the circle respectively parallel to the diameters.

Suppose this figure to be turned about any line XX' so that its plane is inclined to the plane of the paper, and the northogonally projected; the shape of the projection will be as shown to the right, corresponding points being denoted by corresponding letters.

From the principles of orthogonal projection we infer that the square with the inscribed circle touching it at the middle points of its sides will project into a parallelogram with an inscribed ellipse touching the sides at their middle points. It also follows that ef and hg, the tangents at the ends of the diameter jj', are parallel to the diameter $i'i$; similarly the tangents eh and fg at the ends of the diameter $i'i$ are parallel to the diameter jj'.

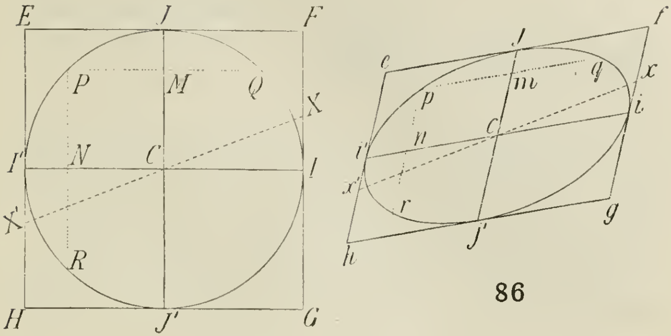
And since the chords PQ and PR are bisected by JJ' and II' at M and N respectively, so also are the chords pq and pr bisected by jj' and ii' at m and n respectively.

Such diameters of an ellipse as ii' and jj' are said to be **conjugate**; the tangents to the ellipse at the ends of either diameter are parallel to the other diameter; each diameter bisects all chords parallel to the other.

87. PROBLEM. — Having given a pair of conjugate diameters of an ellipse, to determine the principal axes.

Let II', JJ' be the given conjugate diameters. Draw JG perpendicular to and equal to CI. Join CG, and on CG as diameter describe a circle, centre O. Join JO, intersecting the circle in b and produced in a. Join Cb, Ca, and on these lines produced take CA and CA', each equal to Ja, and CB, CB', each equal to Jb.

Then AA' and BB' are the major and minor axes of the ellipse.



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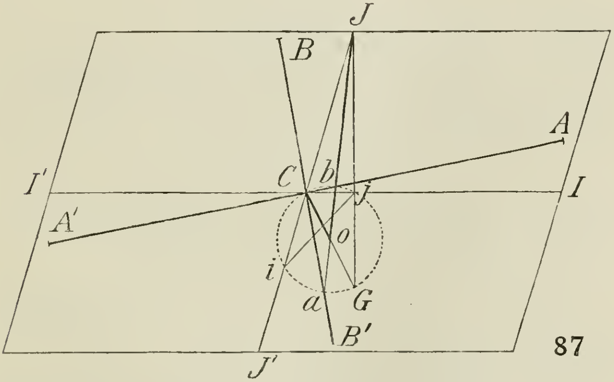

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For, join *ij*, then by comparing with Prob. 85 it is seen that *Jij* is the triangular trammel which will give the required ellipse; *CO*, *OJ* is the double crank, and *Jba* the principal straight line trammel equivalent to the triangular trammel in Prob. 85.

Example.—The lengths of two semi-conjugate diameters of an ellipse are 4″ and 3″, and the included angle 60°; determine the principal semi-axes, and the angle which the major axis makes with the larger conjugate diameter.
Ans. 4.40″, 2.35″, 17.5°.

88. PROBLEM.—**To describe an ellipse, having given a pair of conjugate diameters, or to inscribe the principal ellipse in a given parallelogram.**

Let *DEFG* be the given parallelogram, or *II′*, *JJ′* the given conjugate diameters.

First Method.—On two adjacent sides of the parallelogram, *DG*, *DE*, as diameters, describe semicircles, and divide each into the same number of equal parts, say *six*.

From the points of division draw lines perpendicular to the sides, intersecting the latter in points marked 1, 2.

From the points on *DG* draw lines parallel to *DE*; and from the points on *DE* draw lines parallel to *DG*.

These two series of lines will intersect in a series of points 1_1, 2_1 on the required ellipse.

Second Method.—Divide *CI* and *EI* each into the same number of equal parts, say *three*, and draw lines from *J′* and *J* to the points 1, 2, as shown.

The intersections of corresponding lines will give points 1_1, 2_1, which are on the required ellipse.

Proof.—The above two constructions are readily seen to be true for a square with inscribed circle, of which the figure may be regarded as the projection. Hence they are true for the projection.

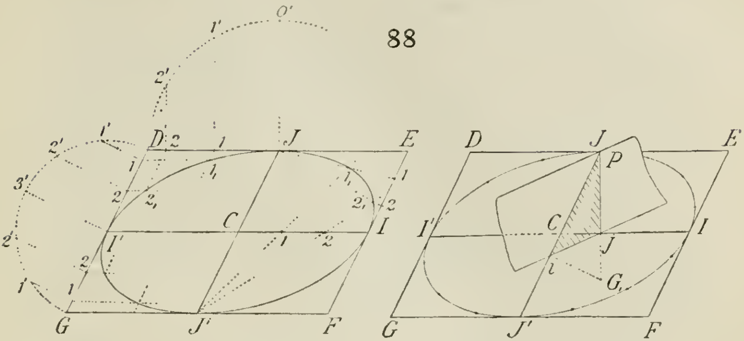
Third Method.—By a triangular trammel.

Draw JjG_1 perpendicular to *CI*, and make $JG_1 = CI$. Draw G_1i perpendicular to *JJ′* and join *ij*. Then *Jji* is the shape of the required triangular trammel.

Take a piece of tracing-paper for the trammel and mark the three points, *J*, *j*, *i* on it; then if *i* and *j* move on *JJ′* and *II′* respectively, *P* (or *J*) will trace the ellipse.

It will be noticed that the angle *iJj* (or *iPj*) is equal to the complement of the angle *ICJ*, and that the lengths of *Pi* and *Pj* are respectively equal to the perpendicular distances of *I* and *J* from the conjugate diameters.

Proof.—The point G_1 is the instantaneous centre of rotation of the trammel when moving through the position shown; hence the ellipse described by *P* touches *DE* at *J*. It can be shown in a similar manner that the ellipse touches *EF* at *I*.



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89. PROBLEM.—**Having given the curve of an ellipse, to determine the principal axes.** (No figure.)

First Method.—Draw any two chords, PP', QQ', parallel to each other. Draw the line bisecting these chords and meeting the curve in I, I', then II' is a diameter, and C, its middle point, is the centre of the ellipse.

Through C draw the diameter JJ' parallel to PP'.

Two conjugate diameters are thus determined, and the major and minor axes can be found as in Prob. 87.

Second Method.—Proceed as in the first method so far as to find C.

Then with centre C and any suitable radius, describe a circle cutting the ellipse in P, Q, R, S (taken in order).

Draw the diameters PR, QS, and bisect the angles between them. The bisecting lines terminated by the curve are the principal axes.

Examples.—**1.** Two conjugate diameters of an ellipse are $4\frac{1}{2}''$ and $3\frac{1}{2}''$ long and make 60° with one another; describe the ellipse by each of the three methods of Prob. 88.

2. Rule any two intersecting lines in ink on drawing-paper. Mark any three points on thick tracing-paper. Let two of the points move one on each line, and prick off points on the ellipse traced by the third point. Draw a fair curve through the points. Find the principal axes of the ellipse by each of the methods of Prob. 89. Confirm the result by the construction of Prob. 87.

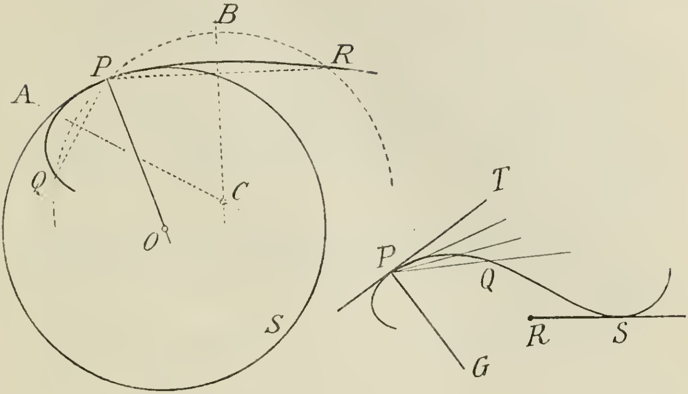
90. Tangent and normal to any plane curve. Curvature. Let P, Q be two points in any given curve, and suppose the point Q to move on the curve and to approach indefinitely near to P, then the chord PQ will turn round P and tend to a definite direction PT, becoming in the limit the *tangent* to the curve at P. The line PG, perpendicular to PT, is called the *normal* to the curve at P.

Definition 1.—*The* tangent *at a point in a curve is the straight line passing through the point and through a second point in the curve indefinitely near the first. The* normal *to the curve at the point is the straight line passing through the point at right angles to the tangent.*

This general definition of a tangent does not indicate a practical method of accurately drawing it at any given point in the curve. In order to be able to draw the tangent in the right direction it would be necessary to know something more with regard to it, such as the position of a second point in it at a *finite* and sensible distance from the first point (or a line to which it is parallel, perpendicular, or inclined at a known angle, or a second curve which it touches, etc.), and this would require special knowledge of the particular curve under consideration.

If it be required to draw a tangent *from* a point R not in the given curve, this may be done with all attainable precision, by applying a straight-edge, adjusting its position and then drawing the line RS to touch the curve as shown. The exact point of contact at S would be still undetermined, and if this point, as well as the line of the tangent, were required, an additional construction would be necessary, such as the drawing of a line intersecting the tangent at the required point. This additional construction would require to be based on some known property of the given curve.

Let Q, P, R be any three points on a plane curve; bisect the two chords QP, PR at right angles by the lines AC, BC meeting at C; then a circle with centre C passing through P will also pass through Q and R. Suppose now the two points Q and R to move along the curve and to approach indefinitely near to P. Then in the limit the two bisectors AC, BC become normals to the curve; they intersect in a definite point O, called the *centre of curvature.* The circle through P with centre O is called the *circle of curvature* at P, and OP is the *radius of curvature.*



Definition 2.—*The* circle of curvature *at any point in a curve is the circle passing through the point and through two other points in the curve indefinitely near the first. Its radius is the* radius of curvature, *and its centre, called the* centre of curvature, *is the point of intersection of the normal at the point on the curve, with a second normal at a point indefinitely near the first.*

The centre and radius of curvature cannot be accurately determined by directly applying the above construction; a special construction is required, depending in each case on the nature of the curve.

Any circle through P with its centre on the normal at P would touch the curve, the two would have a common tangent, or have *two* consecutive points in common; but the circle of curvature coincides more closely with the curve at the point than any other circle which can be drawn; as we have seen, it has *three* consecutive points in common with the curve. On examining the figure it will be noticed that the circle of curvature *crosses* the curve at P; on one side of P the curve gradually becomes flatter, and so falls away from the circle on the outside; on the other side of P the curvature gradually increases, and so the curve bends away from the circle towards the inside. It is thus seen that the amount of curvature at P is correctly measured by the circle of curvature; and the latter always crosses the curve unless the curvature is a maximum or minimum, as, for example, at the ends of the principal axes of an ellipse; in this case the curve and the circle of curvature have *four* consecutive points in common.

91. Focal and tangential properties of the ellipse.

Let PT, PG be the tangent and normal to the ellipse at any point P on the curve.

Draw FH, $F'H'$ perpendicular to the tangent. Draw the focal lines FP, $F'P$, and produce one of them.

Then it may be shown that the angle $FPG =$ angle $F'PG$, and that the angle $FPT =$ angle KPT. Also that $CH = CH' = CA$. Or—

Theorem 1.—*The* normal *at any point P in an ellipse bisects the angle between the focal lines PF, PF', and the* tangent *bisects the angle between one of the focal lines and the other one produced.*

Theorem 2.—*The feet of the perpendiculars from the foci to any tangent to an ellipse lie on the major auxiliary circle.*

92. Problem.—To draw the tangent and normal to a given ellipse at a given point on the curve.

First Method.—Let P be the given point. Join FP, $F'P$; produce one of these, say $F'P$ to K, and bisect the angle FPK by the line PT; then PT is the tangent at P.

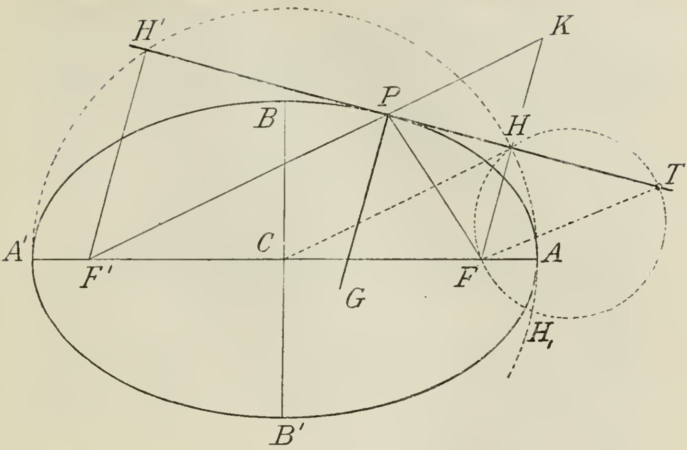
Draw PG perpendicular to PT, or bisect the angle FPF'; then PG is the normal at P.

Second Method (refer to figure on p. 95).—Let Q be the given point. Determine Q_0, the corresponding point on the major auxiliary circle. Draw the tangent Q_0R, intersecting the major axis in R; join RQ; then RQ is the required tangent.

Or if R be too remote, determine Q_1, draw the tangent Q_1S; then SQ is the required tangent.

The normal at Q is not shown in the figure, but it passes through Q, and is perpendicular to RS.

Third Method (see Fig. 84).—Let P be the given point. With centre P and radius equal to CA, describe an arc intersecting the *minor* axis in a. Join Pa, intersecting the major axis in b. Through a and b draw lines (not shown) perpendicular to axes BB', AA', intersecting in G; then PG is the required normal.



For G is the instantaneous centre of the trammel Pba, and therefore the tracing-point P is moving in the direction at right angles to PG.

93. PROBLEM.—**Having drawn a tangent to a given ellipse, to find the point of contact.**

First Method.—Let the tangent be the one shown in the figure above. From one focus F draw FHK perpendicular to the tangent intersecting it in H, and make $HK = FH$. Join KF' intersecting the tangent in P; then P is the point of contact.

Second Method (see the figure on p. 95).—Let the tangent intersect the major and minor axes in R and S.

From R draw a tangent RQ_0 to the major auxiliary circle, and determine the point of contact Q_0 by drawing CQ_0 perpendicular to RQ_0. Draw QQ_0 perpendicular to AA' to meet the tangent in Q. Then Q is the required point of contact.

If the point R is not available, make the corresponding construction for the minor auxiliary circle, *i.e.* draw SQ_1, CQ_1, Q_1Q.

Example.—Draw the above figure one-half larger.

94. PROBLEM.—**Having given the principal axes (or the foci and one axis) of an ellipse, to determine the two tangents to the curve from a given external point (without drawing the curve), and to find the points of contact.**

First Method (figure, last page).—Let T be the given point. Join T to one of the foci, say F. On TF as diameter describe a circle, and draw also the auxiliary circle on AA'. Let these circles intersect in H, H_1; then TH, TH_1 are the required tangents.

To find the points of contact, produce FH to K, and make $HK = HF$. Join KF', intersecting the tangent in P: then P is the required point of contact for one of the tangents; that for the other is found in a similar manner.

Second Method (figure opposite).—This method is based on the projective properties relating to the ellipse and the auxiliary circles. See Art. 82.

First let the given point be R, on the major axis.

Draw RQ_0 to touch the major circle, and draw CQ_0 perpendicular to RQ_0, thus determining the exact point of contact Q_0, and also the point Q_1 on the minor circle.

Through Q_0, Q_1 draw the lines parallel to the axes, meeting in Q. Then RQ is a tangent, and Q the point of contact.

Next let the given point be S, on the minor axis.

Draw SQ_1, tangent to the minor circle, then draw the perpendicular CQ_1Q_0, thus determining Q_1, Q_0, from which Q may be found as in the last case.

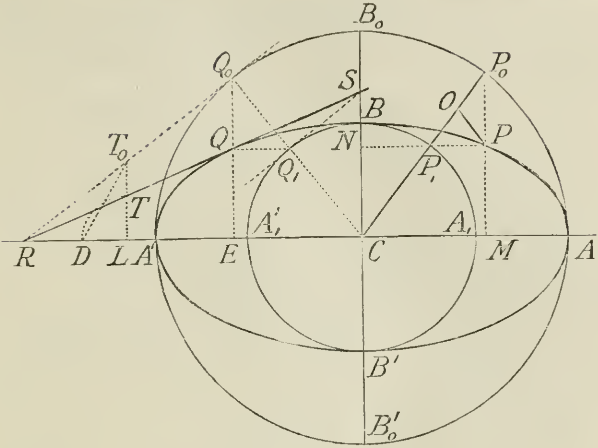
Lastly, let the given point be T, on neither axis.

Draw TL perpendicular to AA'. Make $LD = LT$. Draw DT_0 parallel to $A_1'B_0$ (or perpendicular to AB), intersecting LT produced in T_0. Then T_0 is a point in the auxiliary tangent to the major circle.

$$\text{For } T_0L : TL = B_0C : BC = Q_0E : QE.$$

This tangent T_0Q_0 is then drawn, and the points Q_0, Q_1, Q found as before. The required tangent from T is TQ, and Q is the point of contact.

The second tangent would correspond with the second tangent (not shown) from T_0 to the major circle.



Examples.—1. Draw the above figure double size.

2. Using tracing-paper for a trammel, construct an ellipse whose principal axes are $4\frac{1}{2}''$ and $3''$ long. Select any point P on the curve, and determine the tangent and normal at P by each of the methods of Prob. 92.
3. In Ex. 2 select any point Q external to the ellipse, and through Q draw a tangent to the curve. Then determine the point of contact by each of the methods of Prob. 93.
4. Draw two lines AA', BB', $4''$ and $3''$ long, bisecting each other at right angles in C. Along CA, CB produced, set off CR, CS equal to $3.6''$ and $2.8''$; and mark a point T such that $AT=.7''$, $BT=2.7''$.

   By each of the methods of Prob. 94 draw from R, S, and T tangents to the ellipse of which AA', BB' are the principal axes, without drawing the curve, and find the points of contact.
5. Find the eccentricity of the ellipse of Ex. 2, and the distance apart of the directrices. *Ans.* $0.745 : 6.04''$.

   Hint.—See Theorem 2, Art. 77.
6. Having given one focus F and two tangents TP, TP' with their points of contact P, P'; to find the other focus F'.

   Hint.—See Fig. 91. Draw FHK and join KP. Similarly obtain $K'P'$. Then F' is at the point where KP and $K'P'$ meet.

95. Problem.—**Having given the major and minor axes of an ellipse, to find, without drawing the ellipse, the points (if any) in which a given line intersects the curve.**

The method adopted is based on the projective properties of the auxiliary circles described in Art. 82.

Draw the auxiliary circles on the given axes AA' and BB' as diameters. Let the given line intersect the major auxiliary circle in L, M.

Join CL, CM, cutting the minor auxiliary circle in l, m. Through L, M draw lines parallel to AA' to meet in L_1, M_1 lines through l, m drawn parallel to BB'.

Join L_1M_1 intersecting the minor auxiliary circle in P_1, Q_1. Through P_1, Q_1 draw lines parallel to AA' to meet LM in P, Q; then P and Q are the required points of intersection of LM and the ellipse.

If the given line be perpendicular to the major axis, let H be the point in which it meets the major circle; join CH, cutting the minor circle in D_0, and draw D_0D perpendicular to BB' to meet the given line in D; then D is one of the required points of intersection.

If the given line be perpendicular to the minor axis, let K be the point in which it meets the minor circle. Join CK and produce it to meet the major circle in E_0; draw E_0E perpendicular to AA' to meet the given line in E; then E is one of the required points of intersection.

96. Problem.—**To describe an ellipse having given its centre C and three points P, Q, R on the curve.**

Join CQ and PR intersecting in H. With centre C, radius CQ, describe a circle, and through H draw the chord pr of this circle, such that $pH : Hr = PH : HR$; and draw the radius Cs perpendicular to CQ. Join sr (or sp) intersecting CQ in K. Join and produce RK (or PK) to meet in S a line drawn through s parallel to Rr or Pp, and join CS.

Then CS, CQ, are conjugate semi-diameters of the

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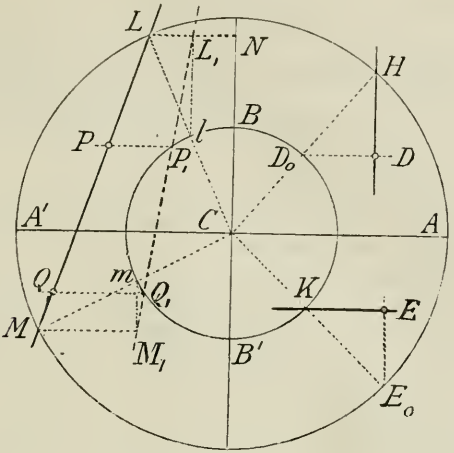

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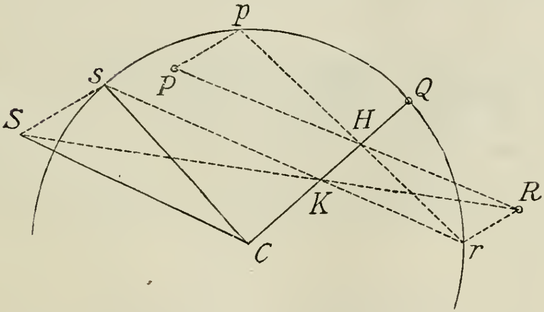


required ellipse. The principal axes can be determined by Prob. 87.

This construction is based on some theorems of projection. The required ellipse is supposed to be projected into a circle. A figure similar to the projection is drawn of such a size and so placed that *CQ* coincides with its own projection. The first part of the construction locates *p* and *r*, the projections of *P* and *R*. The second part locates *S*, a point which projects into *s*, where *Cs* has been drawn conjugate to *CQ* in the projection.

Examples.—**1.** In Ex. 4, p. 95, join RT. Also draw a line DD parallel to AA' at a distance of .5″; and a line EE parallel to BB' and distant .7″ therefrom. By the method of Prob. 95 determine the points in which the lines RT, DD, EE would cut the ellipse, if the latter were drawn.

2. Work Prob. 96, taking for the data $CP=1.5''$, $CQ=1.7''$, $CR=2.2''$; $PCQ=72°$, $QCR=34°$. Complete the figure by drawing the ellipse through P, Q, R.

97. PROBLEM.—**To construct an ellipse, having given the foci F, F′ and a tangent.**

(See figure, Art. 91.) From one of the given foci, F, draw a perpendicular to the given tangent meeting it in H; then H is on the major auxiliary circle.

Bisect FF' in C, and with centre C and radius CH describe the circle cutting FF' produced in A, A'; then AA' is the major axis.

The minor axis may now be found as in Prob. 79, and the ellipse constructed by one of the methods already given.

98. PROBLEM.—**Having given one axis of an ellipse and a tangent, to construct the curve.**

If the major axis be given, describe the auxiliary circle on it, and from the two points in which the given tangent cuts the circle, draw lines perpendicular to the tangent, to meet the major axis in F, F', the foci (see figure, Art. 91). The minor axis can now be found and the ellipse constructed.

If the minor axis be given, describe the minor circle on it and let S (figure, page 95) be the point in which the given tangent cuts the given axis (produced if necessary).

Draw the tangent from S to the circle, and from C draw CQ_1 perpendicular to the tangent.

Draw $Q_1\ Q$ perpendicular to BB' to meet the given tangent in Q, and from Q draw QQ_0 parallel to BB' to meet CQ_1 produced in Q_0; then Q_0 is a point on the major auxiliary circle, and hence the major axis is known and the ellipse can be constructed.

THE PARABOLA

99. Problem.—**To construct a parabola, having given the focus F and directrix DD′.**

It has been explained in Art. 75 that the section of a cone by a plane parallel to a tangent plane is a parabola. And in Art. 76 it was shown that this curve may also be defined as the locus of a point which moves in a plane in such a manner that its distance from a fixed point, *the focus*, is always equal to its distance from a fixed line, *the directrix*; this property enables us to set out the curve.

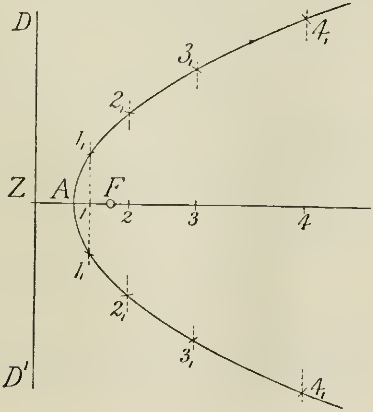
Through F draw a line perpendicular to DD', meeting the latter in Z. Bisect ZF in A.

Then A is on the parabola, for $AF = AZ$. A is called the *vertex*, and the perpendicular ZF the *axis* of the parabola. The latter is a line of symmetry for the curve.

To draw the curve. Select a series of points 1, 2, 3 . . . on the axis as shown, and through these points draw lines perpendicular to the axis.

With F as centre, $Z1$ as radius, describe an arc to intersect the perpendicular through 1 in 1_1.

Repeat this construction for the other points, and draw a fair curve through the points A, 1_1, 2_1, 3_1 . . . thus found.



100. Properties of the parabola.—In the upper figure PN and PM are perpendicular to the axis and directrix. PT is the tangent and PG the normal at P, and AY is the tangent at the vertex. PN is called the *ordinate*, and NG the *subnormal* for the point P. The double ordinate LL_1 through the focus is called the *latus rectum*.

In the lower figure let PQ be any chord of a parabola; PT, QT the tangents at P and Q; TN a line *parallel to the axis*, intersecting the curve at A; and let XY be the tangent at A.

The following theorems are proved in works on pure mathematics. The student should commit them to memory, and test them by setting out the figures to scale.

Theorem 1.—*The tangent PT bisects the angle FPM.*

Theorem 2.—*A line drawn through the focus perpendicular to a tangent, meets the latter at a point which is on the tangent at the vertex.*

Theorem 3.—*AT and AN are equal.* (In both figures.)

Theorem 4.—*The latus rectum LL_1 is equal to $4AF$.*

Theorem 5.—*The length of the subnormal NG is constant, and equal to half the latus rectum, or to $2AF$, or to FZ.*

Theorem 6.—*PN^2 is proportional to AN. Or the square of the ordinate is proportional to the abscissa. For the upper figure $PN^2 = LL_1 \cdot AN = 4AF \cdot AN$. Also for the lower figure $PN^2 = 4AF \cdot AN$, where F is the focus.*

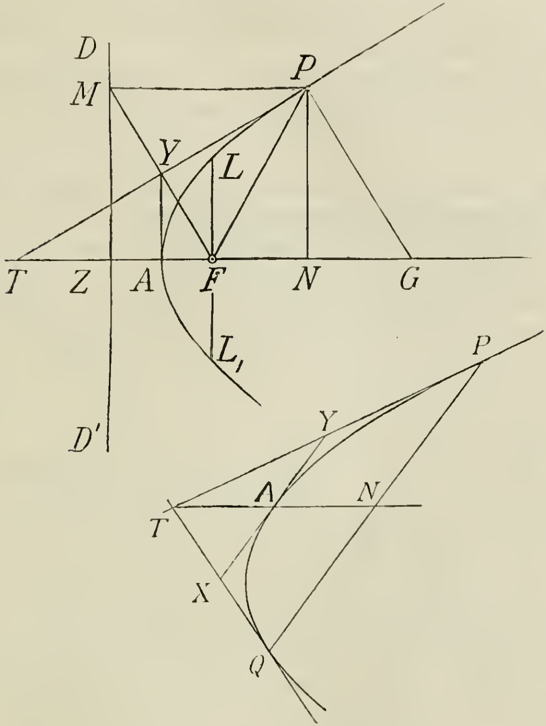
Theorem 7.—*A line which bisects any system of parallel chords is called a diameter, and is parallel to the axis, which is a particular diameter.*

Theorem 8.—*The line TN, through T, parallel to the axis, bisects the chord PQ, and TN is bisected at A. That is, $PN = NQ$, and $TA = AN$.*

Theorem 9.—*The tangent XY at A is parallel to PQ, and $XY = \frac{1}{2}PQ$.*

Theorem 10.—*Tangents which meet in the directrix are at right angles to each other.*

Theorem 11.—*The area $ALPNA$ is two-thirds the circumscribing rectangle $AN \cdot PN$.*



101. Problem.—**To draw the tangent and normal to a given parabola at a given point P on the curve.**

(See the upper figure.) Join P to the focus F, and draw PM parallel to the axis; bisect the angle FPM by the line PT. Then PT is the tangent at P.

Or, make $FT = FP$ and join PT.

Or, make $AT = AN$ and join PT.

To draw the normal, first draw the tangent as above, and then draw the normal PG perpendicular to it.

Or, make $NG = FZ$, and join PG.

102. PROBLEM.—**To draw the tangent to a parabola from an external point T.**

Join *T* to the focus *F*, and on *TF* as diameter describe a circle to cut the tangent at the vertex in *H* and *H'*. Then *TH*, *TH'* are tangents to the parabola.

103. PROBLEM.—**To describe a parabola, having given two tangents with their points of contact.**

Let *PT*, *QT* be the given tangents, intersecting in *T*, and let *P*, *Q* be the given points of contact.

Join *PQ* and bisect *PQ* in *N*. Join *TN* and bisect *TN* in *A*. Draw through *A* a line *Rt* parallel to *QP*.

Then *TN* is parallel to the axis of the parabola, *A* is a point on the curve, and *RA* is a tangent at *A*.

First Method.—Shown on the left of *TN*. Draw *QR* parallel to *NA*, so that *QRAN* is a parallelogram.

Divide *NQ* and *RQ* into any and the same number of equal parts, say four. From the points of division 1, 2, 3 on *NQ* draw lines parallel to *NA*; and from the points of division 1, 2, 3 on *RQ* draw lines through *A*.

Let the parallel lines intersect the corresponding radial lines in the points I. II. III., as shown. A fair curve through these latter points is the parabola required.

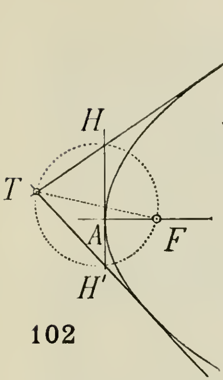
The curve may be extended beyond *Q* by continuing the divisions 5, 6, . . . on *NQ*, *RQ*, and through them drawing the parallels and radials, intersecting in V. VI. . . .

The same construction might be used for the portion of the curve *AP*. Instead we give the following:

Second Method.—Shown on the right of *TN*. Determining *A* as before. Let the tangent at *A*, which is parallel to *QP*, intersect *PT* in *t*.

Join *PA*, and through *t* draw a line parallel to *AN*, intersecting *PA* in *n*. Bisect *tn* in *a*. Then *a* is a point on the curve, and the tangent at *a* is parallel to *PA*.

This construction for determining *a* is the same as the one that was used for finding *A*. If desired, further points between *Aa* and *aP* can be found by repeating the process.



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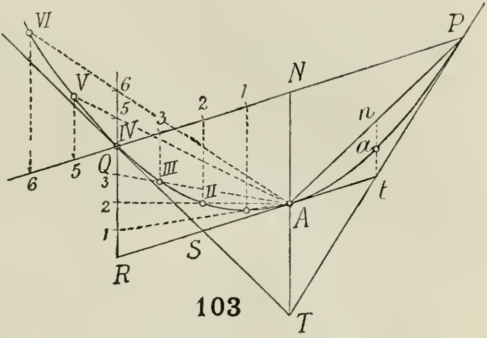

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104. Examples on the parabola.

1. Construct a parabola for which the distance of the focus from the directrix is $\frac{5}{8}''$. Measure the latus rectum. *Ans.* $1\frac{1}{4}''$.

2. In Ex. 1 take a point P on the curve distant $2''$ from the focus, and draw the *ordinate*, the *tangent*, and the *normal* at P. Measure the length of the subnormal. *Ans.* $\frac{5}{8}''$.

3. In Ex. 1 select any point external to the parabola, and draw a tangent to the curve. Determine the point of contact.

4. Draw a triangle TPQ, having given $PQ = 3''$, $PT = 2\frac{3}{4}''$, $QT = 2''$. By each of the methods of Prob. 103 construct the parabola which touches PT and QT at P and Q.

5. Determine the *axis*, *vertex*, *focus*, and *latus rectum* of the parabola of Ex. 4. *Ans.* Length of the latus rectum $= 2.24''$.

6. Draw any parallelogram, and in it inscribe a parabola which touches one side at its middle point, and passes through the ends of the opposite side. Determine the latus rectum.

7. A jet of water issuing from a horizontal nozzle strikes a point 1 foot below the orifice and 3 feet distant horizontally. Draw the path of the jet; scale $\frac{1}{12}$. Find the latus rectum of the path, and obtain the horizontal velocity of the water. *Ans.* 9 feet; 12 feet per second.

Hint.—It is known that (neglecting air resistances) the path of an object moving freely under the action of gravity is a parabola, with axis vertical and latus rectum in feet equal to $v^2 \div 16$, where v is the horizontal velocity in feet per second.

THE HYPERBOLA

105. Properties of the curve.—We explained in Art. 75 that a hyperbola was obtained when a cone was cut by a plane so taken as to penetrate both portions of the complete surface; and in Art. 76 it was shown that the curve might be defined as the locus of a point which moves, so that its distance from a fixed point bears a constant ratio (greater than unity) to its distance from a fixed line.

The curve is set out to scale in the figure; it is seen to consist of two parts detached from each other, each part having two branches; these branches extend to infinity on the complete curve.

Although so different in appearance from the ellipse, the two curves have many closely allied relations. Both have two axes of symmetry, two foci, two directrices, and each has a centre. Each curve, however, has properties peculiarly its own.

The hyperbola is not treated as fully in this work as is the ellipse, on account of its minor importance in the arts. We shall define the terms, state some of the properties, and give the more useful problems. The student should illustrate the subject by drawings to scale. Proofs may be found in works on pure mathematics.

Definitions.—The points F, F' are the *foci*.

C is the *centre*, chords through which are bisected at C.

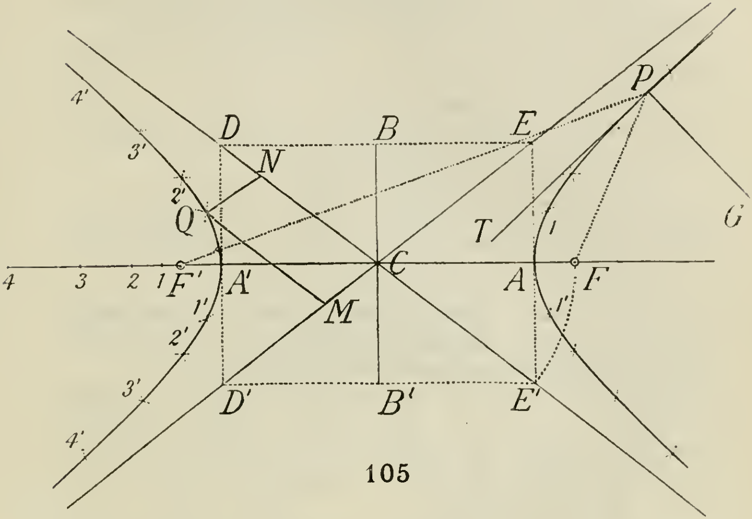
A and A' are the *vertices*, and AA' is the *transverse axis*.

BB', perpendicular to AA', is called the *conjugate axis*, being determined by the condition that $AB = AB' = CE = CE' = CF$.

The diagonals DE', $D'E$, produced indefinitely both ways, are *asymptotes* to the curve.

Asymptotes are lines which, as they recede to infinity with a curve, *approach nearer and nearer to the latter without limit, but never actually coincide with it.*

Properties.—Let P be any point on the curve. Join P to F and F'; then $P'F - PF = AA'$; or



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Theorem 1.—*The difference of the focal radii is constant, and equal to the transverse axis.*

Observe that in the ellipse it is the *sum* which is constant.

Theorem 2.—*The tangent PT at any point P bisects the focal radii PF, PF'.*

In the ellipse it is the *normal* which bisects this angle.

From any point Q on the curve, draw lines parallel to the asymptotes, meeting the latter in M or N. Then

Theorem 3.—*The product* $QM \cdot QN$ *is constant.*

This important property is characteristic of the hyperbola.

Let X be the point where a directrix cuts CA (not shown), then it may be shown that

Theorem 4.—*The eccentricity* $FA : AX = CA : CX = CF : CA$, *and* CX, CA, CF *are in geometrical progression.*

Compare Theorem 2, Art. 77, for the ellipse.

106. PROBLEM.—**To construct a hyperbola, having given the focii F, F′, and the transverse axis AA′** (Fig. 105).

Select points 1, 2, 3, 4 . . . on the axis, outside the foci, say to the left of F'.

With radius $A1$, centres F and F', describe arcs; and with radius $A'1$, centres F' and F, describe arcs intersecting the first arcs in the points marked 1′.

Repeat this construction for the points 2, 3, 4 . . . obtaining the points marked 2′, 3′, 4′ . . . A fair curve drawn through the latter points is the hyperbola required.

This construction is based on Theorem 1, Art. 105.

107. PROBLEM.—**Having given the foci F, F′ and transverse axis AA′ of a hyperbola, to determine the asymptotes DE′, D′E, and the conjugate axis BB′** (Fig. 105).

Bisect AA' in C. With centre C and radius CF describe arcs (only half of one shown) intersecting the perpendiculars to the axis through A and A' in E, E', D, D'.

Then the lines, of indefinite length, through D', E and D, E' are the asymptotes.

To obtain the conjugate axis draw a perpendicular through C, and cut this perpendicular in B, B' by an arc drawn with A as centre, CF as radius. Then BB' is the conjugate axis.

Or make CB and CB' each equal to AE.

Note that in the ellipse $FB = CA$, and in the hyperbola $AB = CF$.

108. PROBLEM.—**To draw the tangent and normal to a hyperbola at any point P on the curve** (Fig. 105).

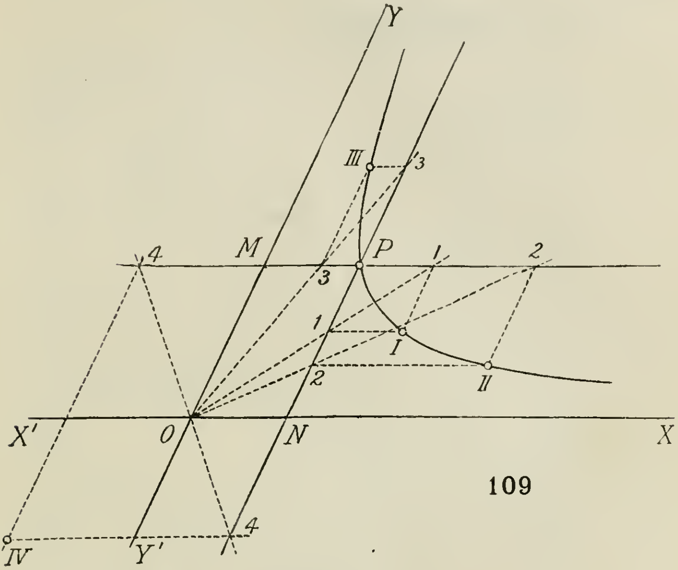
Join PF, PF' and bisect the angle FPF' by the line PT, then PT is the tangent at P.

Draw PG perpendicular to PT, then PG is the normal.

Example.—The transverse axis of a hyperbola is $2\frac{1}{4}''$ and the distance between the foci is $2\frac{3}{4}''$. Determine the conjugate axis, the asymptotes, and draw a portion of the curve.

Draw the tangent at any point and illustrate that this point bisects that part of the tangent between the asymptotes.

Find the eccentricity and the distance apart of the directrices. *Ans.* 1.58″, 1.22, 1.84″.



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109. Problem.—**To construct a hyperbola, having given the asymptotes XX′, YY′, and a point P on the curve.**

Through the given point *P* draw lines *PM*, *PN* of indefinite length, parallel to the asymptotes.

Through the point *O*, where the asymptotes intersect, draw a series of radials cutting the lines *PM*, *PN* in the points 1, 1; 2, 2; 3, 3; 4. 4.

From the points 1 and 1 draw lines parallel to the asymptotes meeting in I.; repeat this for the points 2, 2, obtaining II., and for the points 3, 3, obtaining III., and so on.

A fair curve through the points I. II. III. . . . is the hyperbola required.

This construction is based on Theorem 3, Art. 105.

Example.—Copy Fig. 109 half as large again as shown.

110. Miscellaneous Examples.

***1.** CD, CP are conjugate semi-axes of an ellipse. PM is drawn perpendicular to CD, and PN (on PM or MP produced) is made equal to CD. Show by construction that if the line MN slides between the lines CN and CD, the point P on it will trace out the ellipse. (1889)

Hint.—Mark the points N, P, M on tracing-paper; as this template moves in the given manner, prick through at P.

***2.** O is the centre of an ellipse, whose major axis lies on AB and is 3·30″ in length. CD is a tangent to the ellipse. Draw the half-curve above AB. (1895)

Hint.—Find the foci by applying Theorem 2, Art. 91.

***3.** OH, OK, are tangents to an ellipse at G and E respectively. F is one of the foci. Find the axes and draw the curve. (1897)

Hint.—See Ex. 6, p. 95.

***4.** Determine the axes of the ellipse which would touch ab and have F and F' as foci. Without drawing the ellipse find the points where the line through F perpendicular to ab would cut it.

***5.** Construct the ellipse which has the given points F and F' as foci, and which touches the given line ab. (1890)

Hint.—Use Theorem 2, Art. 91.

***6.** S is one focus of an ellipse. Q and R are two points on the ellipse, and PT is the direction of the major axis. Find the other focus (geometrically) and draw the ellipse. (1896)

Hint.—On RQ produced find D such that $DQ : DR = SQ : SR$. Draw DZ perpendicular to ST, then DZ is the directrix. The eccentricity may now be found, and therefore any number of points on the ellipse. Thus the vertices may be found.

***7.** F is the focus of a parabola, AB is a tangent to the curve, and A is on the directrix. Find the axis and directrix, and draw a sufficient portion of the curve to show that AB is truly tangent to it. (1893)

Hint.—Apply Theorems 2 and 10, Art. 100.

8. Draw two lines, AB, AC, including an angle of 30°. On AC set off a point P, so that $AP = 3''$. AB is the axis of a parabola; AC is a tangent to the curve at the point P. Find the focus and directrix and draw the curve. (1894)

Hint.—Use Theorems 3 and 5, Art. 100.

***9.** The directrix, and two points C and D on a hyperbola, are given. If the eccentricity be $\frac{4}{3}$, draw the branch of the curve on which the points lie. (1898)

Hint.—A focus is readily found.

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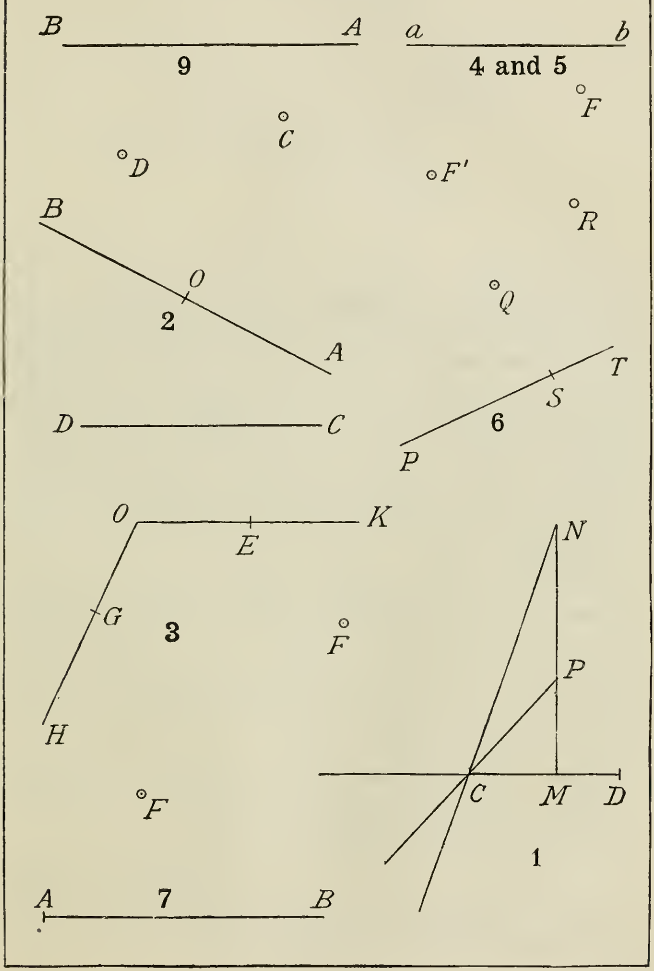

CHAPTER V

SPECIAL CURVES

111. Roulettes. Definitions and construction.—The last chapter dealt with the very important curves known as the conic sections. We have now to consider some other well-known curves, confining attention to those which have some application to the work of the engineer and architect. Taking the curves in the order of their importance from this point of view, the first perhaps to call for discussion are those which are generated by the rolling of one curve on another. The teeth of wheels generally have profiles thus formed.

Definitions.—If two curves roll upon one another without sliding, any point connected with one traces upon the plane of the other a curve which is called a *roulette.* One curve is generally fixed or regarded as fixed, and is known as the *base or fixed curve* or *directing curve;* the other, which then rolls over the base, is called the *generating curve* or the *rolling curve.*

General method of construction.[1]—Let AA be the rolling or generating curve carrying the tracing-point P, and let BB be the fixed curve or base.

By the use of a French curve, or otherwise, draw the curve BB in ink on the paper; draw the other curve AA

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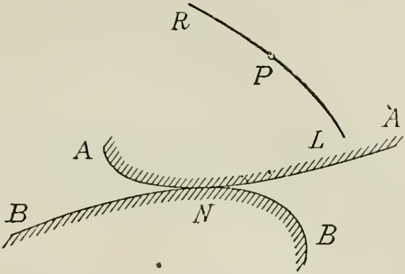
[1] See paper by Mr. W. I. Last, on the "Setting out of Wheel Teeth," *Min. Proc. Inst. C.E.* vol. lxxxix. 1887, p. 335.



in ink on stout tracing-paper, or with a needle-point on thin transparent celluloid, and mark any point P on the tracing-paper or transparent template. Now adjust the template until the curves touch each other, and prick off the point P. Next insert the pricker at the point N where the curves appear to touch, and rotate the template through a small angle into a new position, and again prick through at P. Then insert the pricker at the new point of contact N, rotate slightly, and prick off P. The locus of P is the roulette RL, and the above process is continued until any desired portion of this curve is obtained.

The procedure may be varied by drawing BB on the transparent template, and AA and P on the paper. Then, operating as before, the roulette will appear on the moving transparency. This example affords a good illustration of relative motion.

The method just described does not ensure absolutely pure rolling of the curves AA and BB on one another; but by taking the steps sufficiently small, the errors come well within the limits specified in Art. 2, and are not measurable. Practically, the result is found to be very perfect, and the roulette is obtained with a quickness and precision far exceeding that given by any ordinary geometrical construction.

Note.—If the template be long and of tracing-paper, a strip or drawing-paper glued to it will prevent change of shape by buckling.

112. Problem.—**To draw the normal at any point P of a roulette, and to find the centre of curvature.**

The normal and centre of curvature have been defined in Art. 90 in reference to any curve.

In the present case let the transparency be placed with the tracing-point coinciding with the given point P on the roulette, and let this coincidence be maintained with the pricker inserted at P, while the template is turned until the curves AA, BB come into contact at N. Then PN is the *normal* at P.

For at the instant that the roulette at P is being described the pricker is at N, and the template is then turning about this point. N is called the **instantaneous centre of rotation.** The tracing-point at this instant is thus moving at right angles to PN.

The centre of curvature must be on the normal PN. To find its position, let CD be the common normal to the curves in contact at N. Let C be the centre of curvature for the point N on AA, and D that for N on BB. Join PC, and through N draw a line perpendicular to PN. Let these lines intersect at Q. Join QD to intersect the normal PN in O. Then O is the centre of curvature for the point P on the roulette RL.

113. Problem.—**(a) To find the length of the given circular arc AB. (b) To set off a circular arc AB equal to the given line AE. (c) To mark off a circular arc AD equal to the given circular arc AB.**

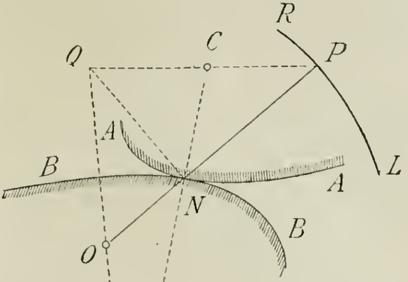
The line AE must *touch* both the arcs at A.

(*a*) Divide the arc AB into four equal parts at 1, 2, 3. Set off AR on the tangent equal to the chord $A1$. With centre R, radius RB, describe the arc BE. Then AE is equal in length to the arc AB nearly.

(*b*) Make AR equal to $\frac{1}{4}$ AE. With centre R, radius RE, describe the arc EB. Then AB is the arc required.

(*c*) Find R as in (*a*). With centre R, radius RB, draw arc BD. Then arc $AD =$ arc $AB = AE$ nearly.

Note.—These constructions are only approximate. They should not be used for arcs which subtend angles greater than 90°.



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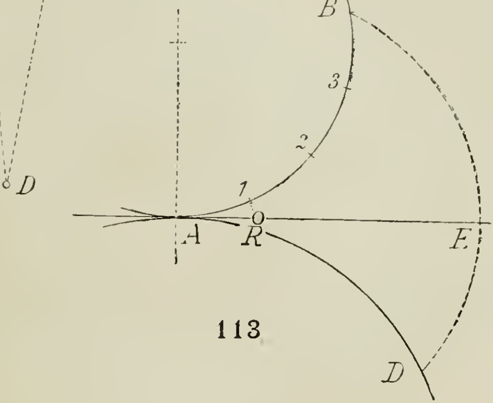

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Examples.—1. Draw a *fine line in ink* on drawing-paper, and a circle about 2″ diameter with a *fine ink line* on stout tracing-paper. Roll the circle exactly once round on the line, operating with the pricker. Measure carefully the lengths of the circumference thus obtained, and the diameter of the circle, and calculate the ratio of the two. *Ans.* 3.14.

2. Repeat this, but draw the line on the tracing-paper, and the circle on the drawing-paper. *Ans.* 3.14.

3. Draw a circle 3″ diameter. Find the length of $\frac{1}{4}$ of its circumference by Prob. 113. Measure this length and that of the diameter, and calculate the ratio of the two. *Ans.* .785.

4. Draw a circle of 3″ radius. By Prob. 113 set off on it an arc equal to the radius, and measure the angle subtended by the arc at the centre of the circle. *Ans.* 57.3°.

114. Cycloidal Curves.—When the two curves which roll on one another are circles, the roulettes form a class known as *cycloidal* curves.

If the tracing-point be on the circumference, we have: a **cycloid** when the circle rolls on a straight line; an **epicycloid** when it rolls on another circle *externally;* and an **hypocycloid** when it rolls *internally.*

If the tracing-point be *not* on the circumference, then, when the circle rolls on a straight line, we obtain a *prolate* cycloid when the point is inside, and a *curtate* cycloid when the point is outside the rolling circle. These two varieties are also called **trochoids.** Thus *epi-* or *hypo-trochoids* result when the circle rolls *outside* or *inside* a fixed circle.

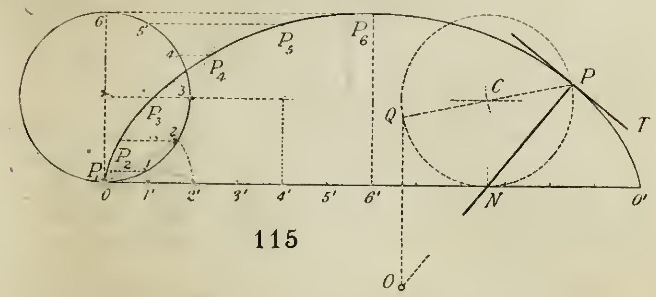
We shall now give some of the usual geometrical constructions for setting out these curves. It should be understood, however, that they are distinctly inferior, as regards expedition and accuracy, to the general method explained in Art. 111, in which a transparent template is used.

115. PROBLEM.—**To describe a cycloid, having given the rolling circle.**

Let the circle to the left of the figure be the rolling circle in its initial position, the tracing-point P being then at 0.

Divide its circumference into a number of equal parts, say twelve, and draw a tangent to the circle at its lowest point. By Prob. 113 make 02′ equal to the circular arc 02, and step off 02′ six times from 0 to 0′. Bisect these lengths, thus obtaining twelve equal divisions, each equal to one of the twelve equal arcs round the circle.

As the circle rolls to the right, the points of division on its circumference occupy, in turn, the positions indicated by the corresponding points of division on the tangent 00′. Also the tracing-point P will ascend and will cross, in turn, the horizontal dotted lines drawn through 1, 2, 3, . . . 6, afterwards descending.



115

For example, when rolling takes place, the point 2 *descends* to $2'$, and at the same time the tracing-point P *ascends* to the level indicated by the dotted line through 2.

But $2'P_2 =$ chord 02, hence the construction is as follows:—

Take the points 0, $1'$, $2'$, . . . $6'$, in turn, as centres, and describe arcs to intersect the horizontal lines through the corresponding points on the circle, the radii being the lengths of the chords from 0 to the corresponding points on the circle.

The second half of the curve may be readily obtained from the first half from considerations of symmetry. The two halves are symmetrical with respect to the line $P_6 6'$.

116. Examples on Problems 114 to 119.

1. Construct a cycloid, the diameter of the rolling circle being 2.5″. Select any point P on the curve and draw the normal at P and find the centre of curvature.
2. Construct the epicycloid and the hypocycloid, the diameters of the fixed and rolling circles being $4\frac{1}{2}''$ and $1\frac{1}{2}''$ respectively. Select a point on each curve, for which determine the normal and centre of curvature.
3. Two circles of 4″ and 2″ diameters have internal contact. By the method of Art. 111 set out the curves traced by a point in the circumference of one on the plane of the other. Also confirm the fact that *any* point in the plane of the 2″ circle will trace an *ellipse* on the plane of the 4″ one.

117. PROBLEM.—**To construct an epicycloid, having given the radii of the rolling and fixed circles.**

Let C_1 be the centre of the fixed circle, and C_0 that of the rolling circle in its initial position, the point of contact o being the initial position of the tracing-point *P*.

The construction given for the cycloid, Prob. 115, applies to this problem with the exception that in place of the straight lines drawn through the points of division on the circumference of the rolling circle, there will be a series of circular arcs concentric with the fixed circle.

118. PROBLEM.—**To construct a hypocycloid, having given the radii of the rolling and fixed circles** (Fig. 117).

Let C_1 be the centre of the fixed circle, and C_2 that of the rolling circle in its initial position.

The construction, being identical with that of the last problem, except that the rolling circle is now inside the fixed circle, is not shown, but the curve is drawn.

119. PROBLEM.—**To determine the tangent, normal, and centre of curvature for any point P on a cycloid, epicycloid, or hypocycloid** (Figs. 115 and 117).

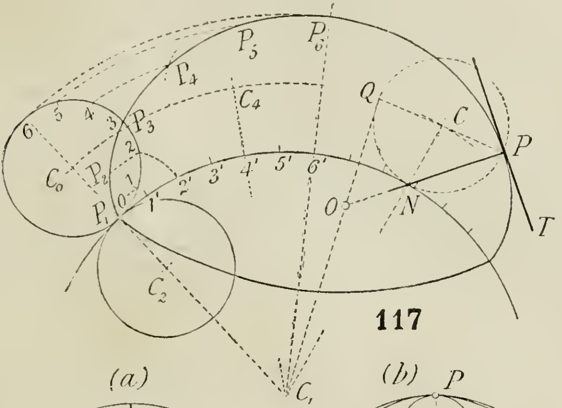
Art. 112 should be read again.

First determine the position of the rolling circle when the tracing-point is at *P*. To do this, with centre *P*, radius $C_0 6$, describe an arc to intersect the locus of the centre of the rolling circle in *C*. With *C* as centre, draw the rolling circle through *P*, and find *N*, its point of contact with the fixed line or circle. Join *PN*, and draw *PT* perpendicular to *PN*. Then *PN* is the normal and *PT* the tangent at *P*.

To determine *O*, the centre of curvature, draw from *P* the diameter *PCQ*.

For the cycloid draw through *Q* a line perpendicular to the fixed line to intersect the normal *PN* produced in *O*.

For the other curves join *Q* to the fixed centre C_1 to itnersect the normal in *O*.



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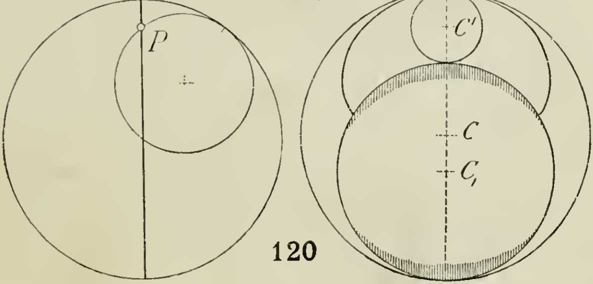


(a) (b)

120

120. Special cases of cycloidal curves.—There are two special cases with which the student should be acquainted.

In the first (a) the *hypocycloid* is a *straight line*, being a *diameter* of the fixed circle. This occurs when the diameter of the rolling circle is *half* that of the fixed circle.

In the second case (b) the circles have internal contact, but the rolling circle is *larger* than the fixed circle. The resulting curve is an *epicycloid*, identical with that which would have been described by a circle of diameter equal to the *difference* of the two diameters, rolling *outside* the fixed circle, as shown at C'.

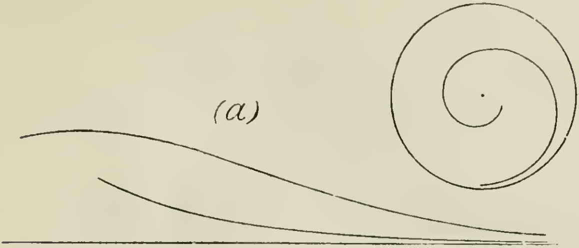
121. Peculiarities exhibited by curves.—We shall now illustrate some special features which the student may find in the curves which he has occasion to draw, and give the names by which such are known to mathematicians.

(*a*) *Asymptotes.*—A straight line is said to be *asymptotic* to a curve when, if the two recede to an infinite distance, they get nearer and nearer together without limit, but never actually coincide. Similarly two curves are asymptotic when they approach nearer and nearer together without limit, but never actually coincide, as the lengths of the curves increase without limit. Thus a spiral may be asymptotic to a circle, see the figure.

(*b*) *Nodes.*—If two branches of a curve cross one another, the point where they cross is called a *node.* Thus in the figure the two branches cross at the node and unite to form the loop. There are two tangents to the curve at the node, one to each branch. A node is otherwise known as a *double point.* If three branches intersect at a point, we have a *triple point*, and so on. See the figure.

(*c*) *Cusps.*—If a point when tracing a curve come to a place where it stops for an instant and then returns on itself, so as to generate two branches which have a common tangent at the point, we have a cusp. Two varieties of cusps are shown in the figure.

(*d*) *Points of inflexion.*—If a curve cross its tangent at any point, the latter is called a *point of inflexion.* Several other alternative and equivalent definitions might be given. Thus a point of inflexion is where three consecutive points on the curve are in a straight line, or where the radius of curvature is infinite, the centre of curvature crossing over from one side of the curve to the other, or where the curve changes from the concave to the convex on the same side, or *vice versa.* Or where the tangent accompanying a point which traces the curve has its direction of rotation reversed, being *stationary* for an instant at the point of inflexion.



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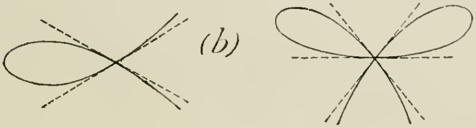

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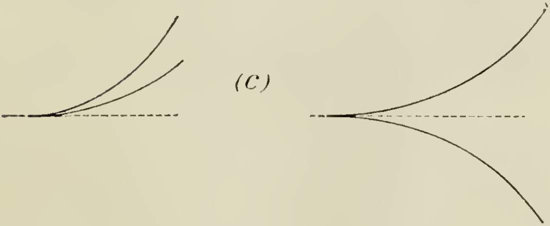

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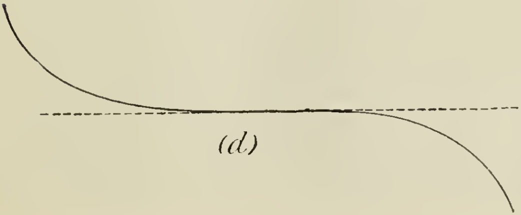


(d)

122. Envelopes.—*If a curve move in any definite manner, there is a certain curve which it always touches; this is called the envelope of the moving curve.*

Thus let successive adjacent positions of the moving curve be those marked 1, 2, 3, 4 . . . Then a fair curve drawn to touch these is the *envelope* of the moving curve.

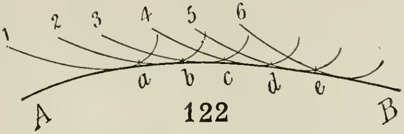
Let the curves 1 and 2 intersect in a; 2 and 3 in b; 3 and 4 in c, and so on. Then the envelope may also be defined as the curve through the intersections a, b, c . . . of consecutive curves, when these are taken very very near to each other, or when the curve is moved by indefinitely small steps.

The reader may acquire a very clear notion as to the nature of an envelope by proceeding thus:—Take a piece of *transparent sheet celluloid*, and shape one edge of it to the form of the moving curve. This is most readily effected by tracing the curve on the celluloid by means of a French curve and needle-point, then bending and breaking the celluloid along the scratched line. Now let the manner in which the curved edge of the celluloid shall move be decided in the following way:—On the celluloid draw some curve, say a circular arc, and on the drawing-paper draw another curve, say another circular arc. Make the first arc roll on the second as explained in Art. 111, but instead of pricking through at P use the curved edge of the celluloid as a template, and by means of it draw a curve on the drawing-paper for each successive position.

One method of setting out *wheel teeth* is by envelopes.

Example.—Find the envelope of a straight line which moves so as to have two points P and Q in the line, 3″ apart, always on two lines pp and qq, which cross one another at 60°.

123. Parallel curves.—If a series of normals of equal lengths be drawn from consecutive points on any curve, the curve through their ends is said to be parallel to the first curve, or the two curves are parallel to one another; they are equidistant at all points when measured *normally*. Otherwise, if a circle move with its centre on any curve, the *envelope* of the circle is a curve parallel to the first curve. This second definition indicates the practical method usually adopted when drawing parallel curves; in applying it, the student will see that the circle has *two* envelopes, thus giving *two* parallel curves on each side of the original curve.



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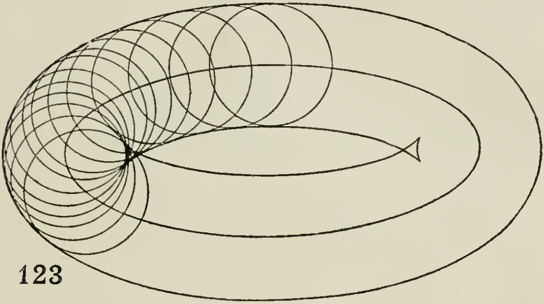

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A curve parallel to a circle is a concentric circle, and that parallel to a straight line is a second straight line. In all other cases the parallel curve is of a *different character* from the original one. Thus the parallel curve to an ellipse is *not* an ellipse. This is seen very clearly in the figure. Here the middle curve, an ellipse, is first drawn. Then a series of circles are drawn of constant radius, with their centres on the ellipse, and sufficiently near together to enable the envelope to be drawn freehand. The external envelope is *not* an ellipse, though it might appear so to an unpractised eye. But no one could mistake the inner envelope for an ellipse, with its four cusps and two nodes.

Note.—A cusp occurs on the envelope for a point on the ellipse where the radius of curvature is equal to the radius of the moving circle. The outer envelope never has cusps ; the inner envelope only has cusps when the radius of the circle lies between the greatest and least radii of curvature of the ellipse which occur at the ends of the minor and major axes. If R, r be their lengths, a and b the semi-axes, then r, b, a, R are in geometrical progression. Hence given a and b, we can find R and r.

124. The involute and evolute of a plane curve.

Definition 1.—*If a straight line roll on a curve, the locus of any point on the line is called an* **involute** *of the curve.*

The involute is thus a special case of a roulette, and can be set out in the manner explained in Art. 111.

Definition 2.—*The locus of the centre of curvature of any curve is called the* **evolute** *of the curve.*

Thus let P_1, P_2, P_3 . . . be consecutive points on a curve; P_1O_1, P_2O_2, P_3O_3 . . . the normals; and O_1, O_2, O_3 . . . the several centres of curvature; then the fair curve through O_1, O_2, O_3 . . . is the evolute of the curve P_1, P_2, P_3 . . .

It will readily be seen that corresponding to a given curve there is only *one* evolute, but an infinite number of involutes.

It is shown in works on pure mathematics that any normal PO to the curve at P is a tangent to the evolute at O. Also that the difference between any two radii of curvature is equal to the corresponding arc of the evolute; *i.e.*, $P_2O_2 - P_1O_1 = \text{arc } O_1O_2$; or $P_3O_3 - P_1O_1 = \text{arc } O_1O_3$, and so on. Thus we have the following theorems:—

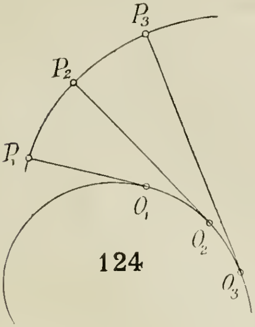
Theorem 1.—*The evolute of a curve is the* **envelope** *of the normals to the curve.*

Theorem 2.—*If the evolute OO of a curve PP be drawn, then PP is* **one of the involutes** *of OO, and might be traced by a point P in a straight line which rolls on OO.*

Theorem 3.—*In a roulette which is traced by a point P in a* **straight** *line which rolls on any curve, the point of contact O is not only the instantaneous centre of rotation, but also the* **centre of curvature** *for P.*

125. PROBLEM.—To draw an involute of a given circle, and to find the tangent, normal, and centre of curvature for any point P on the curve.

Divide the circumference into a number of equal parts, say eight. At the division 0 draw a tangent, and by the aid of Prob. 113 set off 02′ equal to one-quarter the circumference. Step off 02′ along the tangent, and subdivide each of the steps as shown, thus obtaining divisions on the tangent equal to the arcs of the circle.



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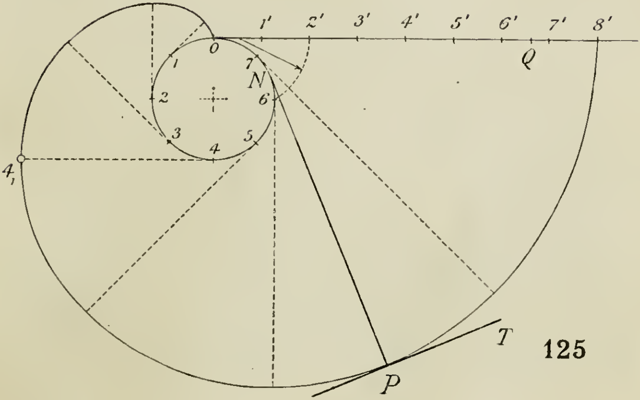

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Draw tangents to the circle at the points, and along each tangent set off the distance of the corresponding point on o8′ from o. For example, make $44_1 = o4'$.

The curve may extend indefinitely outwards.

Let P be any point on the involute. From P draw a tangent PN to the circle, and find N its point of contact.

Then PN is the normal at P, N is the centre of curvature, and PT, perpendicular to PN, is the tangent.

This construction is inferior to the method described in Art. 111.

126. Spiral curves.—Let a straight line, starting from a position *OX*, rotate continuously in one direction about *O*, and at the same time let a tracing-point *P* move continuously along the line always receding from or approaching towards *O*; then the curve generated by *P* is called a *spiral*. The point *O* is called the *pole*; *OX* is the *initial line*; *OP* the *radius vector* (or radius), and the angle *XOP* is called the *vectorial angle*, for the point *P* on the curve.

If during the tracing of the curve the revolving line make one, two, or more rotations, the spiral is said to consist of one, two, or more *convolutions*.

The nature of the curve depends on the law connecting the motions of *P* along the line and the line itself round *O*; there is evidently an infinite variety of form.

We shall give the construction for two of the best known spirals, the laws for which are simple.

In the first, equal amounts of increase in the vectorial angle and radius vector accompany one another, or, if the vectorial angles are in arithmetical progression, so are also the radius vectors. This is the **Spiral of Archimedes.**

In the second, if the vectorial angles increase by equal amounts, that is, form a series in arithmetical progression, the radius vectors form a series in *geometrical progression*, or, the ratio of any two radius vectors differing by the same angle is constant. This is the **Logarithmic Spiral.**

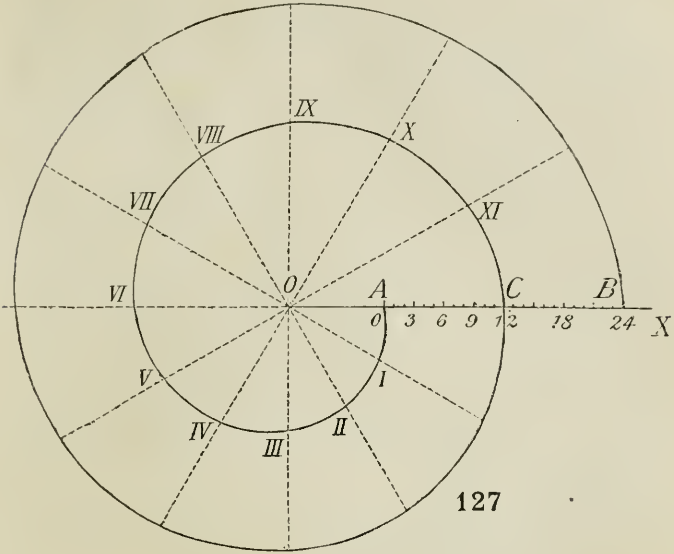
The involute of a circle is a spiral curve.

127. PROBLEM.—**To draw an Archimedian spiral of two convolutions, having given the pole O, and the longest and shortest radii OA, OB.**

Take *OABX* as the initial line.

From *O* draw a series of lines at equal angles with one another, say 30°, for which angle the lines can all be drawn with the 60° and 30° set-square, and there will be twelve radiating lines altogether.

Bisect *AB* in *C*, and divide *AC* and *CB* each into



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twelve equal parts. Then *AC* or *CB* is equal to the *increase* in the radius vector per revolution, and each of the divisions is equal to the increase for 30°.

Therefore make $OI. = O1$, $OII. = O2$, $OIII. = O3$, and so on. Then a fair curve through *A*, I. II. III. . . . *B* is the Archimedean spiral required.

Examples on Problems 126 to 129.

1. Draw two convolutions of an Archimedean spiral, the radius vector increasing from .5″ to 2.5″ in the two turns.
2. Draw a logarithmic spiral of two convolutions, the least radius being $\frac{1}{4}''$, and the ratio of two radii at an angular interval of $22\frac{1}{2}°$ being 1.08.
3. Set out a logarithmic curve, taking 16 equidistant ordinates $\frac{3}{8}''$ apart; the least ordinate being $\frac{1}{2}''$, and the ratio of any two consecutive ordinates 9 : 10.

128. PROBLEM.—**To construct a logarithmic spiral having given the ratio of the lengths of any two radii and the angle between them. Let the ratio be 9 : 10 and the included angle 30°.**

From any pole *O* draw a series of radials making 30° with one another. This is best done with the 30° and 60° set-square and tee-square.

On any line (preferably one drawn with the tee-square) mark off *oa nine* units long on any convenient scale. At *a* erect a perpendicular to *oa*. With centre *o*, radius *ten* units (on the same scale), draw an arc intersecting the perpendicular in *b*. Join *ob* and produce this line.

Now draw in succession: *bc* perpendicular to *ob*; *cd* perpendicular to *oc* . . . ; and *ax* perpendicular to *ox*; *xy* perpendicular to *oy* . . . We thus obtain a series of lines . . . *oz*, *oy*, *ox*, *oa*, *ob*, *oc*, *od*, . . . whose lengths are in geometrical progression, the ratio of any two consecutive terms being 9 : 10.

Let these lengths be set off in succession along the radii from the pole *O*, *i.e.* make $OZ = oz$, $OY = oy$, $OX = ox$, $OA = oa$, $OB = ob$, and so on.

The fair curve through . . . *ZYXABC* . . . is the logarithmic spiral required.

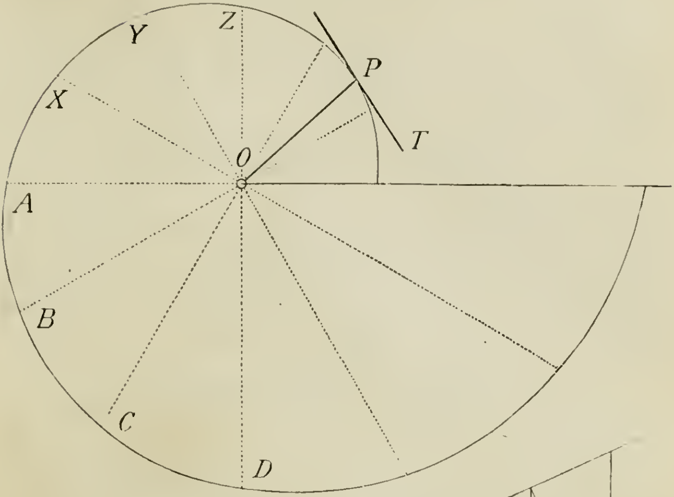
One property of this spiral is that the tangent *PT* at any point *P* makes a constant angle with the radius vector *OP* wherever *P* may be; the curve is therefore also known as the **equiangular spiral.**

129. PROBLEM.—**To draw a logarithmic curve.**

Draw any straight line *OX*, and along it, by applying the scale, or otherwise, mark off a series of points, 1, 2, 3, 4, . . . at equal distances apart, and erect perpendiculars at the points. Set off on the perpendiculars in succession a series of lengths which are in geometrical progression, found as in the last problem.

The fair curve through the ends of these perpendiculars is the logarithmic curve required.

The base *XO* is an *asymptote* to the curve.



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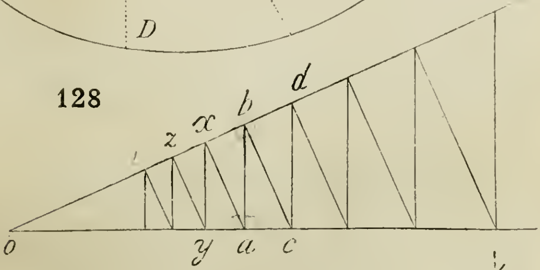

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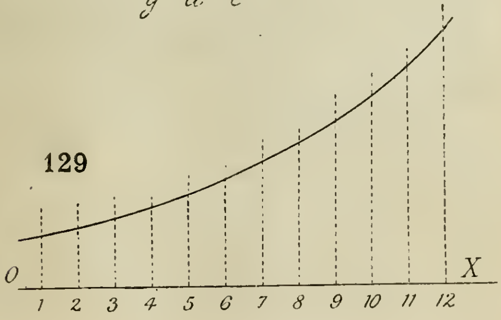


130. PROBLEM.—**To draw a curve of sines of given amplitude.**

With centre *O*, and radius equal to the given amplitude, describe a circle. Draw *OF* and take any two points *D*, *E* on this line.

Divide the circle and *DE* into any and the same number of equal parts, say 16, numbering the parts in each case from 0 to 16.

At any point, say 3, on *DE* erect a perpendicular to meet a line drawn parallel to *OF* from 3 on the circle; this gives *one* point on the required curve; repeat this construction for each of the remaining points, and finally draw the fair curve through the points on the perpendiculars. This is the curve required.

A point of inflexion occurs where the curve cuts *DF*.

This curve is the same as the projection of a helix or screw thread. See Prob. 369.

The ordinate 33 of the curve is proportional to the *sine* of the angle *AO*3, since the sine is equal to the ordinate 33 divided by the radius of the circle. See Art. 13. Hence the name of *sine curve* or curve of sines.

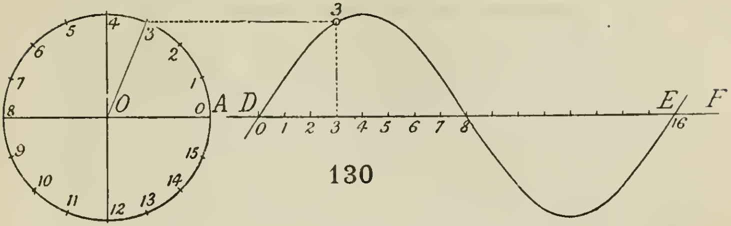
131. Simple harmonic motion. Definitions.

Suppose a point *P* to move at a *constant speed* in a circular path; and let *PM* be a perpendicular from *P* to a fixed diameter *AB*. If this perpendicular move with *P*, its foot *M* is said to execute a *simple harmonic motion* or *simple vibration*. The circle in which *P* moves is called the *directing circle*, and *P* is the *directing point*.

The **amplitude** of the vibration is equal to *CA* or *CP*; that is, to *half* the travel of the vibrating point *M*, or to the *radius* of the directing circle.

The **period** of the vibration is the time which elapses while *P* goes *once round* the directing circle; or it is the time occupied in a complete vibration of *M*, there and back.

The **phase** of the vibration for any position *M* is the



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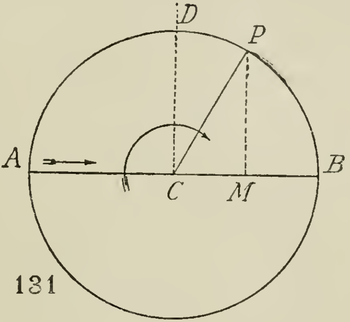

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fraction of the period which has elapsed since the moving point last passed through its middle position in the positive direction. Thus in the figure let from *A* to *B* be the positive direction, and let the rotation of *CP* be clockwise, as indicated. Then the phase for the position of *M* shown in the figure is the angle *DCP*, expressed as a fraction of one revolution. Thus to obtain the phase we might measure the angle *DCP* in degrees and divide by 360°. For a phase of $\frac{1}{8}$th, *DCP* is 45°. If *M* were at *A*, the phase at that instant would be $\frac{3}{4}$.

Examples.—1. Set out a curve of sines the amplitude *OA* being $1\frac{1}{2}''$ and the base *DE* 6″ long.

2. Measure the series of equidistant ordinates or perpendiculars for that portion of the curve situated *above* the base *DE*, Ex. 1. Calculate the mean ordinate by Simpson's second rule, Prob. 43. *Ans.* .95″.

132. Component and resultant motions.

We now consider the path of a point which has two or more motions simultaneously given to it. The motion which the point actually has is called the *resultant*, while the independent motions giving rise to this are named *components*.

In order to fix our ideas, let us suppose that a point moves uniformly towards O over a length PQ of the bar AO, while the bar rotates uniformly about a fixed point O through the angle AOA'; then the point will arrive at Q' by way of a certain curve PBQ'. The point may, however, be supposed to move from the position P to that of Q' in either of the two following ways amongst others:—

First, let the point move from P to Q while the bar remains in the position AO; then allow the bar to rotate through the angle AOA'; thus the point arrives at Q'.

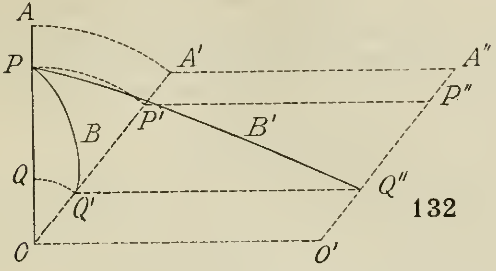
Secondly, let the bar move from AO through the angle AOA' without allowing the point to move along the bar, P will move to P'; then let the point move on the bar from P' to Q'. The point again reaches Q'.

In each of these two latter ways the point receives its component motions *in succession*. Although the paths are different, the final position Q' is the same in all three ways.

Now let us further suppose that, in addition to the two component motions already referred to, the point receives a third component motion, due to O having a uniform motion from O to O' along OO', the three taking place simultaneously; in this manner the point will arrive at Q'' by way of a certain curve $PB'Q''$.

The point may, however, be brought from the position P to that of Q'' by allowing it to receive the first two component displacements in either of the above two ways, and then allowing O to move along OO' to O', the bar moving from OA' to $O'A''$ *without rotation*; thus the point will arrive at Q''. By giving the component displacements in succession in every possible order it is seen that there are *six* ways in which the point may arrive at Q''. From these observations the truth of the following statement will be manifest:—

If a point has a number of simultaneous component motions impressed upon it, it can be brought from any one of its positions to any other, by giving to it its corresponding component displacements in **succession,** *and in* **any order.**



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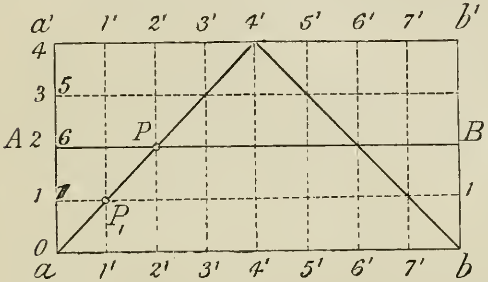

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133. PROBLEM.—**A point P moves backwards and forwards at a constant speed between two points A and B in a straight line, and the line has a similar motion between given limiting positions ab and a′b′. The time of the first oscillation is double that of the second, and the point starts from the position a. Determine the path traced by the point.**

Divide *aa′* into a number of equal parts, say *four*, and divide *ab* into *double* the number of equal parts, that is *eight*. Draw the dotted lines as shown.

From the conditions it is evident that when *AB* occupies the position 11 say, the point *P* will be in the line 1′1′, so that P_1 will be the actual position of *P* at this instant.

Proceeding in this way, the zigzag path *a*4′*b*4′*a* is obtained, as required.

134. PROBLEM.—**A point M has two component simple harmonic motions of the same period in directions at right angles to one another; to trace the curve described by the point, having given the amplitudes, and the phases at any instant.**

Refer to Art. 131 for definitions.

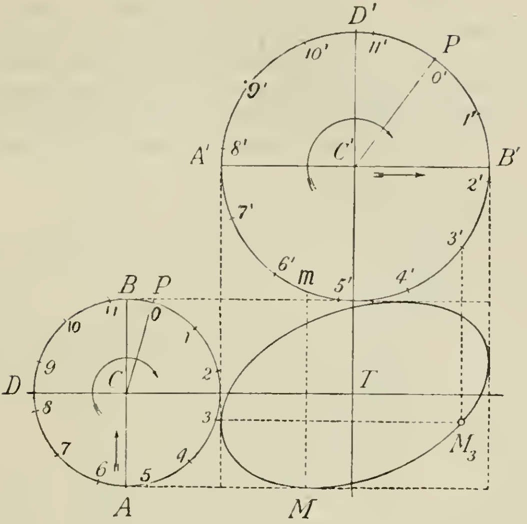
Draw any two perpendicular lines intersecting in T, in which take points C and C'. It is convenient to make TC and TC' each equal to the sum of the given amplitudes.

With centres C and C' describe circles with radii equal to the given amplitudes, and draw the diameters AB, $A'B'$ perpendicular to TC, TC' as shown.

Choosing as positive directions those indicated by the arrows on the diagram, viz. the directions from A to B and A' to B', with clockwise rotations, then D and D' are the positions of the directing points which correspond with zero phase. Now set off the angles DCP, $D'C'P'$, clockwise, fractions of one revolution corresponding to the given phases. In the figure, the angle $D'C'P'$ is for a phase of $\frac{1}{10}$th, and is made equal to $\frac{1}{10}$ of 360° or 36°. The angle DCP is 105°, the phase being $\frac{105}{360}$ or $\frac{7}{24}$.

Next divide the two circles into the same number of equal parts, say 12, P and P' being the zero points, and the points of division 0, 1, 2 . . ., 0′, 1′, 2′ . . ., proceeding clockwise. From corresponding points draw lines respectively parallel to CT and $C'T$; their intersections will give points on the required curve. A pair of these, those from the points 3, 3′, are shown intersecting in M_3, the others being omitted to avoid confusing the figure.

The curve is an ellipse, and the circumscribing rectangle should be drawn as shown, the points where the ellipse touches the rectangle being determined. Thus to find the point of contact M. When the tracing-point is at M, P is at A, between 5 and 6, and P' is at m between 5′ and 6′. Therefore find m by setting off the angle $5'C'm$ equal to the angle $5CA$, then from m draw a line parallel to $C'T$ to intersect the side of the circumscribing rectangle in M.



Examples.—**1.** A point M has two component simple harmonic motions of the same period and equal amplitudes of $1\frac{1}{2}''$, in directions at right angles to one another; set out to scale the curves traced by M, the initial phases being (*a*) 0 and 0; (*b*) 0 and $\frac{1}{4}$; (*c*) 0 and $\frac{1}{12}$.

Ans. (*a*) a straight line; (*b*) a circle; (*c*) an ellipse.

2. A point M has two component simple harmonic motions in perpendicular directions. The amplitudes are $1''$ and $1\frac{1}{2}''$; periods 1 to 2; and initia phases (*a*) 0 and 0; (*b*) 0 and $\frac{1}{4}$; (*c*) 0 and $\frac{1}{8}$. Set out the paths of M.

Ans. (*a*) Double-looped curve having a node and a point of inflection on each branch at the centre; (*b*) a parabola.

3. Determine the curves traced by a point M which has two component simple harmonic motions at right angles as follows: Amplitudes $1''$ and $1\frac{1}{2}''$; periods 2 to 3; initial phases (*a*) 0 and 0; (*b*) 0 and $\frac{1}{4}$; (*c*) 0 and $\frac{1}{12}$.

4. Trace the curves described by a point M which has two component simple harmonic motions in perpendicular directions of amplitudes $1''$ and $1\frac{1}{2}''$; periods 3 to 4; and initial phases (*a*) 0 and 0; (*b*) $\frac{1}{4}$ and $\frac{1}{4}$; (*c*) $\frac{1}{4}$ and 0.

135. Problem.—**A point starting from the position P moves uniformly round the circumference of the circle, centre C, while the circle turns about the fixed point O in such a manner that the diameter OA moves through the angle AOA′ of 90° and back again without stopping with uniform angular velocity; determine the locus of the point P.**

With centre O draw the arc AA'.

Divide the circumference of the circle, centre C, into a number of equal parts, say *twelve*, and the arc AA' into half the number, *six*.

The method adopted is that of finding a series of positions of P, giving a succession of points on the curve, and we shall illustrate the method by determining the position of the tracing-point when it has moved over one-twelfth of the circumference of the circle.

Let the point receive its component motions in succession, Art. 132. First suppose that it receives its circular motion about C; this will carry it from P to 1.

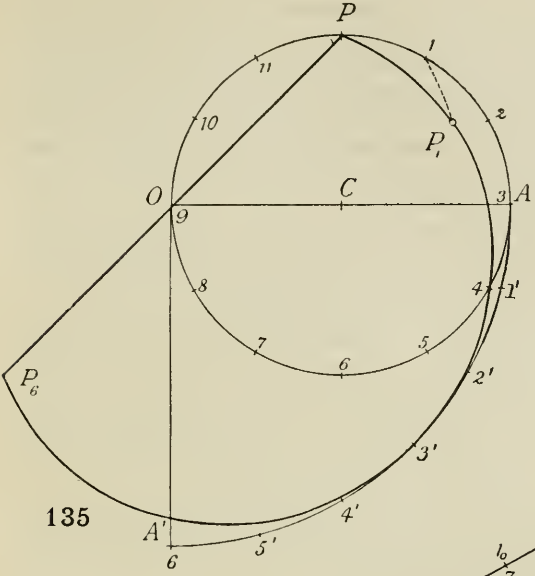
Next, suppose that the circle turns about O until OA arrives at $O1'$; during this second motion the point turns about O, moving from 1 to P_1, where the angle $1OP_1 =$ angle $AO1'$.

P_1 may be determined from the fact that the triangle $O1'P_1$ is equal in all respects to the triangle $OA1$.

Proceeding in this way, the series of points may be found, and a fair curve drawn through them.

The path from P_6 to P is a straight line.

136. Problem.—**OX is a vertical axis, and OA the initial position of a rod which turns about O in a plane containing OX until it has described an angle which is double the angle AOX, the point O during the same time moving along OX to O′. If both of these motions be uniform, determine the locus of a point which starts from O and moves at a constant speed along the rod from O to A and back again in the time that O moves to O′.**


P
1
11
10
2
P₁
O
C
3
A
9
8
4
1'
7
5
6
2'
P₆
3'
4'
A'
5'
6

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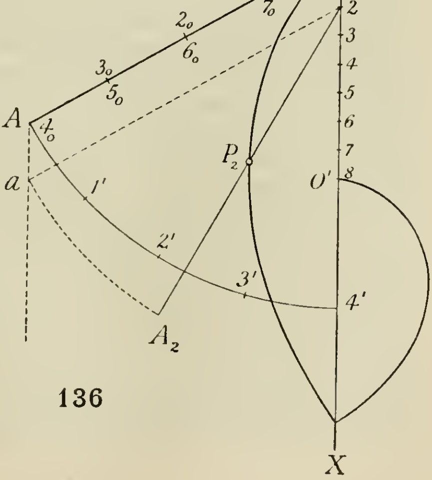
O
0
1
1₀
7₀
2
2₀
3
6₀
4
3₀
5
5₀
A
4₀
6
7
P₂
a
O'
8
1'
2'
3'
4'
A₂
X

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The point has three component motions: (1) a motion along the rod; (2) an angular motion about O with the rod; (3) a motion due to the point O sliding down OO'.

Divide OO' into a number of equal parts, say *eight*: also divide OA into half the number, that is *four* equal parts. With O as centre describe the arc $A4'$, and divide this arc into four equal parts. In each case number the points of division as shown.

We shall show how to determine the position of the tracing-point when one-fourth of the total time has elapsed; that is, when O has moved to the point 2.

Taking the component displacements in succession:

1. Suppose the bar to have its vertical motion bringing it to the position $a2$, where $Aa = O2$.

2. Let the bar then have its angular motion; that is, with 2 as centre describe the arc aA_2, making $aA_2 = A2'$.

3. Now let the point move along the bar; that is, make $2P_2 = O2_0$. Then P_2 is the desired point.

Repeat this construction for the eight positions.

The component motions might have been given in any order, but probably that indicated will involve least trouble.

The path of P is shown.

137. Motions under mechanical constraint.—All the curves and in fact the lines of all the geometrical figures we have yet considered may be regarded as having been described by points moving under some kind of constraint. A straight line, for example, by a pencil point guided by a straight-edge. A circle by a point controlled by a second point (or axis) embedded in the material of the paper and drawing-board. In other cases we have had geometrical conditions imposed on the motion of the tracing-point, and by means of the constraints of ruler and compasses we have found a series of isolated points in the required path, and have completed the curve either by muscular constraint, as when drawing a fair curve by free-hand through the points, or by the use of a template, such as a French curve or bent spline, to guide the pencil.

We shall now give some problems relating to the motions of parts of mechanisms or machines. The constraint in such cases may be described as mechanical. The problems and methods of solution do not differ essentially from those preceding; it is only the form of the data that is new. We are not required to make the mechanism nor a model of it, so as to allow the moving point in it to actually trace its own curve; nor are we even required to make a drawing of, or to be acquainted with, the constructional details of the mechanism.

A mechanical constraint always has its geometrical counterpart, and the first thing is to realise what this is and how it is to be represented. In the majority of cases the mechanical constraints consist only of sliding and turning pairs of elements, represented respectively by straight lines and circles. The moving pieces may also be represented by lines.

We thus set out to scale a line or "skeleton" diagram of the mechanism, putting in only such lines as are essential to geometrically represent the moving pieces and the several constraints. This figure, or a portion of it, is then drawn for a number of positions of the mechanism as the latter moves through all of its possible positions, and the corresponding positions of the point under consideration are noted. A fair curve drawn through the series of points thus determined gives the required path.

This path is always a closed curve for a machine which works continuously, repeating its cycle of operations, since in such a machine no point can go off to infinity. This, however, would not necessarily be the case if we were finding the *envelope* of a moving piece instead of the path of a point.

It will often happen that the piece during its motion assumes *critical* or *limiting* positions, the accurate determination of which is desirable in order to draw the curve to the best advantage. Illustrations will appear in the particular problems to which we now proceed.

138. PROBLEM.—**To find the path of a point in the connecting rod of a direct-acting steam engine.**

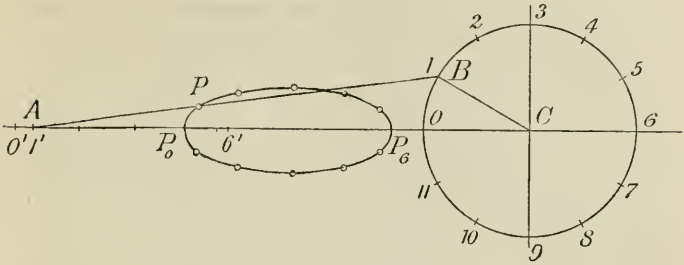
The figure is a skeleton diagram representing the mechanism in one of its positions. *CB* is the crank, turning about *C*, and constraining the crank pin *B* to move in a circle. *BA* is the connecting rod, with the end *B* centred on the crank pin, and the end *A* compelled to keep to the straight line *AC* by means of a slide. *P* is the point in the connecting rod whose locus is required. The problem, stated geometrically, would read thus :—

A given line AB of constant length moves with one end B always on a given circle, centre C, and the other end on a straight line directed through C ; find the locus of a point P in AB.

We must draw the connecting rod *AB*, and thus locate *P*, for a number of positions in the cycle, say twelve positions. To do this in the best way, divide the crank pin circle into twelve *equal* parts, beginning at *O* (most quickly effected by 30° and 60° set-square). With these points, 0, 1, 2 . . . 11 as centres, and radius equal to *BA*, describe arcs cutting the line in which *A* moves, thus obtaining the twelve positions of *A* which correspond to the twelve positions of *B*. The connecting rod *AB* is shown in the figure for one of these pairs of positions, viz. 1, 1′ ; and the point P_1 is marked off at the given distance from one end. In determining the locus of *P* the other positions of the connecting rod were drawn, but for clearness have been omitted in the diagram.

In this and similar problems we might with advantage again make use of a transparent template. Thus draw the circle and the line *AC* on the paper ; then mark the points *A*, *P*, *B* on tracing-paper or celluloid, and adjusting the template in succession to a number of positions, with *A* on the line, and *B* on the circle, in each case prick off the position of *P*. It is now hardly worth while to divide the circle into equal parts.

Examples.—1. In a direct-acting engine the crank is 1 foot long and the connecting rod 4 feet ; find the locus of the middle point of the rod. Scale $\frac{1}{8}$th.



2. The middle point O of a Gooch link of 48″ radius is guided along the centre line of a horizontal engine. The ends A, B of the link are 20″ apart and are connected by "open" rods AP, BQ, 60″ long, to two eccentrics CP, CQ each of radius 4″ and with angular advance of 30°. Find the motion of the valve at half gear, the link-block then moving in a horizontal line DD 5″ below the centre line.

Solution.—Draw the horizontal centre line KC. Set off angles $KCP = 120°$ above KC, and KCQ the same below, and make CP, CQ each 4″. With centre C draw the circle through P and Q. Divide its circumference into *twelve* equal parts, numbered 0 to 11, the divisions 0 and 4 being at P and Q. With these points as centres, radius 60″, draw twelve arcs in *ink*, 0 to 11, extending about a foot above and below K. Draw a horizontal line DD 5″ below K.

On tracing-paper stiffened by a lath draw an arc of 48″ radius, on which mark A, B 20″ apart, and the mid point O of the arc.

Now place this template with its convex side towards C, and adjust it so that its lower and upper points A and B, and its mid point O, lie on the arcs 0, 4, and the centre line respectively. Then prick through the point D_0 where the curve on the template crosses DD.

Again adjust the template, O being on KC as before, but A and B now lying on arcs 1 and 5. Prick through D_1 on DD. Repeat this for the arcs 2 and 6, determining D_2; for 3 and 7, obtaining D_3; and so on.

Measure the displacements D_0, D_1, D_2 . . . D_{11} of the valve from the *average* position of D. *Ans.* 2.71″, 3.08″, 2.64″, 1.55″, 0.13″, −1.34″, −2.46″, −3.01″, −2.83″, −1.84″, −0.21″, 1.51″.

(From which, by Fourier's analysis, $d = 3.07 \sin(\theta + 57°) + 0.14 \sin(2\theta + 103°)$. θ is the crank angle measured from CK.)

139. Problem.—**To set out the complete path of the guiding point in Watt's simple parallel motion.**

This mechanism consists of two "radius rods" or "levers" CA, DB, centred at C and D, having their ends A and B connected by a link AB. The guiding point P is situated in this link, and in such a position that P divides the link into two segments having a ratio inversely as the radius rods. That is, $AP : PB = DB : AC$. The locus of P is a two-looped curve something like a figure of 8, and in the neighbourhood of the node or double point the two branches of the curve are very flat, being almost straight lines. Hence P serves as a point of attachment for a piece which moves to and fro through a limited range, and requires to be guided in an approximate straight line, such as the pencil of a Richard's Indicator, or the top of the piston rod in a beam engine.

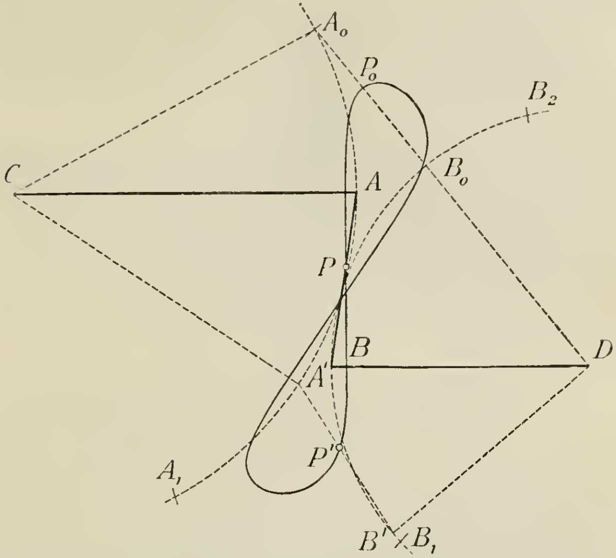
To trace the locus of P. With C and D as centres, describe circular arcs through A and B respectively. Then A and B move in these paths.

Next trace the link APB on celluloid with a needle-point, marking the three points A, P, B by short cross lines. Move this template, placing A in a new position A' on the arc through A. Insert the pricker at A', rotate the template until B comes on the other arc at B', then prick off the point P at P'. Repeat this operation a sufficient number of times to enable the locus of P to be drawn.

Or operate with compasses in the usual manner.

Limiting positions.—Observe that the highest possible position of A is A_0, where DB and BA come into one straight line DB_0A_0, and where, therefore, $DA_0 = DB + BA$. Similarly its lowest position is A_1, where $DA_1 = DB + BA$. In like manner the limiting positions of B are B_1 and B_2, where $CB_1 = CB_2 = CA + AB$.

Thus to find A_0 and A_1 describe arcs with centre D, radius $DB + BA$, to intersect the path of A in A_0 and A_1. To find B_1 and B_2 describe arcs with centre C, radius $CA + AB$, to intersect the path of B in B_1 and B_2.



Note also that A_0B_0 is normal to the curve at P_0, since for this position A_0 is the instantaneous centre of rotation of the link. This statement may be tested by trial with the template, and its truth will then be quite evident. The end A of the link, having come to the limit of its movement at A_0, is there *stationary* for an instant before retracing its path.

Examples.—**1.** In a simple parallel motion the lengths of the oscillating rods or levers DB and CA are 3 feet and 4 feet respectively, and the length of the connecting link AB is 2 feet. The mechanism is so set that when DB and CA are horizontal, AB is vertical. Find the position of guiding point P in the link, and set out the complete locus of P. Scale 1″ to 1′.

Ans. $AP = \dfrac{DB}{CA+DB} \cdot AB = \dfrac{3}{7}$ of 2′ = 0·857′.

Or, geometrically:—Join CD to intersect AB in Q, and make $AP=BQ$.

2. Work Ex. 1 when the centres C and D are moved 0.35′ nearer together horizontally, so as to give the link a slight inclination to the vertical when the rods are horizontal, as in Fig. 139.

Ans. $AP = \frac{3}{7}$ of $2' = 0.857'$ as before.

Note.—The best approximation to a straight line is obtained when the centres C and D are so adjusted that for its working range the link AB deviates from the vertical equally on each side.

3. In Ex. 1 suppose the centres C and D are both on the *same* side of the link, find the guiding point P in the link for an approximate straight-line motion, and trace the locus of P.

Ans. Produce BA to P, such that

$PA : PB = DB : CA$, or $PA : PB - PA = DB : CA - DB$;

that is, $$PA = \frac{3+2}{1} = 5 \text{ feet.}$$

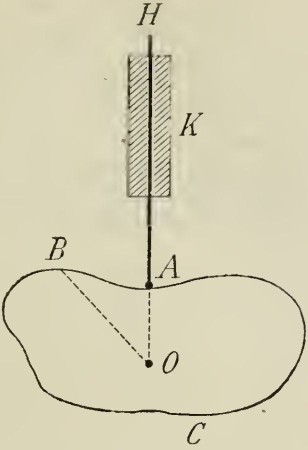
Or, geometrically:—For the position in which CA and DB are parallel, join CD and produce it to intersect produced AB in Q, then produce BA and make $AP = BQ$.

4. In a drag-link coupling the shafts are 6″ apart, the drag link 12″ long, and the cranks each 30″ long. Find the locus of the middle point of the link. Scale $\frac{1}{8}$th.

Or, stated geometrically:—Take two fixed points C and D, 6″ apart, and with these as centres describe circles each of 30″ radius. A line AB of the constant length of 12″ moves with A on the circle C, and B on the circle D. Find the locus of P which bisects AB. Scale $\frac{1}{8}$th.

140. Cams.—We shall conclude this chapter with an example of a *cam.* In previous examples on mechanisms the problem has been the direct one—given the mechanism, to find the consequent motion. We have now the inverse problem, generally more difficult—given the required motion, to design the constraining mechanism. When a machine designer requires a complicated motion, he generally has recourse to a cam or a combination of cams. Numerous examples are met with in textile machinery, printing machinery, and in many other machines.

One form of cam is shown in the figure. It consists of a flat plate with an irregular contour ABC, capable of rotating about an axis O, and so giving a reciprocating motion to a piece AH, which slides in a fixed guide K.



It is obvious that as the cam rotates, the piece *AH* will receive a rectilinear motion, the nature of which will depend on the shape of the curved edge *ABC* of the cam, and on how the latter rotates, whether at a constant or varying speed.

For example, while the cam turns through an angle which brings *B* under the end of the sliding piece, the latter will move upwards through a *distance* equal to the *difference of the radii OB* and *OA*.

Hence, supposing the cam to revolve at a uniform speed, we could (within limits) give any kind of motion to the reciprocating piece by suitably shaping the edge *ABC* of the cam. The motion will be repeated with each revolution of the cam.

In practice the moving piece would require to have a roller pinned to its lower end and bearing on the cam, in order to diminish the friction, and to prevent undue wear.

A spring also is generally required to keep the roller in contact with the cam when the piece is descending. Neither of these is shown in the diagram.

141. Problem.—**It is required that a reciprocating piece guided by a straight slide shall move at the same constant speed both ways, the change of motion being effected without interval. Set out the form of the cam which rotating uniformly will produce the motion, one revolution corresponding to one to-and-fro movement of the piece, and the axis of rotation being in the line of the slide. You are given the diameter of the roller, and the greatest and least distances of its centre from the axis of rotation.**

Let CAB be the line of the slide, taken vertical, C the centre of rotation of the cam, and A and B the extreme positions of the centre of the given roller, which is pinned to the lower end of the reciprocating piece.

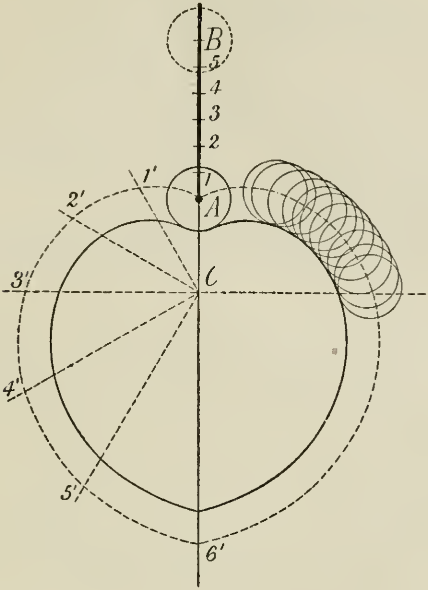
Divide AB into a number of equal parts, say six. Divide the half-revolution into the same number of equal angles; *i.e.* draw lines through C at angular intervals of 30° (with the 30° and 60° set-square). On these set off in succession the lengths $C1'$, $C2'$, $C3'$ . . ., equal to $C1$, $C2$, $C3$ . . . Then if the roller were a mere point the required shape of the cam, on one side, would be the fair curve through A, $1'$, $2'$, $3'$ . . ., which from Art. 127 is seen to be an Archimedian spiral.

To allow for the roller we draw a curve *parallel* to the spiral, determined as the envelope of a circle of diameter equal to the roller, moving with its centre on the spiral. See Art. 123. Some of these circles are shown to the right of the figure.

The right half of the cam is the same shape as the left half, the line of the slide being an axis of symmetry. This cam is known as the *heart-shaped* cam.

Examples.—1. Design a cam to give the following motion to a sliding piece :—Rise of $3''$ at a uniform speed during first quarter revolution; rest during second quarter; uniform fall of $3''$ during third quarter; rest during last quarter. Diameter of roller $\frac{3}{4}''$; least distance of its centre from axis of cam $2''$.

2. It is required that a point P shall move in a straight line with a speed which *increases* uniformly from zero during a vertical rise



of 3″; the motion is then to be suddenly changed, and the point is required to fall 3″ at a *constant* speed, the times of the rise and fall being equal. Design a cam which, while rotating uniformly, will each revolution impart this motion to P, the nearest approach of P to the axis of the cam being 2″.

Hint.—It is shown in mechanics that when the speed increases uniformly, the distance from the position of rest is proportional to the square of the time; that is in this case, for the rise, proportional to the angle turned through by the cam. Now in a parabola the abscissa AN is proportional to the square of the ordinate PN (Theorem 6, Art. 100). Therefore in Fig. 103 take AN 3″ to represent the path of P. Construct the parabola AQ, and from the points I., II., III. draw lines parallel to QN to meet AN in P_1, P_2, P_3. These are the positions of P at intervals of 45° during the rise from A to N.

3. Design a cam which shall give a rise at a uniform speed during the entire revolution, with an instantaneous drop at the end.

142. Miscellaneous Examples.

***1.** Four equal rods, *ab*, *bc*, *cd*, *da*, form a frame jointed at the angular points. The frame is also pivoted at *a*. The point *c* moves on the circumference of the given circle. Draw the curve traced by the point *x* on the bar *bc*. (1887)

***2.** Points *P* and *Q* are constrained to move uniformly along the given lines *ab* and *cd*. While *P* moves from *o* to *a* and back again, *Q* moves from *c* to *d*, and while *P* moves from *o* to *b* and back again, *Q* moves from *d* to *c*. Trace the locus of a point on *QP* produced $2\frac{3}{4}''$ distant from *Q*. (1889)

***3.** *o* and *o'* are fixed pivots about which the bars *oa*, *o'b* can freely revolve ; *ab* is a coupling bar connecting the free ends of *oa* and *ob*. Draw the complete locus traced by the centre point of the bar *ab*. (1893)

***4.** A circle *A*, of diameter *EF*, rolls on the line *CD* with uniform motion from left to right, starting from *E*. Another circle *B*, whose diameter is half that of *A*, rolls inside the circumference of *A*, also with uniform motion, but from right to left, tarting at *E* when *A* begins to move. Circle *B* is in contact with circle *A* at *F* at the same time that *F* reaches the line *CD*. Draw the curve described by the centre of circle *B*. (1897)

***5.** A point *P* revolves round the circle with centre *C* on the line *OD*, with an uniform motion. The point *C* moves from *C* to *D* and back from *D* to *C*, also with an uniform motion, returning to *C* at the same time that *P* has completed one revolution.

(*a*) Draw the path of *P*. (*b*) Supposing that at the same time the line *OD* moves round the point *O* uniformly, making a complete revolution while *P* revolves once round the moving point *C*; draw the path of *P*. (1895)

***6.** Two bars, o_1a, o_2b, are pivoted at o_1 and o_2 respectively. At *i* they pass through a saddle which can travel along o_1a at $\frac{2}{5}$ths the speed at which it can move along o_2b. Trace the locus of *i*. What is this curve? (1888)

7. Two lines meet at an angle of 60°. Find the locus of points in the interior of the angle such that the sum of their distances from the two given lines is constant, and equal to $2\frac{3}{8}''$. (1897)

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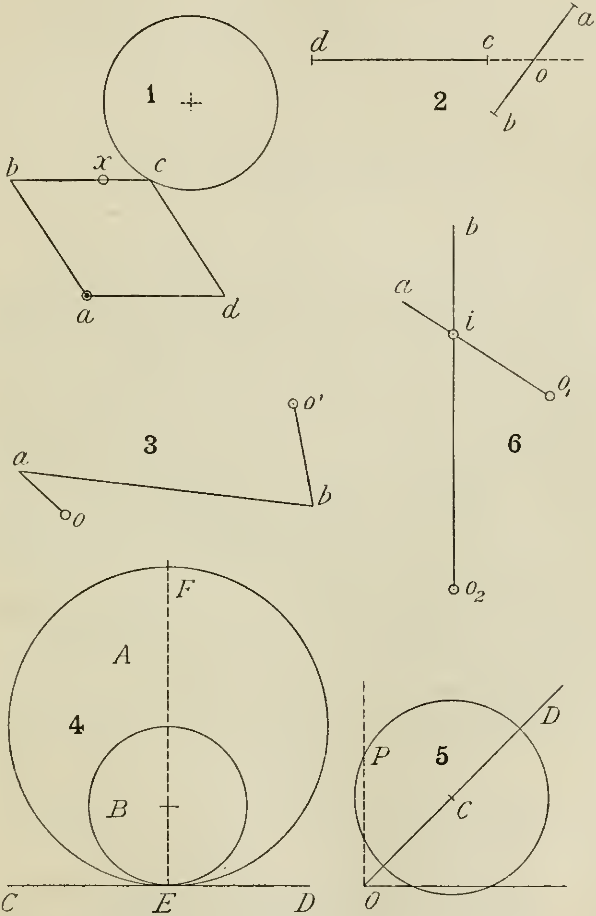


CHAPTER VI

CO-ORDINATES—PLOTTING ON SQUARED PAPER

143. The position of a point in a plane. Co-ordinates of a point.—Our object in this article is to show how the position of a point in a plane may be defined. We shall illustrate the case by means of a concrete example.

Suppose a person wishes to note the position of some simple object such as a boy's marble on the floor of a room, so that he may make a scale drawing which shall exhibit this position. After measuring the room he will require to make *two* further measurements of the position of the object. There is considerable choice in the latter, as appears from what follows.

To fix the ideas, let the room be rectangular, 18 feet long and 12 feet broad, as set out to scale in the figure.

The position in the room of the object P may be observed in the following four ways amongst many others :—

1. By measuring the distances of P from any two adjacent sides of the room. For example,

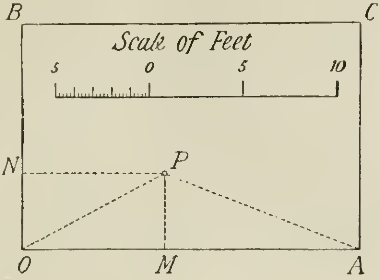
   $PN = 7.6'$, $PM = 4.1'$.

2. By measuring the distances of P from any two corners of the room. Thus

   $OP = 8.6'$, $AP = 11.2'$.

3. By measuring the distances of P from one corner and from one side of the room. Say

   $AP = 11.2'$, $PN = 7.6'$,



4. By measuring the distance of P from one corner of the room, and the angle which the line from P to the corner makes with one of the sides; *e.g.*—

$$OP = 8.6', \qquad AOP = 28.4^\circ.$$

Any one of the above four pairs of measurements completely defines the position of the point on the floor; any one pair being given, the others could be determined by calculation or construction. The two measurements of any of these pairs are called the **co-ordinates** of the point; so we may have different *systems* of co-ordinates.

Rectangular co-ordinates are illustrated by Case 1. This system is the one most generally useful.

Polar co-ordinates are illustrated by Case 4. This method of defining position is frequently employed.

In this chapter we shall confine attention to rectangular co-ordinates. The above illustration does not represent the most general case. It requires to be amplified so as to include points outside the room on the floor-level. This is done by the convention of *positive* and *negative* co-ordinates, defined in the next article.

Example.—The floor of a room $OACB$ is 20 feet square. A small object P on the floor is 14.7 feet from the side OB and 10.3 feet from the side OA. Make a plan of the room showing the position of P to a scale of $\frac{1}{2}''$ to 1′ and measure

(*a*) The distances of P from O and A. *Ans.* 17.9 ft., 11.6 ft.

(*b*) The angles AOP and OAP. *Ans.* 35°, 62.75°.

(*c*) The distances of P from B and C. *Ans.* 17.6 ft., 11.1 ft.

144. Rectangular co-ordinates of a point P in a plane. In the plane draw any two perpendicular lines of reference XX' and YY' intersecting in O. These lines are called the **axes of co-ordinates,** or the *co-ordinate axes*, or simply the *axes*. The point O is called the **origin.**

From P draw perpendiculars PM, PN to the axes. The lengths of these lines are the **rectangular co-ordinates,** or the *co-ordinates* of the point *P referred to the axes.* The horizontal distance NP or OM, measured parallel to OX, is called the **abscissa,** or *x co-ordinate*, and is denoted by x; the vertical distance MP or ON, measured parallel to OY, is the **ordinate,** or *y co-ordinate*, and denoted by y.

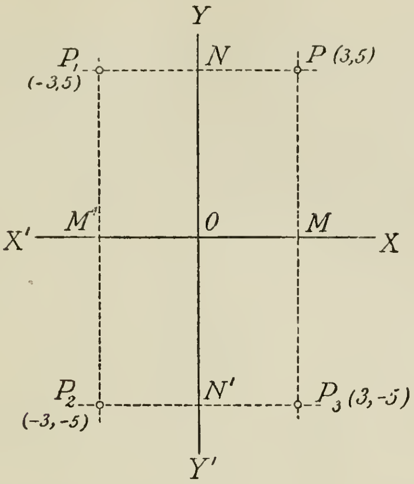
These co-ordinates serve to define the position of the point P in the plane. It is convenient to express this position by writing "*the point* (x, y)." Thus, suppose the abscissa PN or x were 3 units long, and the ordinate PM or y were 5 units, we might speak of P as the point (3, 5), being careful always to write the abscissa first.

Suppose it were required to **plot the point** (3, 5) on paper, this could be done in three ways:—

1. Along OX set off OM 3 units long; draw a perpendicular from M, and on it mark off MP, 5 units *upwards;* or
2. Along OY set off ON 5 units long; from N draw a horizontal line, on which set off NP to the *right*, 3 units long; or
3. Along OX and OY set off OM and ON 3 and 5 units long; through M and N draw lines perpendicular to the axes intersecting in P.

Observe that whichever method of plotting be adopted, in each case we measure from the axes 3 *units to the right* and 5 *units upwards*, in order to arrive at the position of P.

Suppose now the point were at P_2, the distances of 3 and 5 units from the axes being the same, but requiring to be set off *to the left* and *downwards*, instead of *to the right* and *upwards*. In writing down the co-ordinates, how could this case be distinguished from the last?



The convention adopted is to prefix the *minus* sign to the co-ordinates of P_2, writing them -2 and -3, and considering each as a negative quantity. We should thus speak of the point P_2 as the point $(-3, -5)$.

We might plot the point P_2 in any one of the three ways described for P; but in all cases, in order to arrive at P_2, we must mark off 3 units *horizontally to the left*, and 5 units *vertically downwards.*

Thus a *negative abscissa* is measured or set off *horizontally to the left* from the vertical axis; and a *negative ordinate* is measured or set off *vertically downwards* from the horizontal axis.

According to these definitions, the co-ordinates of P_1 are seen to be -3 and 5; and the co-ordinates of P_3 are 3 and -5.

We have thus made complete provision enabling us to define the position of any point on either side of either axis, that is, of *any point in the plane of the axes.*

145. The use of squared paper.—In order to facilitate the plotting of points, and especially of curves determined as a series of points, paper may be used which is covered with horizontal and vertical lines at equal intervals, ruled by machinery, and known as **squared paper.** The lines serve as horizontal and vertical scales, making ordinary scales unnecessary, and points are quickly located. For ordinary work the lines may be one-tenth of an inch apart, every fifth and tenth line being distinguished by a difference in width or colour.

A portion of a sheet of such squared paper is here shown full size; the main divisions are 1″ apart, and these (or the half-inch divisions) may be read as units, tens of units, hundreds . . ., or tenths, hundredths . . ., exactly as described in Art. 5, with regard to decimal scales. With care the position of a point may be plotted to within about the one-hundredth of an inch, if the lines are accurately ruled. Sheets of this paper 18″ by 11″ may be obtained at a very moderate cost, ruled in fine and faint blue lines, every fifth line being broader and more conspicuous than the rest, and ruled alternately in blue and red. The red lines are thus one inch apart, as are also the intermediate broad blue lines.

When using squared paper the axes of co-ordinates may be chosen so as to have any convenient positions on the sheet, according to circumstances, and the paper may be placed with long edges either horizontal or vertical.

Examples.—**1.** Take a sheet of squared paper, and, selecting the origin at a corner of a 1″ square somewhere near the centre of the sheet, plot the following points, taking 1 inch as the unit :—

$(2, 5)$; $(6, 3)$; $(2.5, 4.6)$; $(1.44, 3.55)$; $(-3, 4)$; $(-3, -1.5)$.

Note.—After selecting the origin, mark the divisions to the left and right, and up and down, as for an ordinary scale, but with the additional negative values.

2. Again selecting the origin near the centre, but taking 1 inch to represent 10 units, plot the following points :—

$(25, 36)$; $(-42, 25)$; $(30.4, -40.8)$.

3. Taking 1 inch to represent 100 units, plot the following points :—
(60, 360) ; (55, 400) ; (404, 295).

4. Plot the points (1, 2) and (7, 10) ; join them by a straight line, and *read off* the *ordinates* of those points on the line whose *abscissæ* are 3, 6, 4.2 respectively ; also read off the *abscissæ* of those points whose *ordinates* are 3, 5.2, 9.6 respectively.
Ans. 4.67, 8.67, 6.27 ; 1.75, 3.4, 6.7.

5. Plot the points (0, 3) and (4, 0) ; join them by a straight line, and find the length of the perpendicular drawn to the line from the origin. Take $\frac{1}{2}''$ as the unit. *Ans.* 2.4″.

Note.—The above examples will show the student that considerable care and thought are necessary in selecting the position of the origin and choosing the scales, in order to secure the best results.

146. The equation to a curve.—All curves of known form, such as any of those we have yet considered, have characteristic properties capable of definite expression in some way or other. Take for instance the hyperbola.

In the figure a hyperbola is shown in which the asymptotes XX', YY' are at right angles, and therefore called a *rectangular* hyperbola. Let P be any point on the curve, and draw PM, PN parallel to YO, XO. Now take the asymptotes as axes of co-ordinates, and denote the abscissa and ordinate of P (PN and PM) by x and y, in the usual way. It was explained in Art. 105 that one of the properties of a hyperbola is that the *product of PN and PM is the same for all positions of P on the curve;* or, expressed in the form of an equation,

$$PN \times PM = \text{constant};$$

that is, $\text{abscissa} \times \text{ordinate} = \text{constant},$

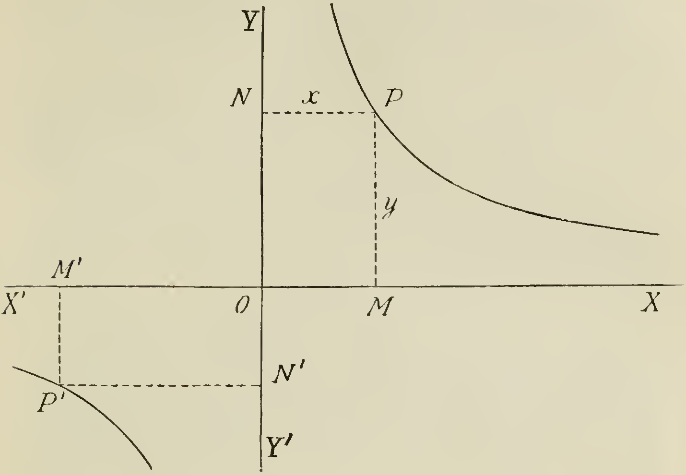
or $$xy = c \quad . \quad . \quad . \quad (1).$$

This way of stating a property which characterises the curve is called the *equation to the hyperbola;* or, more definitely, it is the *equation to the hyperbola referred to the asymptotes* as axes.

Thus the *equation to a curve* is the expression of a characteristic property in the form of an equation in terms of some system of co-ordinates and constants.

If the reader be making an acquaintance for the first time with the idea of an equation to a curve, let him consider well its meaning. Here, in the case before us, x denotes a distance which is *altering* as the point moves along the hyperbola; so also is y altering; but their product does not alter, it is constant so long as we are considering the *same curve.*

Whatever curve we may have to deal with, if there be some *geometrical fact* which is known to be true for any and every point on it, then the method of representing the position of a point by its x and y enables us to express the *same fact* by means of an *algebraical equation.* It is, so to speak, merely a shorthand way of expressing such fact.



Suppose that in the figure $PN = 3$ and $PM = 5$, then the value of c in the equation to the curve will be 3×5, or 15. Accordingly, at the point on the curve where the x is $2\frac{1}{2}$, the y will be 6; at the point where x is 1, y will be 15; if $x = -5$, $y = -3$, and so on.

Thus the equation to any curve involves the idea of a point which moves along the curve, and although the co-ordinates change their values, yet the two co-ordinates are related to each other in a way which does not alter.

If a different system of co-ordinates be taken, the equation to the *same curve* will be *different*. Suppose we adopt system 2, Art. 143, and define the position of a point by its distances from two fixed points. Let the two fixed points be the foci F and F' of the hyperbola. A well-known fundamental property of the curve is that the *difference* of the focal distances is constant (Art. 105);

i.e. $F'P - FP =$ constant, or $r' - r = c$,

where r and r' are now the co-ordinates of P in system 2.

This, therefore, is an equation to the hyperbola. The corresponding equation to the ellipse is

$$r + r' = c.$$

Or take system 3 of Art. 143, and let A (Fig. 143) be the focus, and OB the directrix of the hyperbola. Then if P be any point in the curve, it is known (see Art. 76) that

$AP : PN =$ a constant number greater than unity.

Or writing r and x for the co-ordinates AP and PN in this system, the equation is

$$r = cx;$$

where c is a constant greater than unity.

If c be less than unity, the equation represents an ellipse. If $c = 1$, the equation represents a parabola.

If the student pursues this interesting and important branch of mathematics in works on *Analytical Geometry*, he will find that *any* curve which is of definite form has definite equations which represent it. The *form* of the equation depends on the particular system of co-ordinates which may be adopted as well as on the curve.

It is to be understood that in future we shall keep to the *rectangular* system.

Examples.—1. Find the rectangular equation to a circle, radius a, referred to two perpendicular diameters as axes.

Solution.—From any point P on the circle draw perpendiculars PM, PN on the two diameters or axes OX, OY, where O is the centre of the circle.

Then, by Euc. I. 47, we have the geometrical property,

$$PM^2 + PN^2 = OP^2,$$

which expressed algebraically becomes

$$x^2 + y^2 = a^2.$$

2. Using the property of a parabola stated in Theorem 6, Art. 100, find the equation to the parabola, taking the axis of the curve as the X axis of co-ordinates, and the tangent at the vertex as the Y axis.

Solution.—See Fig. 100. The property in question is

$$PN^2 = LL_1 \times AN,$$

or

$$y^2 = lx,$$

where l is the latus rectum LL_1 of the parabola.

147. To plot a curve, having given its equation.—In the preceding article it was stated that every known curve had some equation which represented it, such equation being only a particular way of expressing the law of the curve.

On the other hand, the converse proposition is true, namely—

Proposition.—Every equation connecting the co-ordinates of a point represents a continuous curve or line of some kind.

We do not attempt to *prove* this general proposition; our object is to *illustrate* it by showing how to plot the curve in an actual example where the equation is given.

Let the equation be $y = 0.2x^2$.

A student acquainted with elementary algebra will recognise in this an *indeterminate* equation; that is to say, one that cannot be solved in the ordinary sense by finding *definite* values for x and y; but there is an *indefinite number* of solutions, or pairs of values of x and y which satisfy the equation.

Now the above proposition is equivalent to saying that if *all* the points given by the unlimited number of solutions, or pairs of co-ordinates x and y, be plotted, such points will not cover the whole plane of the paper, but will be confined to definite curves or lines, and will occupy these lines entirely, without gaps or breaks of continuity. In fact, that the result is the same as if a point were to move in accordance with the law expressed by the equation, and actually trace the continuous curve.

Let us now test this by actually plotting a number of the solutions, and noting the result.

To plot the curve whose equation is $y = 0.2\,x^2$, we take *any* values for *one* of the co-ordinates, and *calculate* the corresponding values of the other. Thus

Take $x = 0$, then $y = 0.2(0)^2 = 0$.
Again take $x = 1$, then $y = 0.2(1)^2 = 0.2$.
And take $x = -1$, then $y = 0.2(-1)^2 = 0.2$.
$x = 2$, $y = 0.2(2)^2 = 0.8$.
$x = -2$, $y = 0.2(-2)^2 = 0.8$.
$x = 3$, $y = 0.2(3)^2 = 1.8$.
$x = -3$, $y = 0.2^2(-3) = 1.8$.
$x = \pm 4$, $y = 3.2$, and so on.

Or we might assign any values to y, and then calculate the values of x. If this is done it will be convenient to transform the equation, and write it

$0.2x^2 = y$, or $x^2 = 5y$; that is, $x = \pm\sqrt{5y}$.

Now take, say, $y = 0$, then $x = \pm\sqrt{5 \times 0} = 0$.
Take $y = 1$, then $x = \pm\sqrt{5} = \pm 2.24$.
Take $y = -1$, then $x = \pm\sqrt{-5} =$ impossible.
$y = 2$, $x = \pm\sqrt{10} = \pm 3.16$.
$y = -2$, x impossible.
$y = 3$, $x = \pm\sqrt{15} = \pm 3.87$.
$y = 4$, $x = \pm 4.47$, and so on.

Before plotting the points, it will be convenient to arrange the results in **tabular form**, somewhat as follows :—

TABLE GIVING SOLUTIONS OF THE EQUATION $y = 0.2x^2$.

Values of x	0	±1	±2	±3	±4	..	±2.24	±3.16	±3.87	±4.47	..
Values of y	0	0.2	0.8	1.8	3.2	..	1	2	3	4	..

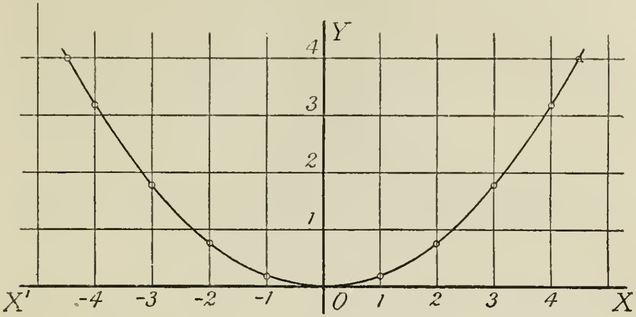
In plotting these points on squared paper, remember that the abscissa x is set off horizontally from OY, to the right if positive, to the left if negative; and that the values of the ordinate y are marked off vertically from OX, upwards if positive, downwards if negative.

The result of plotting the above seventeen points is seen in the figure.

Note.—The subdivisions of the squared paper are omitted in the figure.

If *all* possible intermediate pairs of values of x and y had been calculated and plotted, the result would have been the curve shown in the figure.

It will be found that the co-ordinates of all points on the curve satisfy the given equation, or are solutions; and that the co-ordinates of any point *off* the curve do *not* satisfy the equation. That is, the points occupy *the curve*, the *whole curve*, and *no place but the curve.*



The curve does not extend below XX', since negative values of y lead to impossible values of x. The co-ordinate axis OY is an axis of symmetry of the curve. The upper branches extend to infinity, x and y becoming indefinitely great together.

The given equation $y = 0.2x^2$, or $x^2 = 5y$, merely expresses the law of the curve that one of the co-ordinates is proportional to the square of the other. Now referring to Art. 100, it is seen that this is one of the distinguishing properties of a parabola; the curve therefore is a parabola. On further comparison of the equation with the properties stated in Art. 100, it is readily inferred that OY is the axis of the parabola, XX' the tangent at the vertex; that the length of the latus rectum is 5; and the distance of the focus from the vertex O is $\frac{1}{4}$ the latus rectum; that is, $1\frac{1}{4}$.

Examples.—Plot the following curves on squared paper, after having first calculated and tabulated the co-ordinates of a sufficient number of points on each:—

1. $y - 2 = 0.2x^2$.
2. $y = 0.2(x-3)^2$.
3. $y - 2 = 0.2(x-3)^2$.
4. $y = 0.2x^2 - 1.2x + 11$.
5. $x = 0.2y^2$.
6. $y = 0.1x^3$.
7. $x^2y = 4$.
8. $y^2 = 0.3x^3 + 2.4$.

Note.—The first five equations represent the same parabola, but differently placed as regards the axes. In 7 the co-ordinate axes are asymptotes to the curve.

148. The linear equation $Ax + By + C = 0$.—This equation is said to be *linear*, because it can be proved that it always represents a line which is *straight*. It is also said to be of the *first degree*, because it contains no terms, like x^2, xy, y^2, x^3 . . . ; it involves x and y to the *first* power only.

The student can easily *illustrate* by actual plotting, on squared paper, that the above equation represents a straight line. Take, for example, the linear equation

$$3x - 2y + 6 = 0.$$

Dividing by 2, and transposing for convenience of calculation, we may write it thus—

$$y = 1.5x + 3.$$

Now give to x any convenient series of values, calculate the corresponding values of y, and tabulate the series of solutions as described in the last article. Thus—

TABLE GIVING SOLUTIONS OF THE EQUATION $y = 1.5x + 3$.

Values of x	-3	-2	-1	0	1	2	3	...
Values of y	-1.5	0	1.5	3	4.5	6	7.5	...

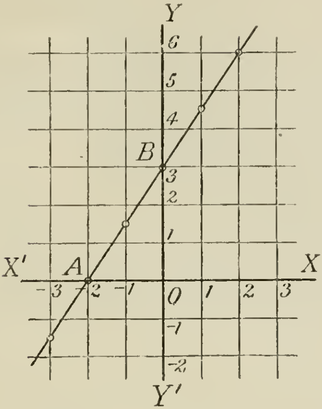
The figure shows these points plotted, with the straight line drawn through the points.

One readily observes from the table or the figure that as x increases by 1, y increases by $1\frac{1}{2}$; or that y increases one and a half times as fast as x. This fact shows the locus to be a straight line.

We give one more example. Let the general linear equation $Ax + By + C = 0$ be transposed into the form

$$\frac{x}{a} + \frac{y}{b} = 1,$$

then a and b are the distances from O of the points A and B in which the line intersects the axes OX and OY. These distances are called the *intercepts* of the line. Take, for instance, the equation previously considered,



$$3x - 2y + 6 = 0,$$

or $$3x - 2y = -6.$$

Dividing throughout by -6, this becomes

$$\frac{3x}{-6} - \frac{2y}{-6} = 1,$$

or $$\frac{x}{-2} + \frac{y}{3} = 1.$$

Now refer to the figure, in which this line is drawn; it will be seen that the intercept a or OA is -2, and the intercept b or OB is 3.

Examples.—Calculate, tabulate, and plot eight points from each of the following equations. Observe that in each case the series of points lie on a straight line.

1. $x = y$.
2. $x + y = 0$.
3. $x + y = 1$.
4. $x + y + 1 = 0$.
5. $y = 2x - 1$
6. $y = -\frac{1}{2}x + 1$.
7. $y = 2x$.
8. $y = -\frac{1}{2}x$.
9. $x = 2y$.
10. $\frac{x}{3} - \frac{y}{2} = 1$.
11. $\frac{x}{2} + \frac{y}{3} = 1$.
12. $2.35x + 3.17y - 4.86 = 0$.

149. PROBLEM.—**To plot a straight line, having given its equation, say $4x + 3y - 12 = 0$.**

First Method.—Select any pair of convenient values of x (or y) and calculate the corresponding values of y (or x); plot the points on squared paper; draw the straight line through the two points thus plotted.

Thus in the given equation

Put $x = 0$, then $y = 4$.

Put $x = 4$, then $y = \frac{1}{3}(12 - 16) = -1.33$.

In the figure the points $(0, 4)$ and $(4, -1.33)$ are plotted, and the required line is drawn through them.

Second Method.—Transform the equation into the shape $\frac{x}{a} + \frac{y}{b} = 1$; the intercepts a and b are then given without further calculation.

Thus the given equation may be written

$$4x + 3y = 12,$$

or

$$\frac{x}{3} + \frac{y}{4} = 1.$$

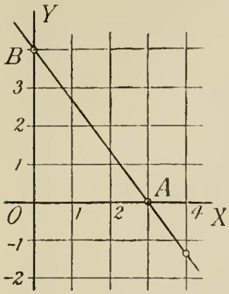
The intercepts are now seen to be 3 and 4, and these lengths are marked off from O on the diagram, viz. at A and B.

150. PROBLEM.—**Having given a straight line and the axes of co-ordinates (on squared paper), to determine the equation to the line.**

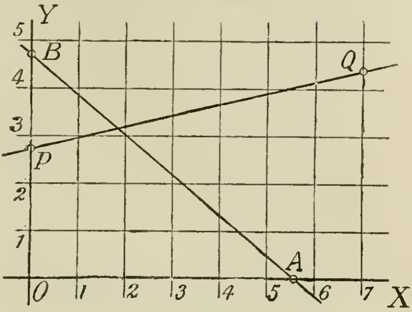
First Method.—Assume any linear equation for the line, say, $y = mx + c$, where m and c require to be found. Read off the values of x and y for *any* two points on the given line. Insert these pairs of values of x and y in succession in the given equation. We thus have two equations from which to determine m and c. An illustration will make this clear.

Let the given line be PQ in the figure. Select any two points on the line, say, P and Q.

Read off the co-ordinates of P, viz. $x = 0$, $y = 2.7$; and the co-ordinates of Q, $x = 7$, $y = 4.4$.



149



150

Now insert these pairs of values in the assumed equation $y = mx + c$, and we get

$$2.7 = 0 + c,$$
$$4.4 = m \times 7 + c.$$

From which

$$c = 2.7,$$
$$m = \frac{4.4 - c}{7} = \frac{4.4 - 2.7}{7} = 0.243\ ;$$

and the required equation to the line is

$$y = 0.243x + 2.7.$$

Second Method.—If the points where the given line intersects the axes are available, and not too close together, read off the intercepts a and b, and insert them at once in the equation,

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Thus, if the given line were AB (Fig. 150), the intercept OA or a is 5.6, and the intercept b or OB is 4.7; therefore the equation to the line AB is

$$\frac{x}{5.6} + \frac{y}{4.7} = 1.$$

151. Examples.—**1.** Plot the following straight lines on squared paper. In each case adopt the method of plotting which seems the most suitable.

(*a*) $y = 3x - 4$.
(*b*) $y = 3x$.
(*c*) $y = 2x - 3$.
(*d*) $y = -\frac{1}{2}x$.
(*e*) $\frac{x}{4} + \frac{y}{5} = 1$.
(*f*) $\frac{x}{5} - \frac{y}{4} = 1$.
(*g*) $x = 2$.
(*h*) $y = 3$.
(*k*) $x = 0$.
(*l*) $y = 0$.
(*m*) $2.7x + 3.9y - 6.2 = 0$.
(*n*) $3.5x - 1.1y + 2.3 = 0$.

2. Measure from your diagrams for Ex. 1 the co-ordinates of the points of intersection of the lines *c* and *d*; also of *e* and *f*; and of *m* and *n*. In each case verify the result by calculation; that is, by solving the three pairs of simultaneous equations.
Ans. (1.2, − 0.6); (4.39, − .488); (− 0.13, 1.69).

3. Determine the equations to the ten lines given in the figure on the opposite page. *Answers*—

1. $\frac{x}{4.8} + \frac{y}{9.1} = 1$.
2. $y = .5x + 5.5$
3. $y = 6.43x - 23.1$.
4. $y = 2.9x + 8.4$.
5. $y = .8x + .8$
6. $y = 1.125x$.
7. $y = -.83x$.
8. $y = 1.21x - 4.6$.
9. $y = x - 5.2$.
10. $y = -2.33x + 2$.

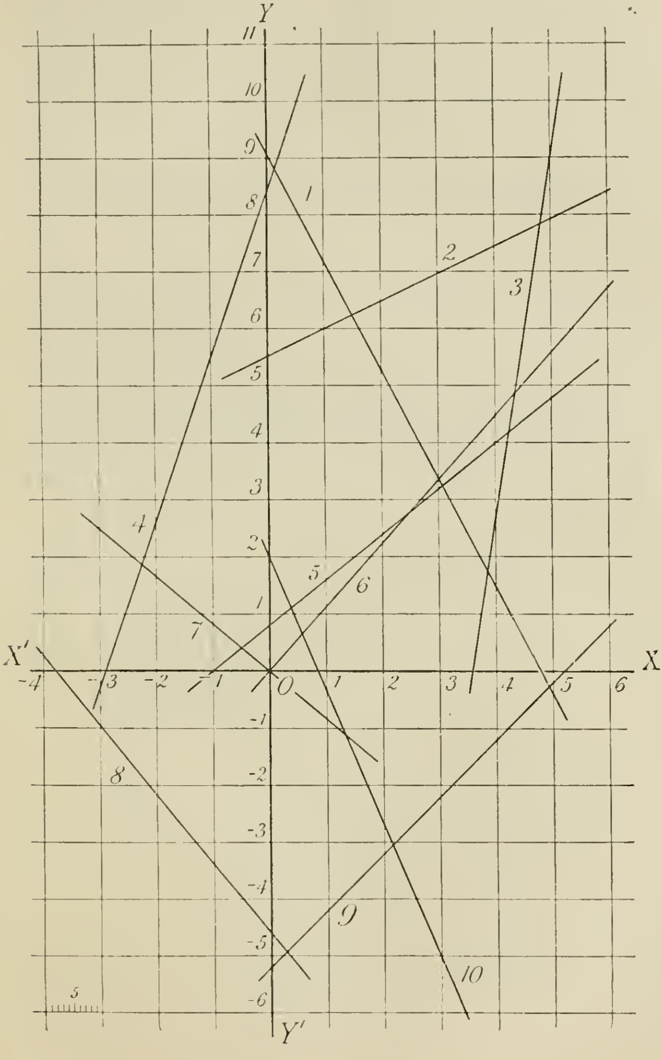
4. Read from the diagram opposite the co-ordinates of the point of intersection of lines 1 and 2. Confirm the result by solving the simultaneous equations which are given as the answers to 1 and 2, Ex. 3. *Ans.* (1.5, 6.25).

5. Measure, from the diagram opposite, the "slope" of the line 6, that is the tangent of the angle which the line makes with *OX*. Find the angle from the table of tangents on p. 20. *Ans.* 6.7 in 6 or 1.16 in 1; 49.3°.

6. Find the slope or gradient of the steepest portion of the road shown in Fig. (*b*), p. 167. *Ans.* 860 feet in 2 miles; *i.e.* 1 in 12.3, *or* 0.0813 in 1, or 4.7°.

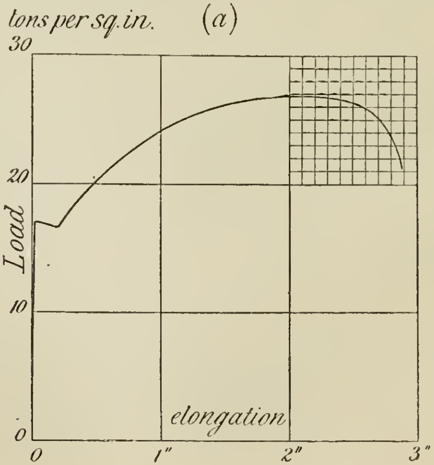
7. After having plotted the lines of Ex. 1, measure the angle (1) between lines (*a*) and (*b*); (2) between lines (*c*) and (*d*); and (3) between (*e*) and (*f*). *Ans.* (1) 0°; (2) 90°; (3) 90°.

Note.—Observe (1) that m is the slope of the line $y = mx + c$; (2) that the lines $y = mx + c$, and $y = mx + c'$ are parallel; (3) that the lines $y = mx + c$ and $y = -\frac{1}{m}x + c'$ are perpendicular.


Y
X
X′
Y′
0
1
2
3
4
5
6
7
8
9
10
11
-1
-2
-3
-4
-5
-6
10

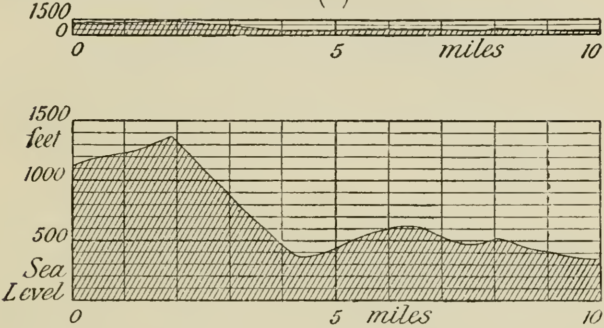
152. Plotting the results of observation and experiment.—As will be explained in a future chapter, any quantity capable of numerical measurement may be represented graphically by a finite straight line set out to a specified scale. The co-ordinates of a point, being lengths, may be taken to represent two such quantities. We may extend this to two quantities of *variable* magnitude, which are mutually related in some manner, and thus exhibit the nature of the relationship by means of a curve. From many familiar examples we select the following:—

(*a*) *Load-elongation diagram.*—In testing the tensile strength of a specimen of mild steel, one square inch cross sectional area, if the loads and the corresponding elongations are noted at intervals during the operation, and subsequently plotted as ordinates and abscissæ respectively, the *load-elongation* diagram is somewhat like Fig. (*a*), its exact shape depending on the nature of the steel. Many testing machines are arranged so that the load-elongation diagram may be drawn automatically.



(*b*) *Contour road-map.*—A cyclist may wish to know the elevation above the sea-level, or the gradient at any point of his route; this is well given by a *contour road-map*, Fig. (*b*), in which the abscissæ are miles travelled, and the ordinates are heights in feet above the sea-level.

(*b*)



(*c*) *Price chart.*—A manufacturer who studies closely the variations in the market prices of materials day by day would be interested in diagrams like that of Fig. (*c*), which shows the fluctuations in the price of the metal tin during the month of June 1898, the abscissæ being market-days, and the ordinates the prices in pounds per ton, as quoted in the London metal market.

(*c*)

153. Choice of scales.—In plotting curves such as those described in the last article, it is important that the scales be judiciously chosen. The main points to be attended to are now described.

The horizontal and vertical scales may be chosen quite independently of each other, even when both co-ordinates represent actual lengths, as in the case of the contour road-map of Fig. (*b*), page 167.

The zero points of the scales need not be on the sheet. Thus in Fig. (*c*), page 167, the vertical scale begins at £67 per ton, as it would be useless to show anything lower than this. Note also that in Fig. 155 the efficiency scale begins at .40.

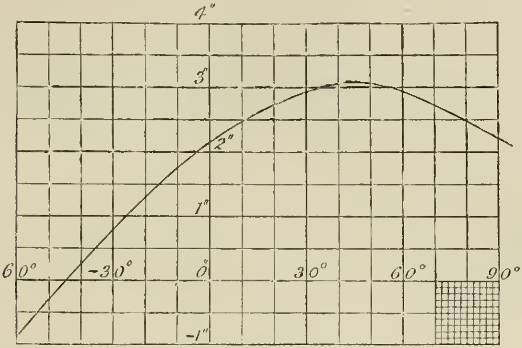
The scales should be so chosen that the curve extends well over the sheet, from bottom to top, and from extreme right to extreme left, and is thus not dwarfed either way.

If the plotted points lie approximately on a straight line, the best result is obtained when the line is about equally inclined to both axes. See the line in Fig. 155. We should avoid adopting scales which would cause the line to be inclined at an angle less than say 30° to either axis.

We may point out **some of the bad effects** which result from the neglect of these precautions. At the top of Fig. (*b*), page 167, the contour of the road is drawn in its true horizontal and vertical proportions; so drawn it would not show with sufficient distinctness the variations in height and slope of the road, to be of practical use to the cyclist. The road would appear to be nearly flat notwithstanding that for a couple of miles there is a gradient of 1 in 12, almost too steep to be ridden down. In the lower figure the vertical heights are *magnified twelve times.*

Again, in Fig. 155 at the top, we have set out the efficiency-resistance curve with the vertical scale reduced to $\frac{1}{10}$th, to show that under these circumstances the curve might easily be mistaken for an approximate straight line.

Or again, in Fig. (*a*), page 166, the first portion of the diagram appears to be a perfectly straight line, nearly vertical; in order to know whether the line is actually straight or not, it would be necessary to enlarge the horizontal scale for this portion of the figure. The enlargement would require to be from fifty to one hundred times, in order to comply with the condition stated above.



154. Interpolation. Maxima and minima.—When we are given the values of a quantity for a series of intervals, we can readily find any *intermediate* value by plotting the given series. This is called **interpolation.** We can also determine in a similar manner the **maximum** or **minimum** value of a quantity, where such exists, if we are given a set of values of the quantity ranging on each side of the maximum or minimum. The following example should make this clear :—

Example.—In a steam engine the displacements of the slide valve from its mean position, corresponding to a series of angular positions of the crank, are given in the following table : determine

(*a*) the displacement of the valve when the crank angle is 10° ;
(*b*) the angle of the crank when the valve displacement is zero ;
(*c*) the maximum displacement of the valve.

Angle of crank . .	– 60°	– 30°	0°	30°	60°	90°
Displacement of valve	– .91″	.80″	2.17″	2.93″	2.95″	2.23″

In the figure above these positions are shown plotted on squared paper, with crank angles as abscissæ, and valve displacements as ordinates. A *fair curve* is drawn freehand through the six points thus obtained. The required results are then read off from this curve. Thus we obtain : *Ans.* (*a*) 2.49″ ; (*b*) – 43.8° ; (*c*) 3.06″.

155. Laboratory test of a crane.—The following numbers were obtained by **testing a small crane** under different loads :—

Load raised by crane, in lbs. or Resistance, *R* .	0	20	40	60	80	100	120	140	
Force exerted at handle, in lbs. or Effort, *E* . .	4·5	10.7	15.6	21.7	27.0	32.8	38.1	43.6	
Efficiency, *f* (calculated) . .	0	·47	.64	·71	·74	·76	·79	.80	

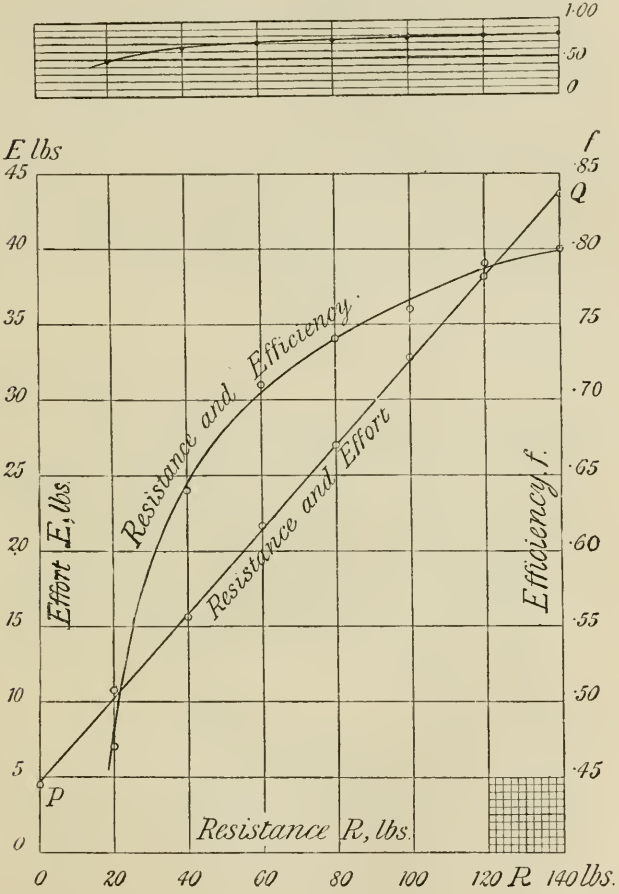
The gearing of the crane was such that, but for friction, the force required at the handle would have been $\frac{1}{4}$ the load raised. The efficiency is calculated by dividing this force (without friction) by the actual force. Thus when 80 lbs. is being raised, the effort required if there were no friction would be 20 lbs., whereas the actual effort from the table is 27 lbs. The efficiency for this load is therefore $20 \div 27 = .74$, or 74 per cent.

Now, plotting the efforts and resistances as co-ordinates on squared paper to suitable scales, we obtain points which are seen to lie very nearly on a straight line.

The *best average position* amongst the points for this line is readily judged by applying to the diagram a piece of *black thread*, or a *piece of tracing-paper* on which a straight line has been ruled.

The efficiency-resistance curve is shown in the same figure. The points on it are plotted, and a *fair curve* drawn so as to lie evenly amongst them, so far as can be judged by sight. This curve may be used **to correct errors of observation.**

The two curves may be said to be set out on a resistance *base ;* the horizontal scale of resistance is marked in the figure. The efforts and efficiencies are plotted as ordinates, their respective scales being shown on the left and right of the diagram.


1·00
·50
0
E lbs
f
45
40
35
30
25
20
15
10
5
0
·85
·80
75
·70
·65
·60
·55
·50
·45
Q
P
Resistance and Efficiency.
Resistance and Effort
Effort E, lbs.
Efficiency, f.
Resistance R, lbs.
0
20
40
60
80
100
120 R
140 lbs.

155

156. Examples on Plotting.

1. Plot the following values of p, the pressure in pounds per square inch, and t the temperature Fahrenheit, of dry saturated steam; join the points by a fair curve, and read off the value of p when t is 293°. *Ans.* 60.4 lbs. per sq. in.

t	212	230	248	266	284	302	320	338	356	374	392
p	14.7	20.8	28.8	39.2	52.5	69.2	89.7	115.1	145.8	182.4	225.9

2. The values of A, the average attendance (in thousands) at the primary evening schools in England for the years 1887 to 1897 are given; obtain a curve showing these values.

Year	1887	1888	1889	1890	1891	1892	1893	1894	1895	1896	1897
A	30.6	33.3	37.1	43.3	52.0	65.6	81.1	115.5	129.5	147	179.6

3. Find one root of the cubic equation $2x^3 - 3x - 16 = 0$.

Method.—Let $y = 2x^3 - 3x - 16$. Give to x the values 1, 2, 3, etc., and calculate the corresponding values of y. Plot these values of x and y on squared paper and draw a fair curve through the points so obtained. Read off the values of x where y is zero, that is, for which $2x^3 - 2x - 16$ zero.

To obtain a more correct solution, after observing that one required value of x lies between 2 and 3, take closer values of x, say 2.1, 2.2, 2.3, etc., and after calculating the values of y, plot these new values of x and y to a bigger scale. *Ans.* One solution is $x = 2.25$.

N.B.—In this manner *one* or more solutions to *any* equation may be obtained.

4. The half-ordinates of the load water-plane of a vessel are 12 feet apart, and their lengths are 0.5, 3.8, 7.7, 11.5, 14.6, 16.6, 17.8, 18.3, 18.5, 18.4, 18.2, 17.9, 17.2, 15.9, 13.4, 9.2, and 0.5 feet respectively. Determine (1) the total area of the plane in square feet, (2) the position of the centre of area of the section of the vessel made by the water-plane. *Ans.* (1) 2649 sq. feet; (2) 102.8 ft. from the first ordinate.

Hint.—(1) Find the mean ordinate as in Art. 43, and multiply it by the total length to obtain the area. (2) Multiply each mid-ordinate by its distance from the first ordinate, and add these products together; divide the result by the sum of all the mid-ordinates.

157. Approximate linear laws.—In Art. 155, referring to the crane, it was seen that when the force applied to work the crane, and the load raised—that is, the effort and resistance—were plotted as co-ordinates on squared paper, the points obtained lay very nearly on a straight line; the law connecting the two is therefore said to be *linear*, or approximately linear. The determination of the *law of the crane* is the same as that of finding the equation to this straight line. The line was located by means of a stretched thread or ruled tracing-paper.

We proceed exactly as in Prob. 150. Let the equation to the line be

$$y = ax + b,$$

or

$$E = aR + b,$$

since the ordinate y is the effort E, and the abscissa x is the resistance R.

To find the constants a and b, select any two points on the line, say P and Q, near the ends. Read off the co-ordinates of P and Q. Thus

for P, $E = 4.91$ lbs., $R = 0$;
and for Q, $E = 43.8$ lbs., $R = 140$ lbs.

Substitute these values in the above equation, and we obtain

$$4.91 \text{ lbs.} = 0 + b,$$
$$43.8 \text{ lbs.} = a \times 140 \text{ lbs.} + b;$$

from which

$$b = 4.91 \text{ lbs.}$$
$$a = \frac{43.8 \text{ lbs.} - 4.91 \text{ lbs.}}{140 \text{ lbs.}} = .278.$$

Therefore the equation to the effort-resistance line, or the law of the crane, is

$$E = .278R + 4.91 \text{ lbs.};$$

Or, the force that must be applied at the handle in order to raise any load is equal to 4.91 lbs. plus .278 of the load.

158. Miscellaneous Examples.

1. In a Stephenson link motion the following measurements were made :—

Angle of crank . .	-45°	-15°	15°	45°	75°	105°
Displacement of valve	$-.25''$	$.98''$	$1.86''$	$2.24''$	$2.05''$	$1.32''$

Determine (*a*) the displacement of the valve when the crank angle is zero ; (*b*) the angle of the crank when the valve displacement is zero ; and (*c*) the maximum displacement of the valve.

Ans. (*a*) $1.47''$; (*b*) -39.2° ; (*c*) $2.25''$.

2. The following numbers refer to the test of a crane, determine the linear law of the crane.

Effort E, lbs.	7	14	21	28	35	42	49	56	63
Resistance R, lbs.	4	36	65	93	122	151	183	216	248

Ans. $E = 0.23\,R + 6.54$ lbs.

3. Try if an equation of the form $xy = ax + by$ approximately represents the relation between x and y, pairs of values of which are given in the table below, and if so, determine the mean values of a and b.

Values of y	5	6	7	8	9	10	11
Values of x	18	28	54	133	-455	-111	-65

Ans. $a = 8.7$, $b = -13$, or $xy = 8.7xy - 13$.

Hint.—The equations $xy = ax + by$ may be written in the form of $1 = a\frac{1}{y} + b\frac{1}{x}$. Therefore *calculate* and plot the values of the reciprocals $\frac{1}{y}$ and $\frac{1}{x}$ as co-ordinates. If these are called Y and X, we have then to determine a and b in the approximate linear law (if such exists) $1 = aY + bX$, as in Arts. 150 or 156.

4. A series of pressures p and volumes v of saturated steam, as determined experimentally, are given in the table below. The logarithms of these quantities are also given. By plotting $\log p$ and $\log v$, try whether a relation of the form $\log p + a \log v = b$ (*i.e.* $pv^a = \text{const.}$) holds approximately between them, and if this is found to be the case, determine the best average values for the constants a and b.

PROPERTIES OF SATURATED STEAM

Pressure, p. Lbs. per sq. inch .	1.06	2.88	6.86	14.7	28.8	52.5	89.7	146	226
Volume, v. Cubic feet per lb. .	313	122	53.4	26.4	14.0	7.97	4.82	3.06	2.02
$\log p$ . . .	.020	.462	.837	1.17	1.46	1.72	1.95	2.17	2.35
$\log v$ . . .	2.50	2.09	1.73	1.42	1.15	.902	.683	.483	.305

Ans. $a = 1{\cdot}05$; $b = 2.66$ (or $pv^{1{\cdot}05} = 460$).

5. A log of timber 20 feet long has the following cross-section areas at the given distances from one end. Find the volume in cubic feet.

Distance from one end in feet	0	2.6	5	7.8	12	15	17.6	20
Area in square feet . .	5.0	4.3	3.8	3.6	3.5	3.3	3.1	3.0

Ans. 72.7 cub. ft.

Hint.—It will be observed that the distances between the given sections are *unequal.* In such a case first plot the given numbers on squared paper, and then divide into, say, 10 strips of equal width. Find the mean cross-sectional area by the mid-ordinate method, and multiply by the total length of the log.

SECTION II

PRACTICAL SOLID GEOMETRY, OR DESCRIPTIVE GEOMETRY

CHAPTER VII

POSITION IN SPACE DEFINED AND EXHIBITED

159. Introduction.—Hitherto the points, lines, and figures in the various problems taken have been confined to one plane, represented by the plane of the paper; we now pass on to the more general case and treat of the geometry of space. Our diagrams must now represent the three dimensions of length, breadth, and thickness, whereas previously we were concerned with only two, length and breadth.

It is necessary to distinguish between pure and practical solid geometry.

Pure solid geometry deals with the geometrical relations which exist amongst points, lines, and surfaces in space. *Practical solid geometry* shows how to exhibit these relations by scale drawings which can be measured. We have an illustration of the former in the eleventh book of Euclid, where a number of definitions relating to solid figures are given, followed by the propositions, arranged in strict logical sequence and proved. The diagrams are of secondary importance, being generally semi-perspective sketches, just sufficient to convey an idea of the form of the solid figure. It is necessary that the student be acquainted with the definitions and some of the propositions of Euclid XI., and for convenience, these are given in an appendix at the end of this volume.

In *practical solid geometry* we are concerned not so much with the proofs of propositions, as with the methods whereby the relative positions and forms of figures in space of three dimensions can be exhibited on a surface such as a sheet of drawing-paper, which has only two dimensions, and in such a manner that these positions and forms can be ascertained from the diagrams or drawings by **direct measurements to scale.**

As in the preceding section, we shall endeavour to enforce upon the student the importance of good draughtsmanship. A leading idea throughout should be that in making practical computations, graphical processes may with advantage often be employed in preference to other methods. One is only properly equipped for the work when accuracy in execution has become habitual.

The special difficulty which the beginner experiences is to be able to conceive clearly the connection between the figures drawn on paper and the actual figures in space which they represent. This difficulty is greatly lessened in the initial stages by making **free use of models.** The problems of this introductory chapter are arranged so that the student may easily make his own models for all of them. It has been found that by their use very clear and precise notions are formed at the start, and an excellent method of working is introduced.

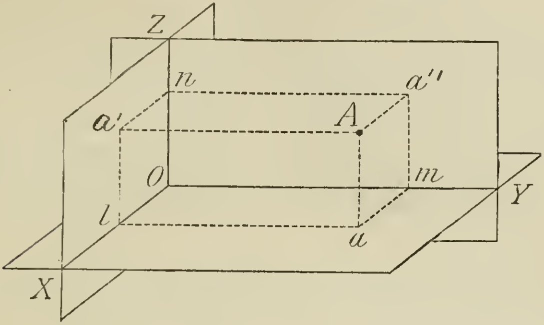
The problems relate to ways of exhibiting the positions of points, lines, and planes in space, and of finding the distances and angles between them. They form the basis of the method by which solid *form* is defined by drawing.

160. The position of a point in space defined by rectangular co-ordinates.—We have seen in Art. 144 how the position of a point *in a plane* may be defined by reference to *two* perpendicular *axes*. In like manner the position of a point *in space* may be defined by reference to *three* mutually perpendicular *planes*. For example, the situation of a small object in a room is known if we know its height above the floor and its distance from each of two adjacent walls.

Let the *planes of reference* be as shown in the figure, one *horizontal* and the other two *vertical*, and at right angles to each other, the latter being distinguished as the *front* and *side* vertical planes. These planes intersect in three lines, also mutually perpendicular, called the *co-ordinate axes*, one being vertical and the other two horizontal. The point common to the three axes and the three planes is called the *origin*, and is denoted by the letter O, the three axes being labelled respectively OX, OY, OZ, the latter being the vertical one. The three planes of reference may also be designated as the planes of XY, YZ, and ZX respectively.

The position of any point A may now be defined by its perpendicular distances Aa', Aa'', Aa from the three planes of reference. These three distances are called the *rectangular co-ordinates* of the point, and may be denoted by x, y, z. The co-ordinate x is the distance of A from the plane of YZ, measured parallel to the axis OX. Similarly for the other two: y is measured parallel to OY, and z parallel to OZ. Thus x and y are the horizontal co-ordinates, and z is the vertical co-ordinate.

A co-ordinate has a *negative* value when the point is situated on the other side of the plane from which that co-ordinate is measured. Thus z would be *negative* for any point *below* the horizontal plane XY. And x would be *negative* for any point at the *back* of the front vertical plane YZ. Similarly, y would be *negative* for any point to the *left* of the plane XZ.



The student will note that the co-ordinate planes, when extended each way from O, divide the neighbouring space into *eight trihedral angles.* He should also note that when the co-ordinates of a point are given in sign and magnitude, the *signs tell us in which trihedral angle* the point is situated, and the *magnitudes give us the exact situation* in this angle. Thus the position of *any* point in space is completely defined, without ambiguity, by its three rectangular co-ordinates.

The point whose co-ordinates are x, y, z may for shortness be referred to as the *point* (x, y, z).

Examples.—State in which trihedral angles the following points are situated :—

1. The point $(-1, 1, 1)$.
Ans. Behind YZ, to the right of ZX, above XY, or in the back-right-upper angle.

2. The point $(1, -1, 1)$.
Ans. The front-left-upper angle.

3. The point $(1, -1, -1)$.
Ans. The front-left-lower angle.

4. The point $(-1, -1, -1)$.
Ans. The back-left-lower angle.

161. The position of a point in space exhibited by projection.—In descriptive geometry it is not enough to *define* the position of a point in space. It is further necessary to *set out* this position by a scale drawing.

As the drawing is necessarily confined to one plane, and the figure which the drawing represents is *not* plane, some convention is required in order to connect the two, and so render the signification of the drawing precise and definite.

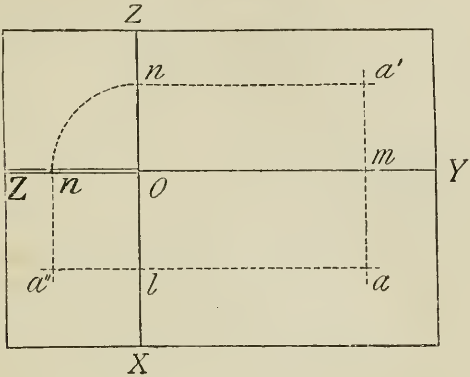
To illustrate, let the student take a quarter of an imperial sheet of drawing-paper, and on it rule two perpendicular lines, distant about $5''$ from the upper and left edges, as shown in Fig. 161 (*a*) at ZY, ZX. Then *cut* the paper along the horizontal line from O to Z, and *indent* it with a blunt instrument from O to Y and Z to X. Now fold the paper along the indented lines, and secure the overlapping parts on the left by means of a paper-fastener. A partial model of the planes of reference, Fig. 161 (*b*), is thus obtained, comprising the front-right-upper trihedral angle, the co-ordinates of points situated in which are all positive.

Suppose now it is required to set out the position of a point A, whose co-ordinates x, y, z are given.

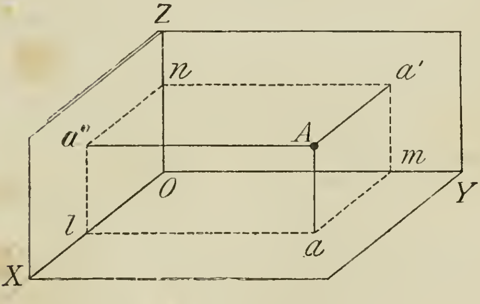
Unfold the model, and along OX and OY mark off Ol and Om equal to x and y, and along OZ, OZ set off On, On, each equal to z. Through l, m, and n draw the lines parallel to the axes, as shown, intersecting respectively in a, a', a''. Then this plane figure, or unfolded model, with its lines and points, is the scale drawing or diagram which, properly interpreted, represents the position of the point A as regards the three planes of reference.

If this figure were given, then in order to determine from it the x co-ordinate of the point A, we should measure the length of Ol, ma, or na''; to ascertain the y co-ordinate, we should measure Om, la, or na'; and for the z co-ordinate we should measure any one of On, On, ma', or la''.

If it were desired to show exactly what the drawing represented, we should fold it along the lines of the axes, 161(a)



161(b)



so as to obtain the model of the planes of projection; and to show the actual point *A* in space and the three perpendiculars from it, we might use a small sphere with three projecting pieces of wire, *Aa*, *Aa′*, *Aa″*.

The points *a*, *a′*, and *a″* are called the three **projections** of the point *A*. But some formal definitions are required, and we give these in the next article.

162. Definitions relating to projection.—If from the various points which may be supposed to constitute any object, straight lines proceed to meet a plane, the object is said to be *projected* on the plane.

The straight lines are called *projectors.*

Parallel projection is the case in which the projectors are all parallel to one another.

Radial or *perspective projection* that in which their directions all pass through one point.

Parallel projection is subdivided into *oblique* and *orthogonal* or *orthographic;* in the former the projectors are *inclined* to plane of projection, in the latter they are *perpendicular.* When the word "projection" is used without qualification, orthographic projection is to be understood.

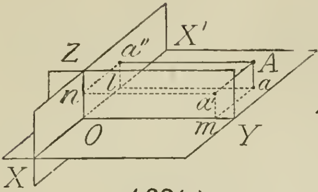
Thus *the projection of a point on a plane is the foot of the perpendicular let fall from the point to the plane.*

Referring again to Figs. 161 (a), 161 (b), the points a, a', a'' are seen to be the three *projections* of A on the *planes of reference.* The projection a on the *horizontal plane* is called *the plan;* the two on the vertical planes are called elevations; a' is the *front elevation,* and a'' the *side elevation.* In Fig. 161 (a) the point A is said to be *represented by its projections.* The planes of reference are also called **planes of projection.**

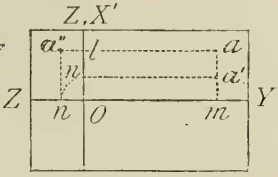
The lines aa' and aa'' in Fig. 161 (a) are also called **projectors.** They are perpendicular respectively to OY and OX.

We may point out that in all cases the three projections show which trihedral angle contains the point. Thus in Fig. 162 (a) the point A is behind the plane of YZ, making the x co-ordinate negative. The point is projected on the planes of reference at a, a', a''. Now in order to derive Fig. 162 (b), the vertical planes of Fig. 162 (a) must be rotated about the axes of X and Y respectively, and *always in the same directions.* These directions of rotation are those given when describing the construction of the model, and are such as to *open out* the positive dihedral angle. The angle in which the point lies and its exact situation are given without ambiguity by Fig. 162 (b).

Notation.—A *capital* letter is used to denote the point in space, and the corresponding *small* letters denote the projections. Thus for the point A the plan is a, the front elevation a', and side elevation a''.



162(a)



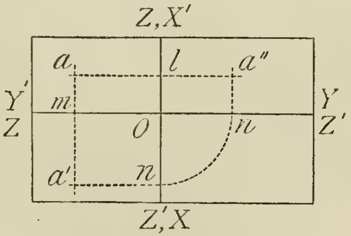
162(b)

Examples.—**1.** The x, y, z co-ordinates of a point A are respectively 3″, 4″, and 2″; draw the three projections of the point and set up the model for this case.

2. Draw the projections of the three points (2.5″, 3″, 4″), (−3″, 4″, 2″), and (3″, −4″, −2″).

3. In which trihedral angle is the point A situated whose projections are given in the figure below? Measure the co-ordinates of the point, scale $\frac{1}{8}$th. *Ans.* −1.6″, −3.6″, −3″.

Note.—OX', OY', OZ', OZ' represent the *negative* directions of the axes. Ol, Om, On are the x, y, z co-ordinates of A, and are seen to be all set off in the negative directions along their respective axes, from which we infer that the three co-ordinates are all negative.



Or thus—Set up the model and in some way represent the point A in its actual position, say by using a lady's hat-pin.

Look vertically downwards on the model; the plane ZOY appears as the line OY, and because the *plan a* is *behind* $Y'OY$ the point A must be *behind* the plane ZOY. Also the plane ZOX appears as the line OX, and since the *plan a* is to the *left* of OX, the point A is to the *left* of the plane ZOX. Now view the model in the direction at right angles to the plane ZOY; the plane XOY appears as the line OY; and because the front elevation a' is *below* OY the point A is *below* the plane XOY. Hence the point A as represented by its projections a, a', a'' is in the *back-left-lower* trihedral angle, and its co-ordinates are found on measurement to be as given above.

163. Problems on the position of a point.—We suggest here some simple problems, and give some examples for the student to work out himself. It is intended that the drawing-paper shall be cut, indented, and folded as explained in Art. 161, and then fixed to the board with the vertical planes YZ, ZX free to be laid flat while drawing, or lifted into position while studying the problems, as often as may be required. All the problems reduce to one or other of the solutions of the right-angled triangle specified in Art. 12.

We give only hints. If formal solutions were written we should defeat the object in view, which is to familiarise the beginner with the method and signification of projections, by encouraging the use of models.

From the lower figure it is seen that the point A and its projections a, a', a'', the points l, m, n, and O form the eight corners of a rectangular prism, and OA is a diagonal of the solid. Diagonals of some of the faces of the prism are shown by dotted lines. We now suggest some problems, having given as data the co-ordinates of the point A.

(a) *To find the distances of A from the axes.*

The distance of A from OX is equal to Al or Oa', that is to the hypothenuse of the right-angled triangle Oma', the two sides of which are two of the given co-ordinates.

(b) *To find the distance of A from the origin.*

Here we must find the length of the hypothenuse OA of the right-angled triangle $Oa'A$, one side Aa' being the given x co-ordinate, and the other side Oa' being known from the above.

(c) *To find the angles which OA makes with the planes of reference.* (See Def. 8, Appendix II.)

The inclination of OA to the horizontal plane is the base angle AOa of the right-angled triangle AOa. The inclination of OA to the front vertical plane YZ is the base angle AOa' of the right-angled triangle AOa'.

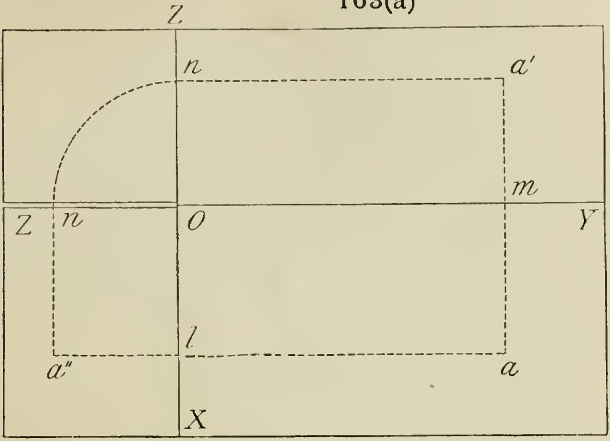
(d) *To find the angles which OA makes with the axes.*

The inclination of OA to the vertical axis OZ is the vertical angle AOn of the right-angled triangle AOn. This angle is the complement of the angle AOa.

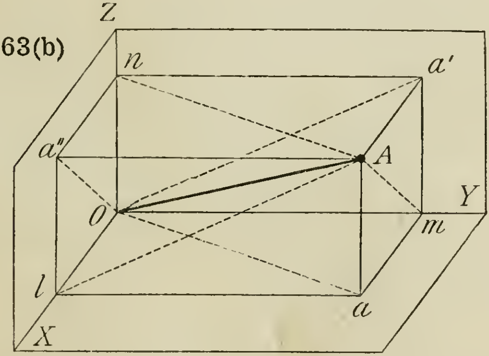
(e) *To draw the three projections of OA.*

These are the lines Oa, Oa', and Oa''.

163(a)



163(b)



Examples.—**1.** The x, y, z co-ordinates of a point A are respectively 3″, 4″, and 2″.

(a) Draw the three projections of A, viz. a, a', and a''.

(b) Draw the three projections of OA, and measure their lengths. *Ans.* 4.47″, 3.61″, 5″ on the planes of YZ, ZX, XY respectively.

(c) Determine and measure the distances of A from the axes of X, Y, and Z. *Ans.* 4.47″, 3.61″, 5.0″.

(**d**) Determine and measure the distance of A from the origin O. *Ans.* 5.39″.

(**e**) Determine and measure the angles which OA makes with the planes of projection. *Ans.* 33.8°, 48.0°, 21.8° with the planes of YZ, ZX, XY.

(**f**) Determine and measure the angles which OA makes with the axes of X, Y, and Z. *Ans.* 56.2°, 42.0°, 68.2°.

(**g**) Determine and measure the *true* distances of the projections a, a', a'' from one another. *Ans.* $a''a = 4.47''$; $aa' = 3.61''$: $a'a'' = 5''$.

(**h**) Set out to scale the *true* shapes of the four triangles $aa'a''$, $Oa''a$, Oaa', $Oa'a''$.

Note.—In g and h the word *true* signifies that the results are to be obtained for the *true* positions of the planes of projection, and *not* for their positions when laid flat.

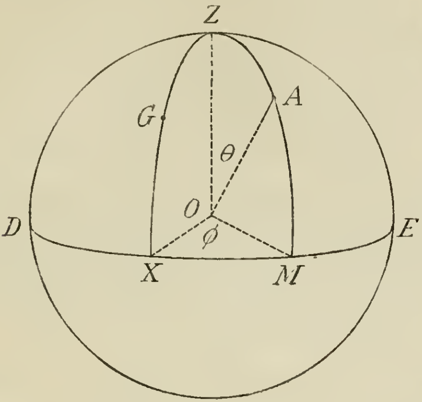
2. The three projections a, a', a'' of a point A are given in Fig. 163 (a). Copy this figure double size, then measure the co-ordinates, and obtain the results c to h of Ex. 1.

3. Draw the figure of Ex. 3, Art. 162, full size, then obtain the results of Ex. 1 above.

164. Polar co-ordinates of a point.—The position of a point in space might be defined in other ways, *e.g.* by its distances from three fixed points. But the only two systems of co-ordinates in general use are the rectangular and the polar.

In the polar method we choose a point O called the *pole*, an axis OZ through the pole, and a plane ZOX containing the axis. The polar co-ordinates, say (r, θ, ϕ) of any point A are then (1) the polar distance OA or r; (2) the angle ZOA or θ; and (3) the angle between the planes of ZOA and ZOX, say ϕ.

In illustration, consider how a place on the earth's surface is located. In the figure O is the centre of the earth, OZ the north polar axis, DXE the equator, ZOX the meridian plane through Greenwich, intersecting the surface in the meridian circle ZGX and the plane of the equator in OX. Let A be the place on the surface, and let the meridian plane ZOA intersect the surface in the meridian circle ZAM, and the plane of the equator in OM. Then the position of A is usually defined by its *longitude*, that is, the angle XOM, and its *latitude* MOA.



Choosing OZX for reference, the polar co-ordinates (r, θ, ϕ) of the point A would be respectively the earth's radius OA, the *co-latitude* ZOA (*i.e.* 90° – latitude), and the *longitude* XOM.

In the figure the polar co-ordinates of A are $r = OA$, $\theta =$ angle ZOA, and $\phi =$ angle XOM.

The problems resolve as before into cases of the solution of the right-angled triangle.

Examples.—**1.** The rectangular co-ordinates of a point A are 3″, 4″, and 2″; find the polar co-ordinates (r, θ, ϕ). *Ans.* 5.39″, 68.2°, 53.1°.

2. The polar co-ordinates r, θ, ϕ of a point A are respectively 3″, 40°, and 55°. Draw the projections of the point, and measure its rectangular co-ordinates. *Ans.* 1.11″, 1.58″, 2.3″.

3. Find the polar co-ordinates of the point A, represented in projection in Fig. 161 (a), drawn $\frac{1}{4}$ size. *Ans.* 5.46″, 70.7°, 66.5°.

4. Taking the latitude and longitude of Rome as 41.9° N. and 12.5° E., represent the position of Rome by a plan and two elevations. Radius of the earth 4000 miles. Scale 1″ to 1000 miles.

5. A line $3\frac{1}{2}''$ long from the origin to a point A makes angles of 40° and 60° respectively with the axes of X and Y. Draw the projections of the line, and measure the rectangular and polar co-ordinates of the point A. *Ans.* 2.68″, 1.75″, 1.41″; 3.5″, 66.2°, 33.1°.

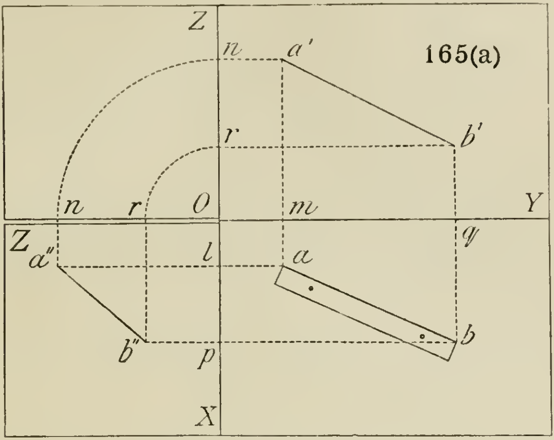
165. The straight line.—In Fig. 165 (*b*), let *AB* be a straight line in space, and *ab*, *a′b′*, *a″b″* its three projections. Then these projections, given as in Fig. 165 (*a*), completely represent and define the line.

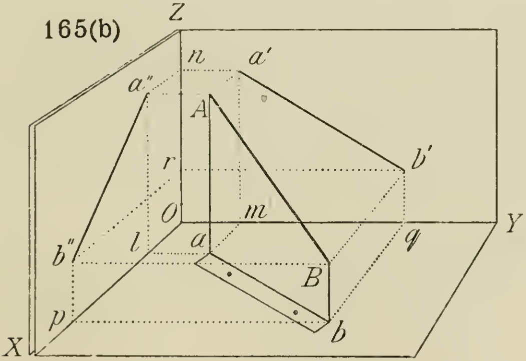
Or these projections could be drawn, if as data we were given the co-ordinates of the two ends of the line.

Again arrange the drawing-paper so that the planes *YZ*, *ZX* may be lifted into the vertical position at any time. The line *AB* in space may be exhibited in the model by cutting a piece of paper to the shape *ABba* and attaching it to the plane *XY*, as shown by a folded margin left for the purpose; in setting out this shape observe that *ab* is equal in length to the plan of the line, and the perpendiculars *aA*, *bB* are equal to the *z* co-ordinates.

Examples.—1. The rectangular co-ordinates of two points *A* and *B* are respectively 2″, 1″, $\frac{1}{2}$″, and $\frac{1}{2}$″, 3″, 1$\frac{1}{2}$″.

- (**a**) Draw the three projections of *AB*, and measure their lengths. *Ans.* 2.24″, 1.8″, 2.5″, on *YZ*, *ZX*, *XY*.
- (**b**) Determine and measure the actual length of *AB*. *Ans.* 2.69″.
  That is, find the length of the hypothenuse of a right-angled triangle, of which the base is equal to the plan *ab*, and the height to the *difference* of the *z* co-ordinates.
- (**c**) Determine and measure the angles which *AB* makes with the planes *YZ*, *ZX*, *XY*. *Ans.* 33.9°, 48°, 21.8°.
  That is, find the angles between *AB* and its three projections *a′b′*, *a″b″*, *ab*. These all reduce to finding the base angles of right-angled triangles.
- (**d**) Determine and measure the angles which *AB* makes with the axes of *X*, *Y*, *Z*. *Ans.* 56.1°, 42°, 68.2°.
  Since *Aa* is parallel to *OZ*, the angle which *AB* makes with *OZ* is equal to the angle *aAB*. Similarly for the other two angles. These angles are determined as the vertical angles of right-angled triangles, and are complementary to the angles of (**c**) above.
- (**e**) Determine the projections of the points where *AB* produced meets the planes of projection *YZ*, *ZX*, *XY*. Measure the co-ordinates of the three *traces* so found.
  Ans. (0, 3.67″, 1.83″), (2.75″, 0, 0), (2.75″, 0, 0).
- (**f**) Determine the true shape of the triangle *OAB*, and measure the angle *AOB*. *Ans.* 72.8°.
  First find the lengths of the sides, and then construct the triangle.





2. Copy Fig. 165 (*a*) double size. Then measure the co-ordinates of *A* and *B*, and obtain the results *b* to *f* of Ex. 1.

3. Find the length of the line which joins two opposite corners of a building brick 9″ × 4½″ × 3″. And determine the angles which this line makes with the edges and faces of the solid. *Ans.* Length = 10½″. Angles with the 9″, 4½″, 3″ edges = 31.0°, 64.7°, 73.4°. Angles with the small, middle, and large faces = 59.0°, 25.3°, 16.6°.

166. The plane.—A plane, of indefinite extent, is located when the positions of any three points in it, not in one straight line, are known. In the rectangular system, the points A, B, C, where the plane meets the axes, are the three most convenient points to choose for the purpose of defining the position of the plane. The lengths OA, OB, OC are called the *intercepts* of the plane on the axes. We may denote the lengths of these intercepts by a, b, and c. The lines BC, CA, AB, where the plane intersects the planes of projection, are called the **traces** of the plane, and serve very well to represent the plane by projection.

Thus, a *plane* is very conveniently *defined in the rectangular system by its intercepts*, a, b, c; *and represented in projection by its traces*, BC, CA, AB.

To the data of Ex. 1 below, let the student make the following model. Cut out in paper a triangle whose sides are equal to AB, BC, CA, Fig. 166 (a), leaving a folding margin along AB for attachment to the horizontal plane as shown in Fig. 166 (b); on it draw CD perpendicular to AB. Next draw OD perpendicular to AB and cut out in paper a right-angled triangle, having OD for base, and OC for height, leaving a margin along OD to be attached to the horizontal plane, underneath the plane ABC, as shown in the lower figure. The first example should now be worked by the student himself, without assistance other than that given in the notes.

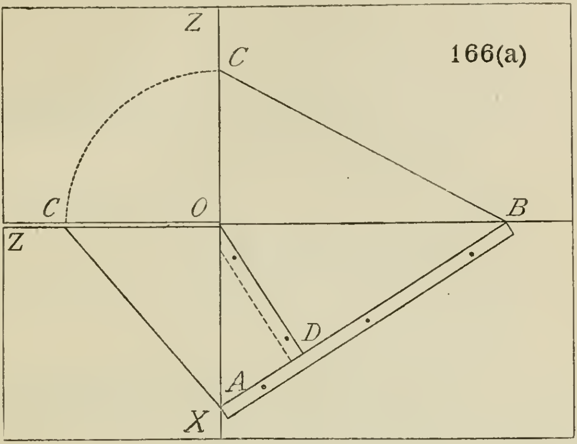
Examples.—**1.** The intercepts a, b, c of a plane on the axes of X, Y, and Z are respectively 4″, 5″, and 3″.

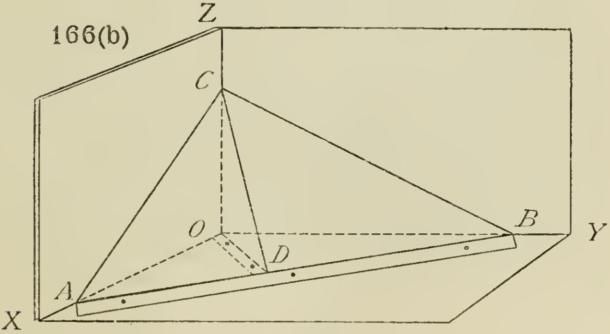
(**a**) Draw the three traces of the plane AB, BC, CA, and measure their lengths.
Ans. 6.4″, 5.83″, 5″.

(**b**) Find the *rabatments* of the triangle ABC into the three planes of projection.

A plane is said to be *rabatted* when it is turned about its trace into the plane of projection. Thus ABC is rabatted into the horizontal plane by rotation about AB; and into the planes of YZ, ZX, by being turned about BC and CA respectively.

(**c**) Determine the rabatments of the triangle COD into each of the three planes of projection.





(d) Determine and measure the angles which the plane *ABC* makes with the planes of *YZ*, *ZX*, *XY*.

Ans. 57.2°, 64.4°, 43.8°.

The inclination of the plane *ABC* to the horizontal plane is measured by the angle *ODC* of the right-angled triangle *ODC*. Corresponding constructions are required for the inclinations to the other two planes of projection. See Def. 9, Appendix II.

(e) Determine and measure the angles which the plane makes with the axes of X, Y and Z.

Ans. 32.8°, 25·6°, 46.2°.

These angles are complementary to the angles of (*e*).

(f) Draw the three projections of OP, the perpendicular, from the origin to the plane. Determine and measure the true length of this perpendicular.

Ans. 2.17″.

The point P is in CD, and OP is perpendicular to CD.

(g) Find and measure the angles which the perpendicular OP makes with the planes YZ, ZX, XY.

Ans. 32.8°, 25.6°, 46.2°.

(h) Find and measure the angles which OP makes with the axes of X, Y, and Z.

Ans. 57.2°, 64.4°, 43.8°.

Note the illustrations of the theorem, that the angle between two planes is equal to the angle between any two lines which are perpendicular to the planes.

2. Copy Fig. 166 (*a*) accurately, double size; then measure the intercepts and obtain the results *b* to *g* of Ex. 1.

3. The latitude and longitude of a place A are 52° N. 0°; and those of a place B are 0° and 40° W. (*a*) Find the angle subtended by the two places at the centre of the earth. (*b*) Find the length of the shortest path on the earth's surface between the two places. Radius of earth = 4000 miles. (*c*) Find the direction of this path at each place. *Ans.* (*a*) 62.4°. (*b*) 4410 miles. (*c*) At A, 46.5° W. of S.; at B 28° E. of N.

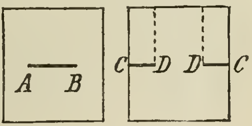
4. The latitude and longitude of Rome are respectively 41.9° N. and 12.5° E., and of St. Petersburg 60.0° N. and 30.3 E., obtain the results of Ex. 3. *Ans.* (*a*) 20°; (*b*) 1440 miles. (*c*) At Rome, 27.4° E. of N.; at St. Petersburg, 43° W. of S.

CHAPTER VIII

FUNDAMENTAL RULES OF PROJECTION

167. Object of chapter.—In this chapter we recall certain fundamental rules and methods of projection, developed in Part I., and which the reader will have learnt from his previous study. A collection of elementary examples is then given to test the knowledge of the student. These should present no difficulty. A well-trained student may generally omit them and pass on at once to the next chapter.

168. The two principal planes of projection.—Since the form of a simple solid is often definitely shown by two projections only, the side vertical plane and side elevation of the last chapter may generally be dispensed with. There remain the two principal planes, the horizontal and the vertical planes of projection. Their intersection is called the *ground line* and will be denoted by XY or xy. These planes, extended both ways, divide the neighbouring space into four dihedral angles.



A model should be made as follows:—Take two pieces of stout drawing-paper, each about 9″ by 6″. Cut one along AB and the other along CD, CD. The latter is then folded along the lines DE, DE, passed through the slit AB of the former, and unfolded. The planes may then be turned into position at right angles to one another.

169. Situation in the four dihedral angles.—Four points situated respectively in the four angles are exhibited by projection in the lower figures. These are derived as indicated in the upper figures, by first projecting the plan and elevation, and then turning the planes into coincidence, about XY, *so as always to open out the front upper angle.*

We may imagine *either* that the vertical plane has been turned into the horizontal plane so that its upper portion moves backwards, *or* that the horizontal plane has been turned into the vertical plane, its front half falling. The result is the same either way.

From the lower figures we can infer the positions of the points. That is, we can measure how much the points are *above* or *below* the horizontal plane, and how much *in front of* or *behind* the vertical plane.

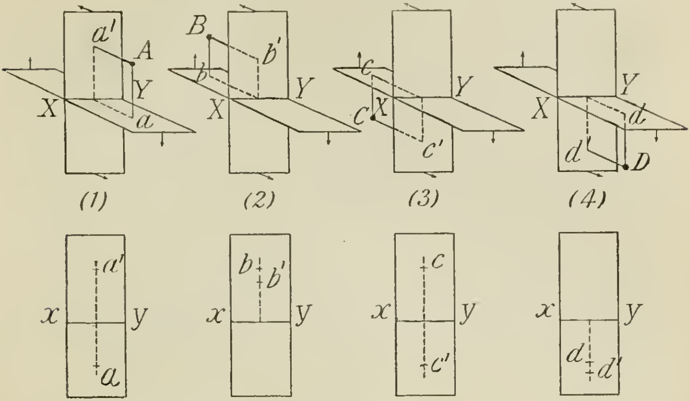
The following conception is useful in the interpretation of drawings.

When considering an elevation, picture the drawing as coinciding with the vertical plane of projection, and think of the ground line xy as being an *edge* or *profile* view of the horizontal plane, just as if the model of the planes were held on a level with the eye and viewed directly from the front. We thus at once recall the rule—

Rule 1.—*A point A is situated* ABOVE or BELOW *the horizontal plane according to whether its elevation a' is* ABOVE *or* BELOW xy. *And the distance of A from the plane is equal to the distance of the elevation a' from xy.*

When considering a plan, picture the drawing as now coinciding with the horizontal plane, and conceive the ground line to be an edge view of the vertical plane, as if the model of the planes were viewed directly downwards from above. This at once leads to the rule—

Rule 2.—*A point A is situated* IN FRONT OF *or* BEHIND *the vertical plane according to whether its plan a is* IN FRONT OF *or* BEHIND xy. *And the distance of A from the vertical plane is equal to the distance of the plan a from xy.*


(1) (2) (3) (4)

This method of reading a drawing is often conducive to clearness of conception, and should be cultivated by the student. Instead of the planes of projection being imagined as turned into coincidence, they are conceived as retaining their horizontal and vertical positions, and it is the drawing-paper which is supposed to be brought to coincide first with one plane, then with the other, accompanied by a corresponding change in the direction of view.

Or thus, if we are reading the plan and elevation of any solid object, and trying thus to imagine the form of the latter, we think of the object as retaining its upright position, and regard the elevation as a picture of the object when viewed horizontally from the front, and the plan as its apparent shape when viewed vertically downwards from above. Thus we may regard *xy* as an elevation of the horizontal plane, *or* a plan of the vertical plane, or as the line of intersection of the two planes.

We add a third rule.

Rule 3.—*The projector drawn between the plan and elevation of a point is always perpendicular to xy.*

170. Rules for drawing auxiliary projections.

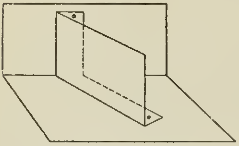
Definition 1.—*An* **auxiliary elevation** *is the projection on any vertical plane not parallel to the principal vertical plane.*

This is illustrated in Fig. (1), where a'' is the auxiliary elevation of A on the auxiliary vertical plane $X'Y'$.

The lower figure exhibits this by projection, and the perspective view indicates how the projection may be derived by turning the two vertical planes backwards into the common horizontal plane.

Observe that $na'' = ma' = aA$, and that aa'' is perpendicular to $x'y'$. Hence the theorem :—

Theorem 1.—*If two or more elevations of a point be projected from one plan, the distances of the several elevations from their respective ground lines are the same in all.*



The following simple and effective model should be made by all students.

Indent and fold a piece of drawing-paper about 10″ × 8″ to represent the principal planes of projection, and attach an auxiliary plane, say 6″ × 4″, by paper-fasteners through folded margins. See the figure. The projections of A may be drawn on this model.

Definition 2.—*An* **auxiliary plan** *is the projection on any plane which is perpendicular to the vertical plane and not parallel to the horizontal plane.*

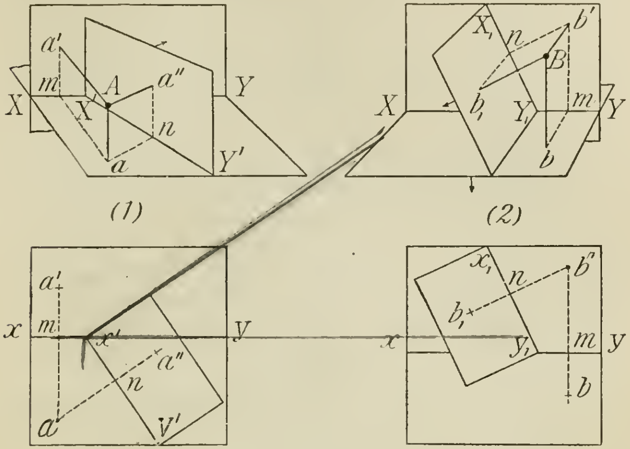
Fig. (2) illustrates this case. The perspective view shows the point B projected on the three planes, and the arrow-heads indicate how two planes may be turned into the common vertical plane so as to obtain the lower figure.

Observe that $nb_1 = mb = b'B$, and that $b''b_1$ is perpendicular to $x_1 y_1$, which leads to

Theorem 2.—*If two or more plans or a point be projected from the same elevation, the distances of the several plans from their respective ground lines are the same in all.*

The previous model will serve to illustrate this case, if it be held in the proper position.

The projections having been drawn on it, the model may be viewed



at right angles to the auxiliary plane, or either hinge may be unfastened for the purpose of rabatment.

Theorems 1 and 2 apply to any object, conceived as an assemblage of points. They may be put in the form of rules. Thus suppose the plan and elevation on xy to have been drawn; then

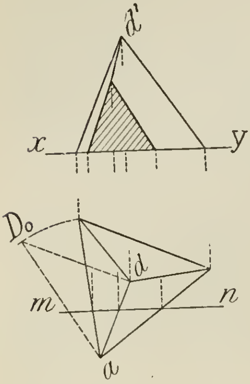
Rule 1.—*To obtain the auxiliary elevation on a new ground line* $x'y'$, *project from the plan perpendicular to* $x'y'$, *and for the new elevation mark off on the projectors the distances of the points above (or below)* $x'y'$ *equal to the distances of the corresponding points of the old elevation above (or below)* xy.

Rule 2.—*To obtain the auxiliary plan on a new ground line* $x_1 y_1$, *project from the elevation perpendicular to* $x_1 y_1$, *and for the new plan mark off on the projectors the distances of the points in front of (or behind)* $x_1 y_1$, *equal to the distances of the corresponding points of the old plan in front of (or behind)* xy.

171. Sectional projections.—An object is often assumed to be cut in two by a *section plane*, in order the better to show its internal form. A projection of either portion on the cutting plane, or one parallel to it, is called a *sectional projection*, the severed parts being indicated by section lines.

The figure shows the plan and a sectional elevation of a regular tetrahedron.

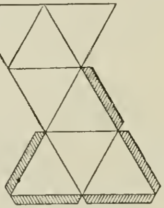
Note.—The height of the solid is found by rabatting the sloping edge *AD* about its plan.



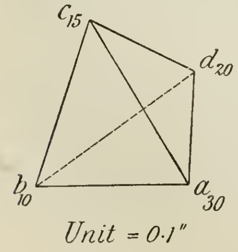
172. Developments of surfaces.—The surfaces of all polyhedra, and also certain curved surfaces, including the cone and cylinder, can be *developed;* that is, unfolded into one plane without any wrinkling or stretching.

A workman who makes objects of sheet metal, or a student who makes paper models of geometrical solids, first cuts out the shape of the development.

The illustration shows a development of a regular octahedron, with a margin left for gluing up the edges of a paper model of the solid.



173. Figured projections.—The diagram shows the projection of an irregular tetrahedron, in which the numbers affixed to the letters at the corners indicate to a given unit or scale the distances of the points from the plane of projection. In this way *one projection* is sufficient to *define a solid form.*



174. Miscellaneous Examples.

1. Explain in your own words the meaning of the terms *projector*, *projection*, *plan*, *elevation*, *section*, *planes of projection*.

2. Represent in plan and elevation four points A, B, C, and D, situated respectively in the four dihedral angles, each point being $1\frac{1}{2}''$ distant from the horizontal plane, and $2\frac{1}{2}''$ from the vertical plane.

3. Draw the plan and elevation of a point A which is in the horizontal plane, and $1''$ behind the vertical plane. Also of a point B, in the vertical plane, and $2''$ below the horizontal plane.

4. A point $1.72''$ below xy is both plan and elevation of a point A; in which of the four dihedral angles is A situated? Find the true distance of A from the ground line. *Ans.* $2.43''$.

5. Draw the plan and elevation of a point $1''$ below the horizontal plane, and $2''$ distant from xy. How many solutions are there? *Ans.* Two.

6. Determine the plan and elevation of a cube of $2\frac{1}{4}''$ edge: (a) when one face rests on the horizontal plane, and an edge makes an angle of 25° with xy; (b) when an edge rests on the horizontal plane perpendicular to xy, and a face is inclined at an angle of 65° to the horizontal plane.

7. A cube of $2''$ edge is pierced by holes $1''$ square through all its faces, so as to form a framed or skeleton cube. Draw the plan and elevation when in the positions of Ex. 6.

8. Draw the plan and elevation of a square pyramid, side of base $2''$, length of axis $2\frac{3}{4}''$: (a) when the base rests on the ground with one side making an angle 30° with xy; (b) when a triangular face rests on the ground, the axis of the solid being parallel to the vertical plane.

9. A point A is $1\frac{1}{2}''$ above the ground, and $2''$ in front of the vertical plane. Determine the auxiliary elevation of A on a vertical plane which makes an angle of 50° with the vertical plane of projection.

10. ABC is a triangle, the heights of A, B, and C above the ground being $1''$, $2''$, $1\frac{3}{4}''$, while the corresponding distances from the vertical plane are $1''$, $2\frac{1}{4}''$, $\frac{3}{4}''$. In plan ab measures $2''$ and bc $1\frac{3}{4}''$. Draw an elevation of the triangle on a vertical plane which makes 50° with xy.

11. A point A is $1''$ above the ground and $2''$ in front of the vertical plane. Draw the auxiliary plan of A on a plane at right angles to the vertical plane of projection and inclined at 60° to the horizontal plane.

12. $ABCD$ is a square. The three corners A, B, C are $2''$, $1\frac{1}{2}''$, and $1''$ above the ground, their distances in front of the vertical plane being $1\frac{1}{2}''$, $2''$, $1\frac{1}{4}''$. In plan ab is $2''$ and bc

1″ long. Draw the plan and elevation of the square, and an auxiliary plan on a plane at right angles to the vertical plane, and parallel to *AD*.

13. A regular tetrahedron rests with one edge (2″ long) on the ground, whilst a face containing that edge is inclined at 40°. Show in plan the section of it made by a horizontal plane 1″ high.

14. A prism, 3″ long, the ends of which are equilateral triangles of $1\frac{1}{2}''$ side, rests with a rectangular face on the ground. Draw the plan and a sectional elevation on a vertical plane which bisects the axis of the prism at an angle of 45°. Draw a development of the portion of the surface of the solid situated on one side of the cutting plane.

15. A cube, 2″ edge, rests with one edge on the ground and at right angles to the vertical plane of projection. A face containing this edge is inclined at 25°; draw the plan and elevation of the cube. Draw also an auxiliary elevation on a vertical plane which is parallel to a diagonal of the solid.

16. Draw a sectional elevation on a vertical plane which bisects that edge of the cube in Ex. 15 which is on the ground, the cutting plane making 70° with the principal vertical plane.

17. A hollow sphere, $2\frac{1}{2}''$ external diameter and $1\frac{3}{4}''$ internal diameter, has a portion cut away by a plane distant $\frac{3}{4}''$ from the centre. Draw a sectional projection of the larger remaining portion, on a plane which divides it into two exactly equal parts.

18. A cube 2″ edge is pierced centrally by holes $\frac{3}{4}''$ square through all its faces. It rests with one face on the ground. Draw a sectional elevation on a vertical plane which contains a vertical edge and bisects a horizontal edge of the cube.

19. A hexagonal pyramid, edge of base 1″, height 3″, rests with the base on the ground, one side of the base making 10° with *xy*. Draw the plan and elevation, and an auxiliary plan on a plane perpendicular to the vertical plane and parallel to a long edge of the pyramid.

20. Suppose the pyramid in Ex. 19 to rest with a side of the base on the ground and perpendicular to *xy*, the base being inclined at 25° to the ground. Draw the plan and elevation, and a sectional plan on a plane which passes through the centre of the base, and is parallel to a long edge of the pyramid.

21. Draw a line *ab* 2″ long, and attach indices of 18 and 25 to *a* and *b* respectively, thus obtaining the figured plan of a line *AB*. Find the true length of *AB* by drawing an elevation on a vertical plane taken parallel to *AB*. Unit = 0.1″.

If *ABC* be an equilateral triangle having its plane vertical, determine the indexed plan of *C*.

22. Copy the figure of Art. 173 three times the size, keeping the indices the same. Draw an elevation of the pyramid on a vertical plane parallel to *AC*. Draw also a sectional elevation of the pyramid on a vertical plane through *A* and the middle point of *CD*. Index the plan of this section.

23. Draw a quadrilateral, *abcd*, making *ab* $2\frac{3}{4}''$, *bc* 2″, the angle *abc* 120°, *cd* 4″, *ad* $3\frac{1}{4}''$; choose *v* within *abcd*, and such that *av* is $1\frac{3}{4}''$, and *vd* $2\frac{1}{4}''$. Attach indices of 12, 8, 18, 24, and 35 to *a*, *b*, *c*, *d*, and *v*. Join *v* to *a*, *b*, *c*, *d*; the result is the figured plan of a pyramid with vertex *V* and base *ABCD*. Determine (*a*) the plan of a section by a horizontal plane, at a level 19; (*b*) a sectional elevation on a vertical plane, bisecting *AV* and *CV*. Unit 0.1″.

24. Draw the plan of a cube, 2″ edge, when a diagonal of the solid is (*a*) vertical, (*b*) horizontal.

25. A building brick has a line joining two opposite corners of the solid vertical. Draw a sectional plan on a horizontal plane passing through the centre of the brick. Size of brick $9'' \times 4\frac{1}{2}'' \times 3''$. Scale $\frac{1}{3}$.

26. An instrument box with the lid open at an angle of 120° rests on the ground; draw its plan and a sectional elevation on a plane parallel to one end. Draw also an elevation on a vertical plane which makes 45° with the plane of one end.

Dimensions of box outside—length 6″, breadth $4\frac{1}{2}''$, depth of box $1\frac{1}{2}''$, depth of lid $\frac{3}{4}''$. Thickness of wood at sides and ends $\frac{3}{8}''$, at top and bottom $\frac{1}{4}''$. Scale $\frac{1}{3}$.

27. Draw a development and make a paper model of each of the following six solids—(*a*) a cube, (*b*) a regular tetrahedron, and (*c*) a regular octahedron, each of 2″ edge; (*d*) a building brick $9'' \times 4\frac{1}{2}'' \times 3''$, scale $\frac{1}{3}$; (*e*) a prism 3″ long, ends equilateral triangles, 2″ side; (*f*) a pyramid 3″ high, base a square of 2″ side.

28. Draw plan and elevation of a regular tetrahedron, 2″ edge, when resting with one edge on the ground and at right angles to the vertical plane, a face containing that edge being vertical.

29. A regular octahedron, $1\frac{1}{2}''$ edge, rests with a face on the ground. Draw its plan and elevation, when one edge of that face is parallel to the vertical plane. Draw a sectional elevation on a vertical plane which bisects any two adjacent horizontal edges.

30. A building brick $9'' \times 4\frac{1}{2}'' \times 3''$ is cut into two unequal portions by a plane which contains three corners. Draw the plans of the two parts when resting on their section faces. Index the plans of the corners of the solid.

***31.** A solid is formed of two equal square prisms—side of base $1\frac{1}{2}''$, height $3\frac{1}{2}''$—the axes of which bisect each other at right angles. The elevation is shown but is not drawn to scale. Draw this elevation the proper size, and deduce the plan, and a second elevation on a new ground line, making 50° with xy. (1878)

***32.** The figure is the elevation of a truncated hexagonal pyramid. Draw the plan of this solid inverted, so that the plane of truncation is on the horizontal plane of projection. (1889)

*** 33.** The position and dimensions of two equal and equally inclined rectangular prisms are indicated by their given projections. Draw the sectional elevation on AB. (1877)

***34.** The figure represents an end elevation of an open trunk. The length is twice the breadth. Draw a front elevation on a plane parallel to one of the diagonals of the bottom. The thickness of wood may be neglected. (1884)

***35.** A spherical segment rests on the top of a truncated hexagonal pyramid. The plan and elevation are partially given. Draw the figure *four* times the given size, and make a sectional elevation of the solid on AB. (1888)

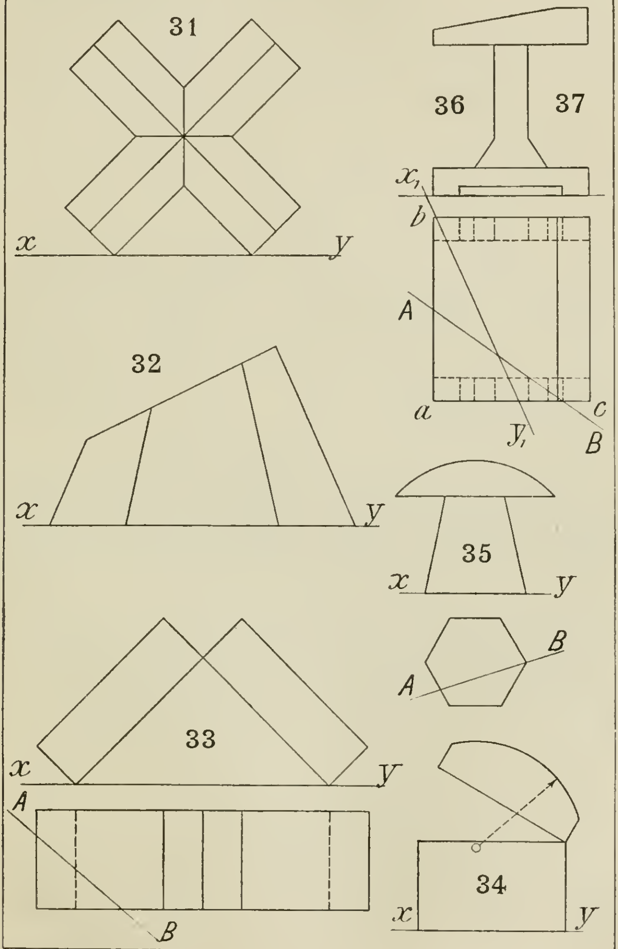
***36.** The plan and elevation of a desk are given. Draw a front elevation on a line parallel to x_1y_1. (1886)

***37.** Make a sectional elevation on AB of the desk. (1886)

38. Draw the plan of a square prism, height $2\frac{3}{4}''$, side of base $1\frac{1}{4}''$, a diagonal being vertical. (1891)

39. ABC is the base of a pyramid, and V its vertex. $AB=2\frac{1}{2}''$, $AC=2''$, $BC=2\frac{3}{4}''$, $AV=CV=3''$, $BV=2\frac{1}{2}''$. Draw the plan of the pyramid when A is $1\frac{1}{2}''$, B $2\frac{1}{2}''$, and C $1''$ above the horizontal plane. (1896)

Copy the figures double size.



CHAPTER IX

THE STRAIGHT LINE AND THE PERPENDICULAR PLANE

175. The possible positions of a line.—The various positions which a line, situated in the first dihedral angle, may have, relatively to the planes of projection, are shown lettered (*a*) to (*n*).

The remaining lines (*o*) to (*t*) are wholly or partially in one or more of the other three angular spaces.

The lines (*a*) and (*b*) are respectively perpendicular to the horizontal and vertical planes; (*c*) is parallel to the vertical plane and inclined to the horizontal plane; (*d*) is parallel to horizontal plane and inclined to the vertical plane; and (*e*) is parallel to both planes.

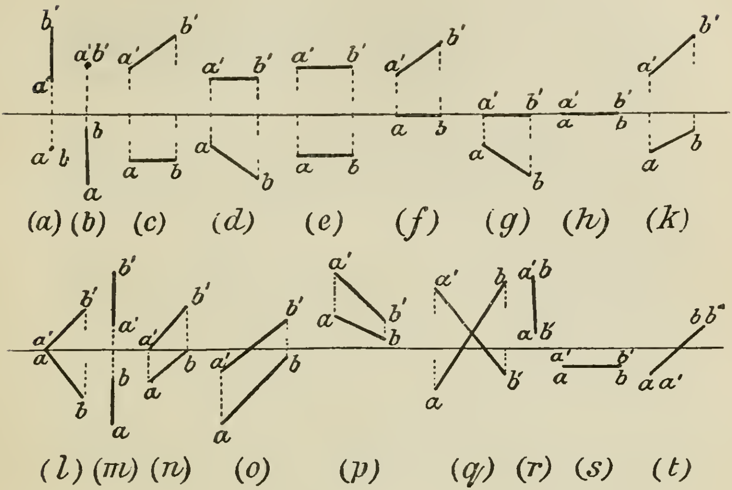
The line (*f*) is in the vertical plane; (*g*) is in the horizontal plane; and (*h*) is in both planes.

The lines (*k*), (*l*), (*m*), (*n*) are inclined to both planes, (*l*) having one end in *xy*, (*m*) being perpendicular to *xy*, and (*n*) having its two ends respectively in the planes of projection.

In studying the position of the lines, let the student take the model described in Art. 168.

Pieces of wire passed through the paper, or held in the hand, may be used to represent the lines, and the wires should be placed in positions so as to agree with the projections given.

When the model is held with the horizontal plane on



a level with the eye, and viewed from the front, the lines should appear respectively as the elevations *a′b′*; when looked down upon vertically, the appearance should be that of the plans *ab*.

Examples.—**1.** Show in plan and elevation: (*a*) a point 2″ from the ground line and $1\frac{1}{4}$″ from the vertical plane of projection; (*b*) a line parallel to and $1\frac{1}{4}$″ from the vertical plane of projection, and inclined at 40° to the horizontal plane.

2. Draw the projections of any four lines, which are situated wholly in the four dihedral angles, one in each.

3. Show by their projections any two equal parallel lines not parallel to either plane of projection.

4. How can you tell from the projections of two lines whether the lines meet or not? Draw the projections of any two lines which intersect, and of two lines which do not intersect.

5. A line 2″ long is parallel to *xy*, and distant 2″ therefrom; is this sufficient data to fix the position of the line? Draw the projections of the line, if it be 1″ in front of the vertical plane.

176. Problem.—**Having given the plan ab and elevation a′b′ of a line not parallel to either plane of projection, to determine the length of the line, and its inclinations, α and β, to the horizontal and vertical planes respectively.**

There are three useful methods of solving this problem.

First Method. By auxiliary projections, Fig. (1).—Take $x'y'$ parallel to ab, and draw the auxiliary elevation $a''b''$. Also take x_1y_1 parallel to $a'b'$, and draw the auxiliary plan a_1b_1.

Then both a_1b_1 and $a''b''$ give the length of AB. The angle marked α is the inclination of AB to the ground; and the angle marked β the inclination to the vertical plane.

Second Method. By rabatments, Fig. (2).—Draw aA_0, bB_0 perpendicular to ab, and equal to ma', nb' respectively. Draw $a'A_1$, $b'B_1$ perpendicular to $a'b'$, and equal to ma, nb respectively. Then A_0B_0 is the rabatment of AB about ab into the horizontal plane; and A_1B_1 is the rabatment of AB about $a'b'$ into the vertical plane.

The length of the line AB is either A_0B_0 or A_1B_1; and the inclinations are the angles marked α and β.

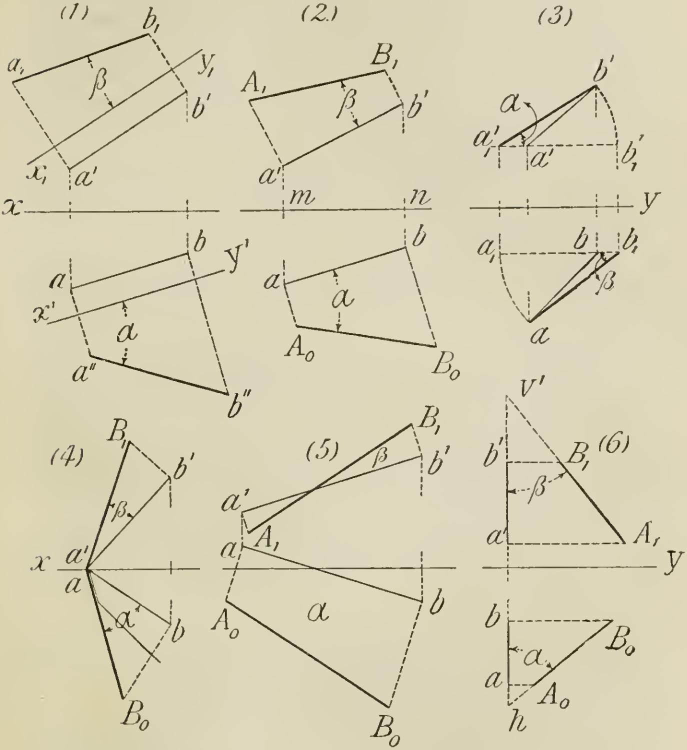
A model should be made, by cutting out the shapes of the two quadrilaterals in paper, leaving a margin along the edges ab and $a'b'$, for attachment to the planes of projection.

Third Method. By turnings about projectors, Fig. (3).—With centre b, radius ba, describe the arc aa_1, to meet ba_1 drawn parallel to xy. Project from a_1 to meet the horizontal through a' in a_1'. Then $a_1'b' = AB$, and $b'a_1'a' = \alpha$.

With centre a', radius $a'b'$, describe the arc $b'b_1'$, to meet the horizontal through a' in b_1'. Project from b_1' to meet the line through b parallel to xy, in b_1. Then $ab_1 = AB$, and $ab_1b = \beta$.

The first construction represents AB turned about Bb, so that AB becomes parallel to the vertical plane. In the second construction AB is turned about Aa', until AB is horizontal.

The lower figure shows three additional examples of this problem, all solved by the method of rabatments.



Examples.—1. The plan and elevation of a line are 2″ and 3″ long. The projectors of its extremities are 1″ apart. Find the true length and inclinations of the line. *Ans.* 3.46″, 54.7°, 60°.

2. One end of a line is .75″ below the horizontal plane; the other end is 1.5″ above it. What is the true length and inclination of the line if its plan measures 2″? *Ans.* 3″, 48°.

3. The elevation of a line 3″ long measures 1.5″; at what angle is the line inclined to the vertical plane? *Ans.* 60°.

177. PROBLEM.—**Having given the projections of a line AB, to find the horizontal and vertical traces H and V.**

The plans and elevations of the traces must be respectively in the plan and elevation of the line (produced if necessary); also the plan of the vertical trace and the elevation of the horizontal trace must both be in *xy*. The construction is therefore as follows:—

Produce the elevation to meet *xy* in *h'*; from *h'* draw a projector to meet the plan produced in *h*. *H* is the horizontal trace required.

Produce the plan to meet *xy* in *v*, draw *vv'* to meet the elevation *a'b'* produced in *v'*. *V* is the vertical trace of the line.

In the right-hand figure the horizontal trace is behind the vertical plane. The perspective diagram underneath illustrates this case.

The construction given in this problem fails when the line is perpendicular to *xy*. In this case, find the rabatments of the line, and, if necessary, produce these to meet the projections produced, as shown in Fig. 6, page 207.

When a line is parallel to a plane, its trace on that plane may be considered at an infinite distance away in the direction of the line.

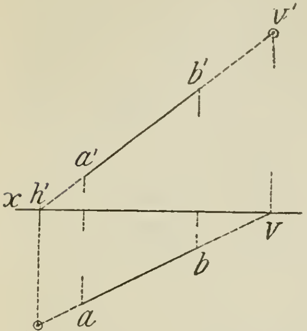
178. PROBLEM.—**Having given the indexed plan of a triangle, to find the true shape. Unit = 0.05″.**

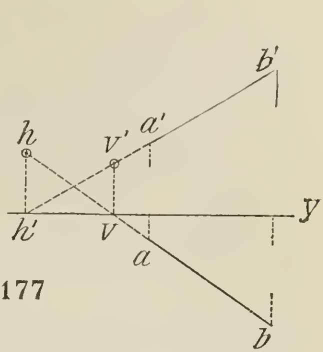
Let *abc*, with the indices, be the given plan.

Determine the rabatments of the three sides, as in the second method of Prob. 176, and draw the triangle $A_0B_0C_0$ with its sides respectively equal to these rabatments.

$A_0B_0C_0$ is the true shape of the triangle, and may be regarded as being the rabatment of the triangle into the horizontal plane—first, by turning the quadrilateral *BCcb* into the ground about *bc*, then by further turning the triangle into the ground about B_0C_0.

Examples.—1. The plan of a line is 2″ long and its elevation 3″. The projectors of its extremities are 1″ apart. If the lower





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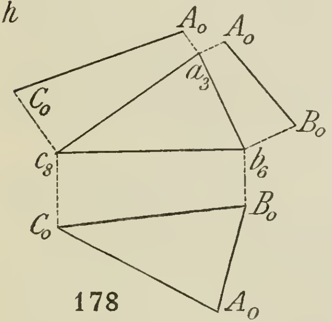

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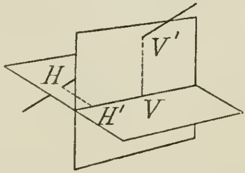


end of the line be 1″ in front of the vertical plane and $1\frac{1}{2}$″ above the ground, determine the horizontal and vertical traces of the line.

Regarding the lower end as fixed, how many lines are there satisfying the above data? *Ans.* Four.

2. Draw a triangle abc having $ab = 2\frac{1}{2}$″, $bc = 2$″, $ac = 3$″. Attach indices of 6, 15, 24 to a, b, and c respectively. Determine the true shape of the triangle ABC by finding the true lengths of its sides. Unit = 0.1″.

   Ans. $AB = 2.66$″, $BC = 2.19$″, $CA = 3.5$″.

3. Determine d_{15} in ac of Ex. 2. Join d and b, then DB is a horizontal line at a level 15. Now draw $x'y'$ at right angles to db, and obtain an auxiliary elevation of the triangle on $x'y'$; this should be a straight line, the *edge elevation* of the plane of the triangle. This is a **useful construction.**

179. Solutions depending on the right-angled triangle.—We have seen that many problems in practical geometry reduce to problems on the right-angled triangle. The nine cases of the latter have already been given in Art. 12, but the importance of the recognition of these analogies seems to warrant a repetition of the cases specially applicable to the problems under immediate consideration.

Let OPp, OQq be two right-angled triangles. Then if OP be the true length of a line, and α its inclination to the ground, Op is the *length* of its plan, and Pp the *difference* in the vertical heights of its ends. Any two of these four quantities being given, the other two may be found by constructing the triangle OPp.

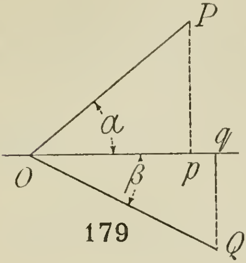
Similarly, if OQ be the length of a line, β the inclination to the vertical plane, Oq is the *length* of its elevation, and Qq the *difference* in the distances of the ends of the line from the vertical plane. Any two of these quantities being given, the remaining two can be determined.

180. Problem.—**A line $2''$ long has one end $\frac{1}{2}''$ high; the plan of the line measures $1\frac{1}{2}''$, and the elevation $1\frac{3}{4}''$. Draw the projections of the line.**

Determine by the method of Art. 179 the *difference* in the heights of the ends of the line, and in the distances of the ends from the vertical plane. That is, in Fig. 179 find Pp and Qq, having given $OP = OQ = 2''$, $op = 1\frac{1}{2}''$, $oq = 1\frac{3}{4}''$. Then proceed as follows:—

Draw any line ab, $1\frac{1}{2}''$ long, as the plan. With centre b, radius Qq, describe a circle. From a draw am, a tangent to the circle, and take xy parallel to am. Then the elevation on xy is the one required.

In drawing the elevation, a' is to be set off $\frac{1}{2}''$ above xy, and b' a distance Pp (Fig. 179) higher than a'. The length of the elevation will then be found to be $1\frac{3}{4}''$, because the difference in the distances of A and B from the vertical plane, being equal to bm, is equal to Qq (Fig. 179).



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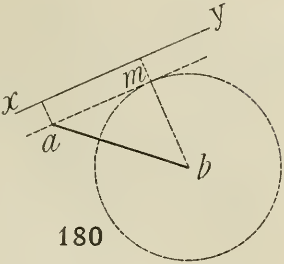

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Examples.—**1.** Take a point a in xy, and draw ab $1\frac{1}{4}''$ long, perpendicular to xy. Let ab be the plan of a line AB, of which B is $1\frac{1}{2}''$ higher than A; find the length and inclination of AB. If A be $\frac{1}{2}''$ above the ground, draw the elevation of AB, and find the horizontal trace of the line.

2. A point A is $2''$ above the ground and $1\frac{1}{2}''$ in front of the vertical plane. AB is a line $2\frac{1}{4}''$ long, whose plan is perpendicular to xy; draw the plan and elevation of AB, when B is in the vertical plane.

3. The plan of a line measures $2''$ and the elevation $1\frac{1}{2}''$. Is this data sufficient to fix the length of the line? If not, add any third condition which will fix the length.

A is a point in the vertical plane $1\frac{1}{4}''$ above the horizontal plane; B is a point in the horizontal plane $1\frac{3}{4}''$ from the vertical plane. The real distance from A to B is $3''$. Draw the plan and elevation of the line AB.

4. A point is $2''$ from the vertical plane and $1.75''$ above the horizontal plane. Determine a point B on the ground line $3\frac{1}{4}''$ from A, and measure the inclination of AB.

5. Two points A and B are respectively $1.5''$ and $1.75''$ from both planes of projection. Their plans are $2.25''$ apart. Determine the projections of a point C in the vertical plane $2''$ from A and $2.75''$ from B. If C has to be $2''$ from A, what is the least possible distance of C from B? *Ans.* $2''$.

6. Determine the projections of a point D in AB, Ex. 5, distant $1.75''$ from A.

7. A line AB is inclined at 35°. The upper end A is $2''$ above the ground, and ab is $2''$ long. Draw the elevation of AB on $x'y'$ which makes 30° with ab.

8. A line $3''$ long has one end $1''$ high, the plan of the line measures $2''$ and the elevation $2\frac{1}{4}''$. Draw the projections of the line.

181. PROBLEM.—**A line AB of given length has its ends in the planes of projection, and the line is inclined at given angles α and β to the horizontal and vertical planes respectively. Draw the plan and elevation.**

Method 1.—By the construction of Art. 179 this problem could be at once converted into the case of the preceding problem, and thus solved.

Method 2.—Draw B_1a' and B_1c inclined at α and β to xy, and make $B_1a' = B_1c = AB$. Draw the projectors $a'a$, cc'.

Then, as in Art. 179, B_1a is the *length* of the plan of the line, and B_1c' the *length* of the elevation.

Make $a'b' = B_1c'$, and from b' draw a projector to meet the circle, with a as centre, and aB_1 as radius, in b. Then ab, $a'b'$ are the required projections.

To explain this construction, suppose in the first instance the line to be in the vertical plane, so that aB_1, $a'B_1$ are its projections. Let the line be then turned about aa' as axis. The path of B is a circular arc of which B_1b is the plan, and B_1b' the elevation. The rotation is continued until the length of the elevation is equal to B_1c'; the line is then inclined at β to the vertical plane. It is also inclined α to the horizontal plane, since the latter inclination remains unchanged during the rotation.

Limits to data.—The sum $\alpha + \beta$ cannot be greater than 90°.

182. PROBLEM.—**Having given the projections of a line AB, to find the shortest distance of the line (produced if necessary) from xy.**

Let ab, $a'b'$ be the given projections.

Draw o_0h and o_0v, an edge view of the horizontal and vertical planes of projection.

Determine a_0, the projection of A in this view, by making $a_0m = a'l$, that is, equal to the height of A above the ground; and $a_0n = al$, that is, equal to the distance of A in front of the vertical plane.

Similarly determine b_0 and join a_0b_0. Draw o_0c_0 perpendicular to a_0b_0. Then o_0c_0 is the shortest distance required.

Obtain c' from c_0, and c from c'; then co, $c'o'$ are the projections of the shortest line between xy and AB. This line is perpendicular to both xy and AB.

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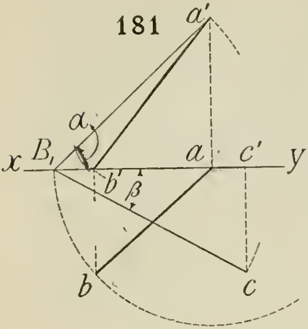

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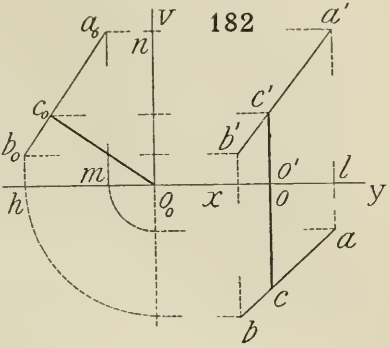


Examples.—**1.** Draw the plan and elevation of any line $2\frac{1}{2}''$ long, which is inclined at 35° to the horizontal plane, and 25° to the vertical plane.

2. A line $2\frac{3}{4}''$ long has one end in *xy*, and is inclined at 30° and 40° respectively to the horizontal and vertical planes of projection. Represent the line.

3. Draw the plan and elevation of any line $3''$ long, which shall be equally inclined to the planes of projection, and not parallel to *xy*.

4. The plan of a line $2\frac{1}{2}''$ long measures $2''$; draw the elevation on a vertical plane inclined to the line at 20°.

5. The plan of a line measures $1''$ and the elevation $.8''$. The inclination of the line to the horizontal plane is 30°; what is the inclination to the vertical plane? *Ans.* 46.2°.

6. Draw a triangle *abc* having $ab = 2\frac{1}{4}''$, $ac = 1\frac{1}{4}''$, $bc = 1\frac{3}{4}''$. Through *c* draw *cd* $3''$ long, cutting *ab* and making an angle of 50° with *ca*. Attach indices of 8, 18, 6, 15 to *a*, *b*, *c*, *d*. Find whether the lines *AB* and *CD* intersect. Unit = $0.1''$.

If they do not, find what the index of *d* must be so that the lines shall intersect. *Ans.* 25.6.

7. Draw the plan of a horizontal line at a level 10 to intersect the lines *AB*, *CD* in Ex. 6.

8. A point *A* is $\frac{1}{2}''$ above the ground, and $1\frac{1}{2}''$ behind the vertical plane. Find the distance of *A* from *xy*. *Ans.* $1.58''$.

9. A point *A* is $1''$ above the ground, and $1\frac{1}{2}''$ in front of the vertical plane. A point *B* is $\frac{1}{2}''$ above the ground, and $2\frac{1}{2}''$ in front of the vertical plane. The projectors of the points are $1\frac{1}{4}''$ apart. Find the shortest distance between *AB* and *xy*. Draw the projections of the shortest line between *AB* and *xy*.

183. The perpendicular plane.—*A perpendicular plane* is one which is at right angles to one or other (or both) of the principal planes of projection; a plane which is inclined to both being called *oblique*. A perpendicular plane may be either *inclined* or *vertical*. An *inclined plane* is one which is inclined to the horizontal plane and perpendicular to the vertical plane. A *vertical plane* is perpendicular to the horizontal plane and may be inclined to the vertical plane of projection.

As was stated in Art. 166, a plane is conveniently represented by its *traces*. We shall now have only two traces, the horizontal and the vertical.

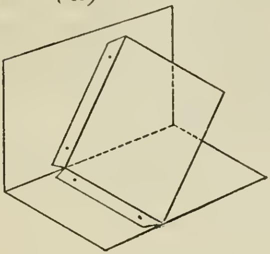
The student should make a model of a perpendicular plane as shown at (a). For the planes of projection take a piece of drawing-paper 9″ square, indented and folded across the middle. For the perpendicular plane, cut a rectangular piece of drawing-paper $6\frac{3}{4}''$ by $5\frac{1}{4}''$, and indent and fold margins $\frac{3}{4}''$ wide on two adjacent edges for attachment by paper-fasteners. By taking out one or other pair of fasteners, the plane can be rabatted into either plane of projection at pleasure; and by placing the model the two ways up in succession, it illustrates both the inclined and vertical plane, as shown at (a) and (b).

The *characteristic feature* of a perpendicular plane is that *one or other of its traces is a profile or edge view of the plane*. In the oblique plane neither trace is so.

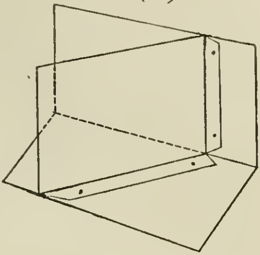
To illustrate this let the student take the model, and holding it in the position (a), let him view it directly from the front; the whole plane will appear as a line coinciding with the vertical trace. The vertical trace is thus strictly an *elevation* of the plane, for it contains the elevations of all points and lines in the plane. The horizontal trace could not be called a *plan* of the plane, for it contains the projection of only one line in it, the trace itself, as will be quite evident from the model.

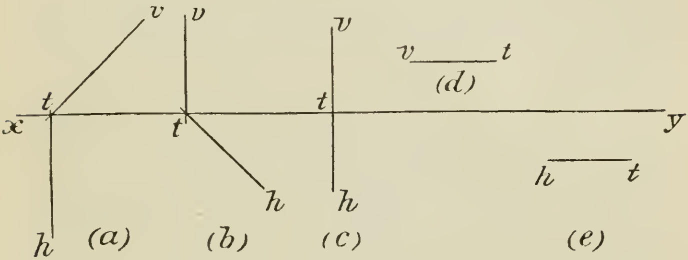
Similarly in (b), it will be at once seen from the model that the horizontal trace is a true *plan* of the vertical plane, for it contains the plans of all points and lines in the plane. But the vertical trace is not an elevation.

(*a*)



(*b*)





Five perpendicular planes are represented by their traces in the lower figure: (*a*) as an inclined plane, (*b*) a vertical plane, and (*c*) a plane perpendicular to *xy*; (*d*) is a horizontal plane, and (*e*) a plane parallel to the vertical plane of projection. In (*a*) *tv* is an edge elevation of the plane; in (*b*) *th* is a profile plan; and in (*c*), (*d*), and (*e*) the traces are all edge views of the planes.

By using the model the student will see that the angles which the planes make with the horizontal and vertical planes of projection are respectively equal to the angles which the vertical and horizontal traces make with *xy*. In an oblique plane this is not the case.

It will also be noticed that the *apparent angle* between the traces is not the *real angle*, for the latter is a right angle.

184. PROBLEM.—**A square pyramid is given by its projections. (a) Determine the plan of the section made by any inclined plane VTH. (b) Draw the plan of the frustum with its section end on the ground. (c) Draw a development of the surface of the pyramid, showing the trace of the cutting plane on the surface.**

(a) The elevations of P, Q, R, S, the points in which the plane cuts the edges, are on tv, because the vertical trace is an edge elevation of the plane.

The plans p, q, r, s are obtained by projection from the elevations.

(b) On tv draw an auxiliary plan of the frustum, as shown in the figure; this is determined according to the second rule of Art. 170, and is the plan when standing on the section end.

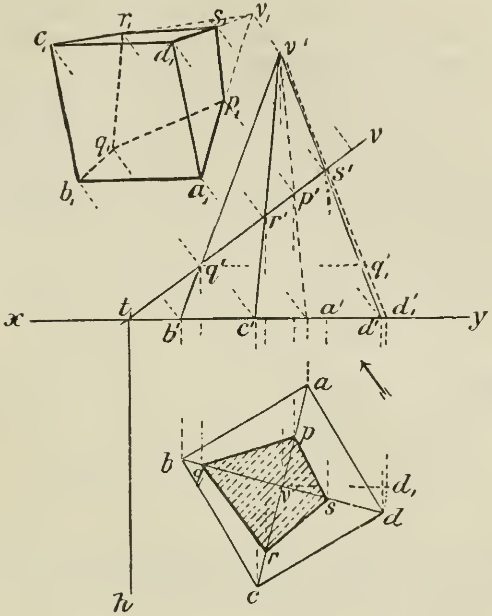
The plan may also be considered as obtained by rabatment of the plane about its vertical trace, the frustum having been first projected on the section plane.

(c) To obtain a development, first find the true length of a sloping edge of the pyramid and the true distances of P, Q, R, S from the vertex. In the figure $v'd_1'$ and $v'q_1'$ are the true lengths of VD and VQ respectively, obtained as in Prob. 176, third method. The lengths of VP, VR, VS may be found in a similar manner.

A development of the surface of the pyramid must now be drawn as shown, and the lengths VQ, VR, VS, and VP set off along the developed edges. The development is then completed by joining QP, PS, SR, RQ, these lines forming the developed trace.

Examples.—**1.** A cube, 2″ edge, has its base on the horizontal plane, one corner being in xy, and the diagonal of the base through that corner making 60° with xy. Draw the plan and elevation showing the section by a plane passing through the centre of the cube, and inclined at 30°. Find the true shape of the section, and draw a development of the frustum.

2. A regular tetrahedron $ABCD$ of 3″ edge rests with ABC on the ground. Draw its plan and show the section by a plane through the centre of the solid, whose horizontal trace is am making 45° with ab.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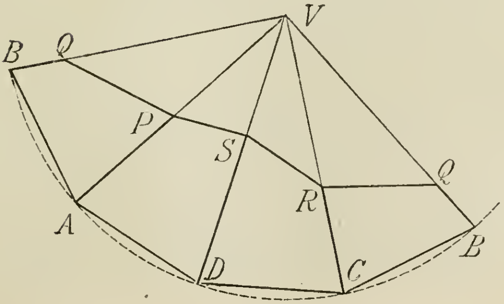
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185. PROBLEM.—**Having given the projections of a cube, and the traces vth of a vertical plane, to determine the plan, elevation, and true shape of the section.**

The horizontal trace *ht* is an edge view of the plane, and the required plan of the section is therefore in the horizontal trace, and is the line *pq*. The elevation of the section is the shaded rectangle obtained by projection from *pq*.

The true shape of the section is the rectangle pQ_0, obtained by the rabatment of the section plane into the horizontal plane, pP_0 being made equal to $p'p'$.

The true shape might be equally well obtained by a rabatment into the vertical plane, about the vertical trace, and the student should make this construction.

Example.—A square pyramid, side of base $1\frac{1}{2}''$, axis $2\frac{1}{4}''$, has its base in the vertical plane, one corner being in *xy*, and the diagonal of the base through that corner inclined at 60°. Draw the plan and elevation, showing the section by a vertical plane, bisecting the axis of the solid, and making 30° with the vertical plane of projection. Find the true shape of the section.

186. PROBLEM.—**Having given the plan of any polygon or plane figure lying in a given inclined plane, to find the elevation, rabatment, and true shape of the figure.**

Let *abc* be the given plan, and *vth* the traces of the given plane.

From the points in plan draw projectors to meet the vertical trace in a', b', c'. Then $a'b'c'$ is the required elevation of the polygon. For, the vertical trace being an edge elevation of the plane, it must contain the elevation of the polygon.

The true shape may be obtained by a rabatment of the plane about its horizontal trace. With centre *t* describe the arcs $a'A_0'$, $b'B_0'$, $c'C_0'$; these are the elevations of the paths of A, B, C; and A_0', B_0', C_0' are the elevations of the rabatments of A, B, C. The plans of the arcs are the lines aA_0, bB_0, cC_0, drawn perpendicular to the horizontal axis of rotation *th*. These intersect the projectors from A_0', B_0', C_0' in A_0, B_0, C_0.

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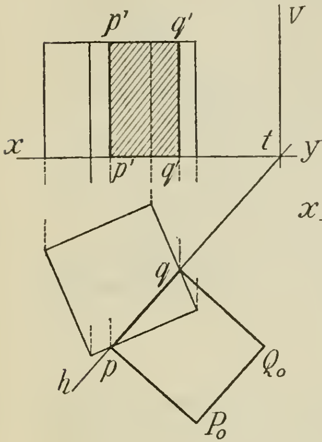

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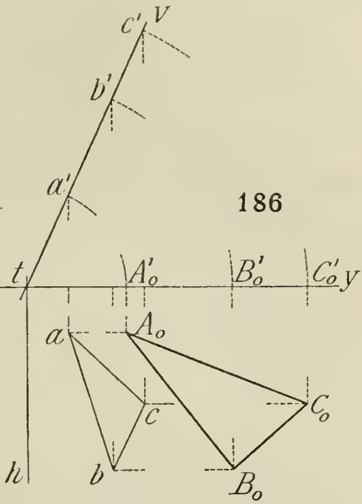


The true shape may also be obtained by a rabatment into the vertical plane; this is equivalent to an auxiliary plan on *tv*, and may be determined by applying the principles of Art. 170.

This is an **important problem** because it occurs so often in other problems; hence the student should make himself quite familiar with the construction.

Examples.—**1.** Draw an equilateral triangle *abc*, 2″ side, with *ab* making an angle of 45° with *xy*. Let this be the plan of a triangle *ABC* lying in a plane inclined at 40°. Find the inclinations of the sides of the triangle.

2. By using the rabatment of the triangle *ABC* in Ex. 1, determine (*a*) the plan of the bisector of the angle *ACB*; (*b*) the plan of the perpendicular drawn from *C* to *AB*.

Note.—Observe that if in the rabatment the angle $A_0C_0B_0$ be bisected by a line which meets the horizontal trace of the plane in *i*, then *ic* is the plan of the bisector. In problems involving rabatments it is a very useful artifice to produce lines to meet the trace in stationary points.

187. Problem.—**To find the point of intersection of a given line AB and inclined plane VTH.**

The elevation of the required point of intersection is p', for reasons stated in previous problems. The plan p is found by drawing the projector from p' to meet ab.

188. Problem.—**To determine a perpendicular from a given point A to a given inclined plane VTH.**

Draw $a'm'$ perpendicular to tv; through m' draw the projector $m'm$, to meet at m a line through a perpendicular to th. Then AM is the required perpendicular.

The student should use the model to illustrate this problem.

189. Problem.—**To determine the plan and elevation of the projection of a given point A on a given inclined plane VTH.**

Proceed as in the last problem. Then M is the required projection.

190. Problem.—**Having given a line AB and an inclined plane VTH, to determine (a) the trace of the line on the plane; (b) the projection of the line on the plane; and (c) the angle between the line and plane.**

(a) The required trace of the line on the plane is the intersection S, and is found as in Prob. 187.

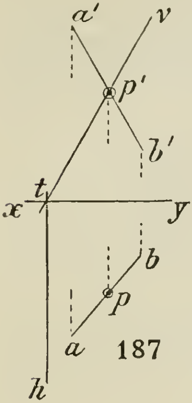
(b) The projection MN of the line on the plane is determined by applying the construction of Prob. 189.

(c) The inclination of the line to the plane is the angle between the line AB and its projection MN on the plane, and is found by a double rabatment as follows:—

First, M_0N_0, the rabatment of MN, is found as in Prob. 186. Then $A_0B_0N_0M_0$, the rabatment of the quadrilateral $ABMN$ about M_0N_0, is determined by drawing M_0A_0, N_0B_0 perpendicular to M_0N_0 and equal to $a'm'$, $b'n'$, the true lengths of the perpendiculars AM, BN.

The inclination of the line to the plane is the angle $A_0S_0M_0$.

The student may readily make a model to illustrate this problem, by cutting out the shape of $A_0B_0M_0N_0$ in paper, with a margin at M_0N_0 for attachment to the model of the inclined plane.



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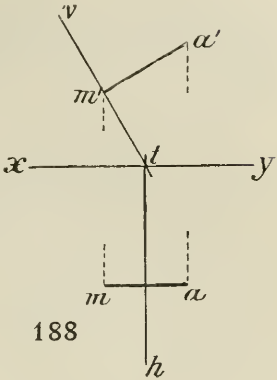

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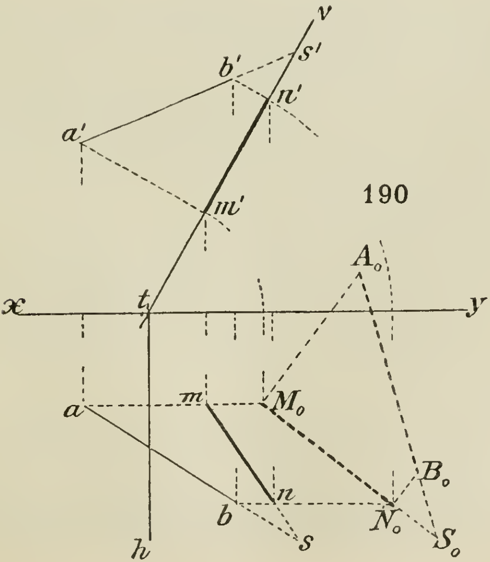

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191. PROBLEM.—**Determine the plan and elevation of a line inclined at α, and lying in a plane inclined at θ. Find the angles which the line makes with the traces of the plane, and the inclination of the line to the vertical plane.**

Draw the traces vth of the plane inclined at θ.

Take any point A in the vertical trace of the plane; its elevation a' will be in tv, and its plan a in xy.

Through A draw a line in the vertical plane inclined at α, with one end B on the ground, so that $a'B_1$ is its elevation, and aB_1 its plan.

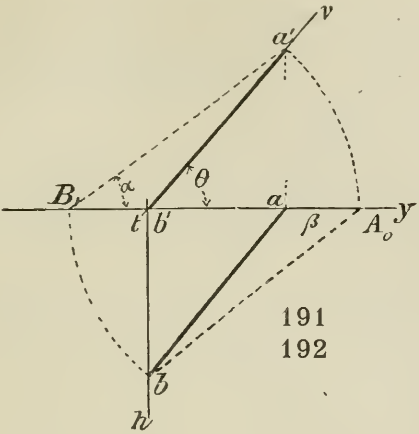
Let this line revolve about aa' until its end B comes into the horizontal trace of the plane; the line itself is then in the plane, because both its ends are in it. The arc B_1b struck with a as centre is the plan of the path of B, and B_1b' is the elevation of the path.

Then ab, $a'b'$ are the projections of the line in the required position; for the inclination of the line remains unchanged during the rotation.

Let bA_0 be the rabatment of the line obtained as in Prob. 186, then the true angles which AB makes with the horizontal and vertical traces are respectively A_0bb' and bA_0b'. The latter angle is also β, the inclination of AB to the vertical plane, since it is equal to the angle between the line and its elevation.

This problem is an important one, and should be studied with the model, until the exact meanings of the five angles referred to are fully understood, and the differences between them are recognised. Students partially acquainted with the construction are apt to take wrong centres for the two arcs.

Limits to the data.—The angle α cannot be greater than θ, but may have any value from 0 to θ. When $\alpha = 0$, the line is horizontal, and therefore parallel to the horizontal trace of the plane. When $\alpha = \theta$, that is, when the line and plane are equally inclined, the plan of the line is perpendicular to the horizontal trace, and the line itself in the plane is also perpendicular to the horizontal trace.



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192. Problem.—**Determine the plan and elevation of a line inclined at β to the vertical plane, and lying in a plane inclined at θ to the horizontal plane. Find α, the inclination of the line to the ground.**

Let *vth* be the traces of the plane inclined at θ.

First draw bA_0 making an angle β with *xy*, and suppose this to be a line in the horizontal plane.

Now let this line be turned about *bt* until it comes into the plane *VTH*. That is, with centre *t* describe the arc A_0a', and draw the projector *a'a*.

Then *ab*, *a'b'* are the projections of the line in the required position.

The inclination α is obtained by turning *AB* into the vertical plane, about *a'a*, as in Prob. 176, third method.

The construction of this problem corresponds to that of the preceding problem worked backwards, and the two problems should be studied together with the model.

Example.—A vertical plane makes 45° with *xy*. Draw its traces and show a line lying in it and inclined at 45°. Find the angle which this line makes with the vertical plane of projection.

193. Plane represented by a scale of slope.—It has already been explained that by a system of indexed plans, points and lines may be represented by one projection only. It has now to be shown how in the same system a plane is represented.

Draw a line through any two points, *P*, *Q*, on a piece of cardboard, and let one edge *AB* of the cardboard be perpendicular to *PQ*. Take the cardboard as the model of a plane, and hold it in an inclined position, with *AB* resting on any flat surface representing the horizontal plane.

The student will then observe that *PQ* is inclined at the same angle as the plane, and that the plan of *PQ* is perpendicular to the horizontal trace *AB*. Also that any line parallel to *PQ*, and lying in the plane, has the same inclination as the plane.

Lines in the plane such as *PQ*, perpendicular to the horizontal trace, are called *lines of slope*, and it is evident that if the figured plans of *P* and *Q*, or of any two points on any line of slope, be given, the position of the plane relatively to the horizontal plane is completely defined.

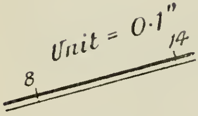
The figured plan of a line of slope is called a *scale of slope*, and is conventionally drawn as two lines, one thicker than the other, as is usual in drawing scales. The double line serves to distinguish the representation of a plane from that of a line.

Fig. 193 shows the representation of a plane by a scale of slope, the unit for the indices being 0.1″.

To represent this plane in the manner previously explained, see the next problem.

194. PROBLEM.—**Having given a line by its indexed plan, $a_{12}b_{5\cdot5}$, and a plane by its scale of slope, 5,15, to determine the indexed plan of the point of intersection of the line and plane.**

Draw any *xy* parallel to the scale of slope, and find *a′b′*, the elevation of *AB*; find also 5′, 15′, the elevations of the two given points on the scale of slope.



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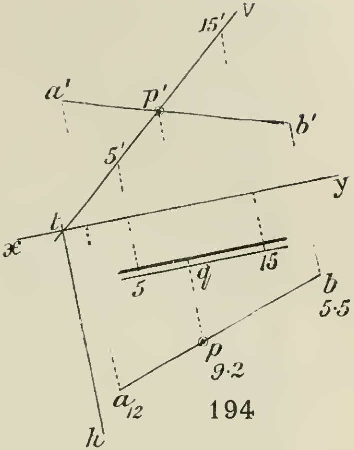

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Then *vt*, drawn through 5′, 15′, is the vertical trace of the given plane as regards *xy*; and *p*′ is the elevation of the intersection of the line and plane, as in Prob. 187.

Project from *p*′ to determine the plan *p*. To find the index of *p*, measure the height of *p*′ above *xy*, *or* read the position of *q* on the scale of slope.

Example.—Draw a line *ab* $1\frac{1}{2}''$ long; draw *ad* and *bc* each perpendicular to *ab* and $\frac{1}{2}''$ and $1''$ in length. Attach indices of 10, 30, 25, and 15 to *a*, *b*, *c*, *d*. Regard *ab* as the scale of slope of a plane and *cd* the figured plan of a line. Determine the indexed plan of *P*, the intersection of the line and plane. Unit 0.1″.

195. PROBLEM.—**Having given the indexed plan of a triangular pyramid, and the scale of slope of a plane, to determine the indexed plan of the section of the solid.** (No figure.)

Draw an elevation of the plane and pyramid on an *xy* taken parallel to the scale of slope, and proceed as in Prob. 184. Measure the heights of the points in the section, and index the corresponding points in plan.

196. Miscellaneous Examples.

1. Draw the traces of a perpendicular plane, inclined to the right at 40° to the ground; draw ab 2″ long anywhere to the right of the horizontal trace. Regard ab as the plan of a line AB lying in the plane, and determine the plan of a square $ABCD$ also lying in the plane.

Hint.—First find the rabatment $A_0B_0C_0D_0$; produce C_0B_0 to meet the horizontal trace in i; join ib and produce it to meet in c a line from C_0 parallel to xy; then complete the parallelogram $abcd$. Observe that by working with i we do not require to use the elevation in order to determine c from C_0.

2. Represent two planes inclined, one at 40° and the other at 70°, and such that their intersection is $1\frac{1}{4}''$ from the ground. Obtain the projections of a horizontal line, $\frac{1}{2}''$ above the ground and $2\frac{1}{2}''$ long; the ends of which are in the two planes. Determine the point of intersection of this line with the inclined plane which bisects the acute angle between the first pair of planes.

3. Determine the traces of a plane inclined to the right at 15°, and perpendicular to the vertical plane. A point A is $1\frac{1}{2}''$ in front of the vertical plane, and 2″ above the horizontal trace, but 1″ to the right of it. Draw the projections of the perpendicular let fall from A on to the plane.

4. Represent a vertical plane making an angle of 40° with xy. In this plane place a line $2\frac{1}{4}''$ long, which shall be inclined to the vertical plane of projection at an angle of 30°.

5. Represent a vertical plane which makes an angle of 40° with xy, and draw the projections of a square lying in this plane, one diagonal being inclined at 70°, the lower end of which is on the ground.

6. An inclined plane makes an angle of 40° with the ground; draw the projections of a line which bisects the angle between the traces.

7. Draw the plans of two lines AB, AC, which lie in a plane inclined at 50°, the lines being inclined at 30° and 45° respectively. Obtain the projections of the bisector of the angle BAC.

8. Draw the traces of a plane inclined at 50°, and in this plane place a line 2″ long inclined at 40°. Find the inclination of the line to the vertical plane.

9. The horizontal trace of a vertical plane makes 42° with the ground line. Determine the elevation of a line lying in this plane, inclined at 30°, and passing through the point where the given plane cuts the ground line.

10. Draw the traces of a vertical plane which makes an angle of 60° with the vertical plane of projection. Represent a line lying in this plane, and inclined at 70° to the vertical trace.

11. A person on the top of a tower 60 feet high, which rises from a horizontal plane, observes the angles of depression of two objects *A* and *B* on the plane to be 20° and 30°, the directions of *A* and *B* from the tower being west and south respectively. Find (*a*) the distances of *A* and *B* from the tower; (*b*) the distance apart of *A* and *B*; and (*c*) the direction of *B* and *A*.

12. Represent a plane inclined at 50°, and in it place two lines, one horizontal and the other inclined at 30°. Find the true angle between these lines.

13. The face of a hill is inclined at 30°, the lines of slope being due east; what is the inclination of a path on the hillside which goes in a north-easterly direction.

14. A plane inclined at 40° contains a point *A*, distant $1''$ and $1\frac{1}{2}''$ respectively from the horizontal and vertical planes of projection. Determine the rabatments of the point about the traces of the plane.

15. A line is inclined at 35° and 50° to the horizontal and vertical planes respectively, its traces are $4''$ apart. Determine its projections.

16. A line *ab* $2''$ long, making 30° with *xy*, is the plan of a horizontal line $1\frac{1}{4}''$ above the ground. Find its elevation, and determine the plan and elevation of an isosceles triangle, having the given line for base, and its vertex in *xy*.

17. Two points are respectively $1.5''$ and $0.75''$ from both planes of projection. Their plans are $2.25''$ apart. Determine a point on the ground line equidistant from the points.

18. A rectangle, sides $2''$ and $3''$, revolves upon one diagonal as a fixed horizontal line until the plan of a right angle opposite becomes 120°. What is then the inclination of the plane of the figure?

19. Two lines inclined to the horizontal plane at angles of 25° and 45° respectively are drawn from a point situated $2''$ from both planes of projection. The plans of these lines make 110° with each other. Determine the real angle between the lines.

20. Given a point $1.25''$ from the horizontal plane, and $1''$ from the vertical plane; obtain the projections of any line $4\frac{1}{2}''$ long passing through the given point and terminated by the planes of projection.

21. A line is inclined at 30° and 40° to the planes of projection; what is its inclination to a plane which is perpendicular to *xy*?

22. Draw the plan of a regular octahedron of $1\frac{1}{2}''$ edge when resting with one face on the ground. Determine the plan and true shape of the section made by a plane which is inclined at 45° and contains one diagonal of the solid.

***23.** Determine the inclination of the line *AB* (Fig. *a*) to each plane of projection.

***24.** Determine the horizontal traces, *H* and *K*, of *any* pair of lines which pass through *P* (Fig. *a*), and intersect *AB* in, say, *C* and *D*. Show in the same figure the true shapes of the triangle *PHK* and the quadrilateral *CDHK*.

***25.** From a given point *p*, *p'* (Fig. *a*), draw a perpendicular on the given line *ab*, *a'b'*. (1877)

Hint.—Proceed as in Ex. 24, and having obtained the true shapes $P_0H_0K_0$ and $C_0D_0H_0K_0$, draw a line from P_0 perpendicular to C_0D_0, meeting H_0K_0 in L_0. The projections of the required perpendicular may be at once obtained from L_0.

***26.** A line parallel to the vertical plane is given by its projections *ab*, *a'b'* (Fig. *b*). Draw the traces of a plane containing this line and perpendicular to the vertical plane. In this plane draw lines passing through *A* and *B* and making 45° with the line *AB*. Draw also an elevation of the three lines on a vertical plane perpendicular to the ground line. (1885)

***27.** *a'b'* (Fig. *c*) is the elevation of a line $2\frac{1}{2}''$ long. Its centre *C* is 2″ from *xy*. Determine (1) the plan of *C*, (2) the difference of the distances of *A* and *B* from the vertical plane of projection.

***28.** *a'b'* (Fig. *c*) is the elevation of a line $2\frac{1}{2}''$ long. Its centre point is 2″ from *xy*. Draw its plan. (1884)

***29.** a_0 (Fig. *d*) is the lowest corner of a cube; *ab* is the plan of one edge, *AB* (real length = 20 units), of a face which lies in a plane, of which *ht* is the horizontal trace. Draw the plan of the cube. Also the plan of its section by a plane parallel to the horizontal plane of projection, and at a height of 12 units above it. Unit = 0.1″. (1896)

***30.** Determine the height of *P*, a point on *CD*, whose figured plan is given (Fig. *e*). Unit = 0.05″.

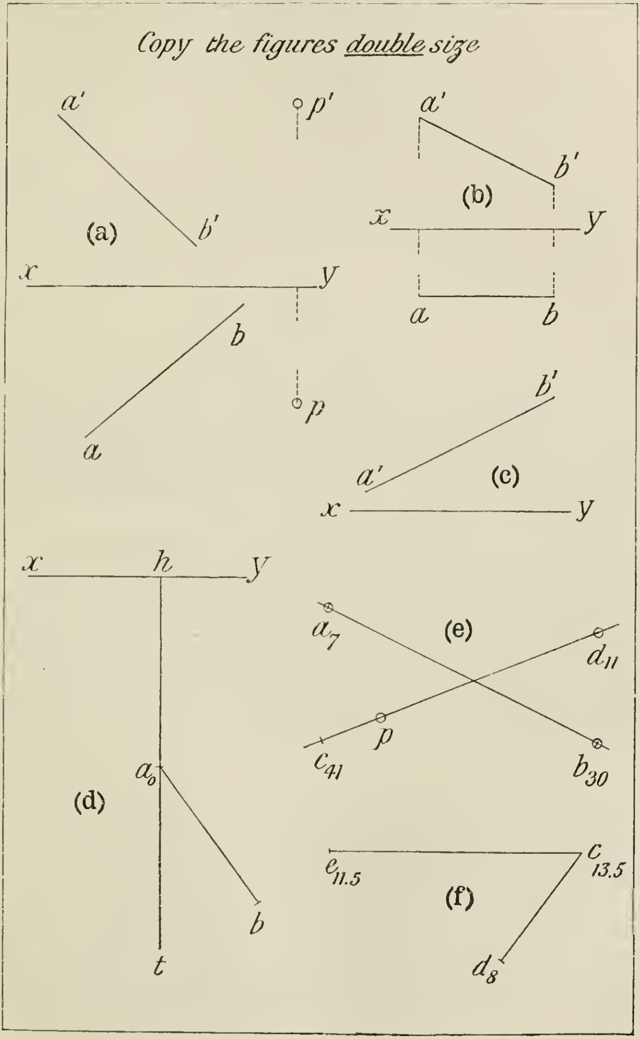
***31.** In Fig. (*e*) *p* is the centre of a sphere $2\frac{1}{8}''$ radius. Determine a sectional elevation of this sphere on a vertical plane whose horizontal trace is *ab*. Unit = 0.05″.

***32.** Two lines are given by their figured plans, Fig. (*e*). From the point *P* on one of them draw a line $2\frac{1}{8}''$ long, terminating on the other. Unit = 0.05″. (1886)

Hint.—Proceed as in Ex. 31, obtaining an elevation of *AB* on the vertical plane whose horizontal trace is *ab*.

***33.** Find the length of the line *CD* in Fig. (*f*), and determine the length of the diagonal of a cube of which *CD* is one edge. Unit = 0.1″.

***34.** Determine whether the triangle *CDE*, Fig. (*f*), is right-angled at *D*.



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CHAPTER X

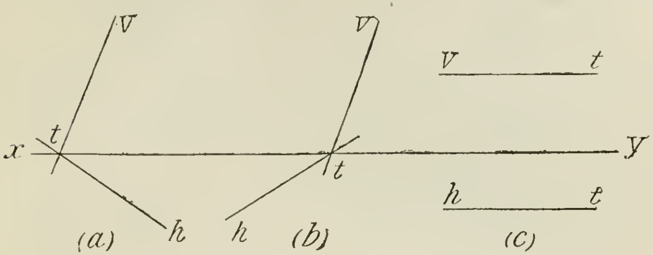
THE OBLIQUE PLANE

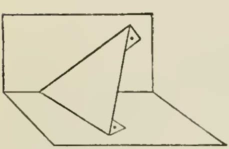
197. Introduction.—As defined in Art. 183, an oblique plane is one which is inclined to both planes of projection. The traces are both inclined to *xy*, or both parallel to it, or both coincide with it; and the true angle between the traces is not a right angle.

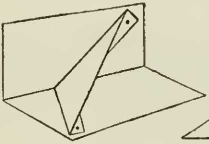
The figure shows four cases of an oblique plane. In (*a*) the real angle between the traces is acute; in (*b*) it is obtuse; in (*c*) the traces are parallel to one another; and in (*d*) the plane contains the ground line, and cannot be represented by its two traces in the ordinary way, since these both coincide with *xy*. It requires to be shown by a side elevation, on the plane perpendicular to *xy*, as indicated in the figure.

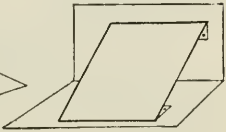
Let the student illustrate these cases by four models, made in the manner explained in Art. 183, and shown in the figure. The planes are attached to the planes of projection by paper-fasteners passing through folded margins; either margin can be unfastened to allow of the plane being rabatted about the other trace.

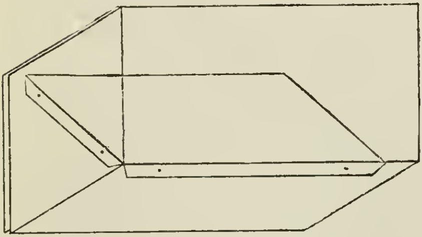
The student should note that the intersection of the horizontal and vertical traces of a plane is always on the ground line *xy*. When one trace is parallel to *xy*, so is the other, and the intersection of the three lines is at an infinite distance away in the direction of the ground line.

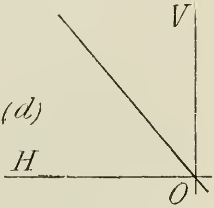












A characteristic feature which distinguishes an oblique plane from one which is perpendicular to one or both of the planes of projection, is that neither trace is an edge or profile view of the plane. On this account the problems are generally more difficult than those of Chap. IX. We shall begin with some cases which may be made to depend on the constructions of Chap. IX.

198. PROBLEM.—**To convert a given oblique plane into an inclined plane, by means of an auxiliary elevation.**

In the diagram VTH is the given oblique plane.

Take Mn, a new vertical plane of projection, at right angles to the given plane. Then $X'Y'$, the new ground line, is perpendicular to TH, the horizontal trace.

With reference to this new plane of projection the given plane is an inclined plane (Art. 183).

Let ON be the intersection of Mn and VTH. Then ON is the new vertical trace, and is also an *edge elevation* of the plane VTH.

The above process has now to be represented by orthographic projection.

Let vth be the given traces of the plane.

Take a new ground line $x'y'$ perpendicular to th, intersecting th in o and xy in n. Draw nn', nn'' perpendicular to xy, $x'y'$, and make $nn'' = nn'$. Thus n'' is the auxiliary elevation on $x'y'$ of the point N. And on'' is the new vertical trace required.

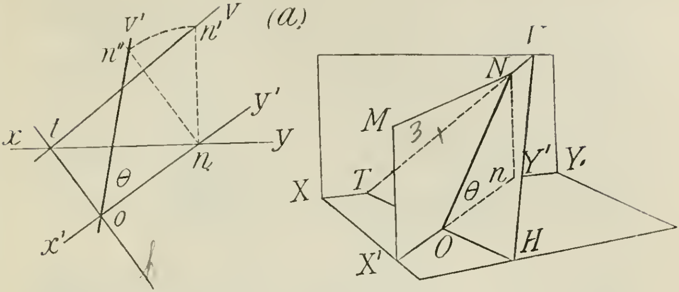
The student should make a special model to illustrate this problem, by cutting a model of the oblique plane along a line NO perpendicular to TH, and inserting the new plane of projection Mn.

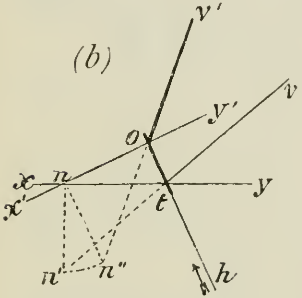
On holding the model so as to look along HT, it is seen that $X'Y'$ is the *new elevation of the ground*, and NO, the *new vertical trace*, is the *edge elevation of the plane*.

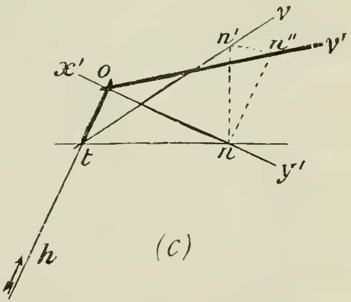
The lines nn', nn'' are the positions taken by nN, the line of intersection of the two vertical planes of projection, when these planes are turned back about their respective ground lines into the horizontal plane.

It should be observed that the angle $v'on$ is the true inclination, θ, of the plane to the ground.

Figs. (b) and (c) show two other examples of this problem. In Fig. (b) the plane vth is the same as in Fig. (a), but $x'y'$ is drawn in a different position, to avoid the overlapping of the two elevations. Fig. (c) shows what form the construction takes when the real angle between the traces is obtuse.







By means of the construction just given, many problems on the oblique plane may be at once converted into corresponding problems on the inclined plane, and thereby simplified. This method of attacking such examples is a most valuable one, on account of its wide applicability, and should become quite familiar to the student. The six problems which immediately follow are worked in this manner.

Example.—Convert the following oblique planes into inclined planes by means of auxiliary elevations, and in each case measure the inclination to the horizontal plane :—

(*a*) The horizontal and vertical traces *ht*, *tv* make angles of 40° and 55° with *xy*.

(*b*) The traces make 120° and 35° with *xy*.

(*c*) The traces are in one straight line making 45° with *xy*.

Ans. (*a*) 65.8° ; (*b*) 39° ; (*c*) 54.7°.

199. PROBLEM.—**To determine the point of intersection of a given straight line AB, and a given plane VTH.**

Take $x'y'$ perpendicular to th, and on $x'y'$ draw elevations of the plane and line as shown.

The problem is thus reduced to that of Prob. 187, c'' being the auxiliary elevation of the point of intersection of the line and plane.

The plan c is obtained by projecting from c''; and the elevation c' by projecting from c.

200. PROBLEM.—**To determine the plan and elevation of the section of a given prism by a given oblique plane VTH. Also to draw a sectional plan of the solid.**

By drawing an auxiliary elevation, this problem is converted into that of finding the section of a solid by an inclined plane, and is then worked as in Prob. 184.

Take $x'y'$ perpendicular to th, and on $x'y'$ draw the auxiliary elevations of the plane and prism as shown.

Then ov' is an edge elevation of the plane, and a'', b'', c'' are the auxiliary elevations of the points where the plane cuts the three vertical edges of the prism.

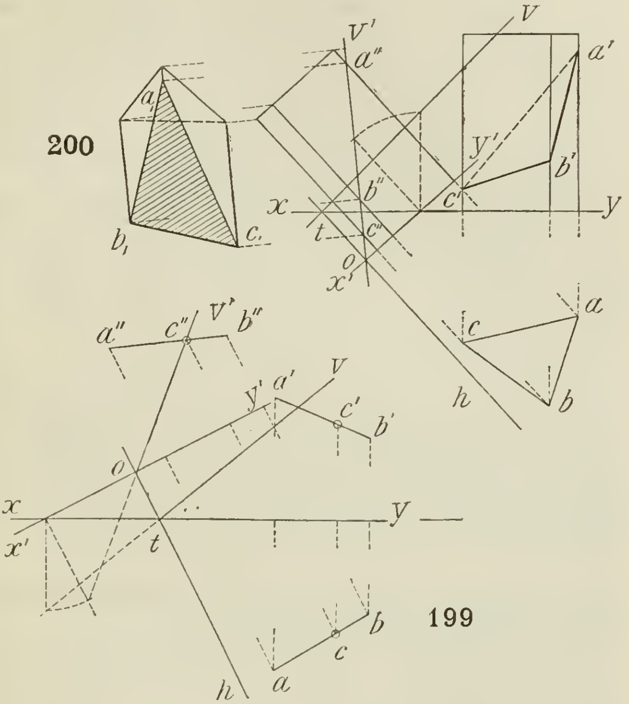
The heights of a', b', c' above xy are now made equal to the heights of a'', b'', c'' above $x'y'$, thus determining the elevation of the section. The plan of the section is abc, coinciding with the plan of the prism.

The required sectional plan is a projection of one portion of the solid on the cutting plane, or on a plane parallel to it.

Take ov' as a second auxiliary ground line, and from the auxiliary elevation project the plan $a_1b_1c_1$, making the distances of a_1, b_1, c_1 from ov' the same as the distances of a, b, c from $x'y'$, according to the rules of Art. 170.

The sectional plan is completed by drawing the plans of the other lines of the frustum, obtained in a similar manner.

Compare this plan with the plan of the frustum of the pyramid in Prob. 184. The two are obtained by the application of the same principle.



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Examples.—1. Draw the traces of the three planes of the example on page 233. Draw the projections of a line parallel to *xy* and $1\frac{1}{2}''$ from each of the planes of projection. Determine the plan and elevation of the intersection of this line with each of the three planes.

2. Draw the plan and elevation of a square pyramid, side of base $2''$, height $3''$, the base being on the ground with its centre $1\frac{3}{4}''$ from *xy*, and one edge making 35° with *xy*. Select a point on *xy*, $3\frac{3}{4}''$ to the left of the plan of the centre of the base. Through this point draw horizontal and vertical traces making 45° and 38° respectively with *xy*. Determine the plan and elevation of the section of the pyramid made by this plane. Draw also a *sectional plan* of the pyramid.

201. Problem.—**To determine the perpendicular from a given point A to a given oblique plane VTH.**

Convert the problem into that of Prob. 188 by drawing ov' and a'', the auxiliary elevations of the plane and point on $x'y'$, taken perpendicular to th.

Through a'' draw $a''\,m''$ perpendicular to ov'; then $a''\,m''$ is the auxiliary elevation of the perpendicular.

Through m'' draw the projector $m''m$ to intersect in m the line through a perpendicular to th.

The required plan, am, is thus determined; and the elevation on xy is at once found by projection from the plan, since the height of M is known.

It may be proved by pure solid geometry that

Theorem.—If a line and plane be perpendicular to each other, the plan and elevation of the line are respectively perpendicular to the horizontal and vertical traces of the plane.

The student may satisfy himself of the truth of this proposition by using a model. The accuracy of the above solution should be tested by ascertaining whether $a'm'$ is perpendicular to tv, as ought to be the case.

202. Problem.—**To determine a plane which shall bisect a given straight line AB at right angles.**

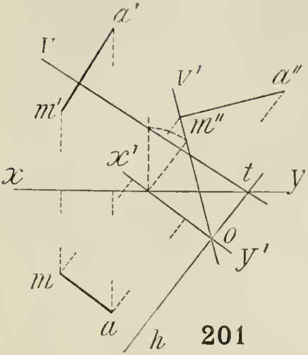
Take $x'y'$ parallel to the given plan ab, and draw the auxiliary elevation $a''b''$. Draw $p''o$ bisecting $a''b''$ at right angles, and meeting $x'y'$ in o. Then $p''o$ is an edge elevation on $x'y'$, of the required plane.

The horizontal trace of the plane is ot, drawn through o perpendicular to ab; and the vertical trace is tv, drawn through t perpendicular to $a'b'$.

203. Problem.—**To determine the true distance between two given parallel oblique planes.** (No figure.)

If the planes are parallel, their horizontal and vertical traces are also respectively parallel (*Theorem* 13, *Appendix*).

The required distance between the planes is that between their edge elevations, obtained as in Prob. 198.



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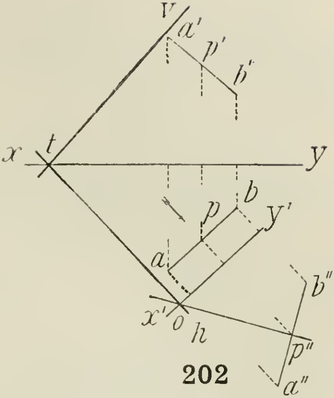


202

Examples.—**1.** Represent an oblique plane, and a line perpendicular to it. Show also an oblique plane and a line lying in it.

2. Draw the traces of the three planes of the example on page 233, and in each case draw the projections of a point *A*, situated 1″ to the right of *t*, $2\frac{1}{2}''$ above the ground, and 2″ in front of the vertical plane. Draw the projections of the perpendicular from *A* on to each plane.

3. The horizontal and vertical traces of a plane make angles of 30° and 50° respectively with *xy*. Take a point *O* in *xy*, 2″ from *t*, and determine the plan and elevation of the line *OM*, drawn at right angles to meet the plane. Determine the angle which *OM* makes with *xy*. *Ans.* 62.3°.

4. A line *AB* 3″ long has its ends $1\frac{3}{4}''$ and $2\frac{1}{2}''$ from each of the planes of projection. Determine a plane which shall bisect *AB* at right angles.

5. Draw the traces of any oblique plane. Take any two points *a* and *b* in the horizontal trace, and any point *c'* in the vertical trace. Find the true shape of the triangle *ABC*.

6. Draw the traces of any oblique plane and those of a plane parallel to it, such that the perpendicular distance between the horizontal traces is 1″. Determine the true distance between the planes.

7. The horizontal traces of two parallel planes are $1\frac{1}{2}''$ apart and make 45° with *xy*; the vertical traces are $\frac{5}{8}''$ apart, and the angle between the traces is obtuse. Find the distance between the planes. *Ans.* 6″.

204. PROBLEM.—**Having given an oblique plane VTH, and the plan of a point A in the plane, to determine the projections of a straight line through A which shall lie in the plane, and make a given angle δ with the horizontal trace.**

Determine ov', the edge elevation of the plane.

Through a draw the projector to meet ov' in a''. Then a'' is the auxiliary elevation of the point A.

Obtain A_0, the rabatment of A about oh, as in Prob. 186. Through A_0 draw A_0b, making the given angle δ with the horizontal trace oh. Then A_0b is the rabatment of the required line.

When the plane goes back into its original position, the point B does not move, being in the axis of rotation. The plan of the required line is therefore ab, and the elevation $a'b'$ can be drawn since the heights of A and B are known.

205. PROBLEM.—**Having given an oblique plane VTH, and a point A in it, to determine the projections of a straight line AB, which shall lie in the plane, and be inclined at a given angle α to the ground.**

The line is first drawn through A parallel to the vertical plane; inclined at α; its lower end on the ground. In this position its projections are $a'B'_1$ and aB_1.

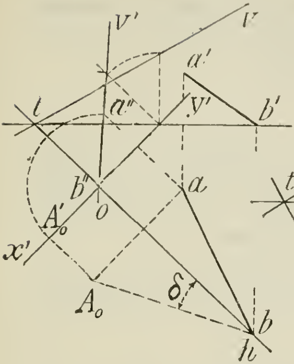
The line is then turned about Aa until B comes into the horizontal trace of the plane. The line is then altogether in the plane; it is still inclined at α; and it passes through A.

The required projections are ab, $a'b'$, the manner of obtaining which is obvious. Compare Prob. 191.

206. PROBLEM.—**To determine a straight line which shall pass through a given point P, have a given inclination α, and be parallel to a given oblique plane VTH.**

First, by Prob. 205, draw *any* line AB *in* the plane, having the given inclination α. Then draw a line through P parallel to AB.

The plan of the line is drawn through p, parallel to ab, and the elevation through p' parallel to $a'b'$.



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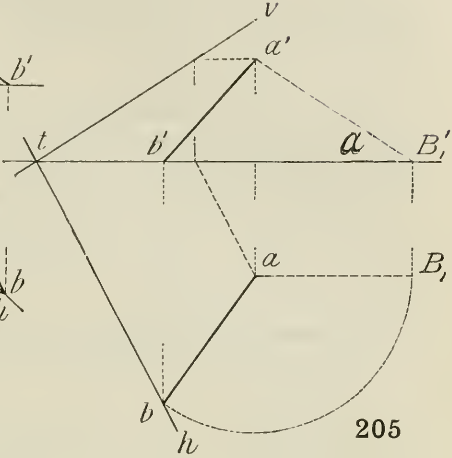

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Examples.—1. The traces vt, th of a plane make angles of 45° and 30° with xy. Select a point a 2″ to the right of t and 1″ below xy; this is the plan of a point A in the plane. Draw the plan and elevation of a line AB $2\frac{1}{2}''$ long, which lies in the plane and makes 30° with the horizontal trace.

Determine also the projections of a line AC 3″ long, lying in the plane, the angle BAC being 50°.

Hint.—Having obtained A_0C_0, produce it to meet the horizontal trace in i. To find the plan c, join ia and produce it to meet in c a line from C_0 perpendicular to ht. If the point i obtained as above be not within the limits of the paper, draw some other line from C_0 to meet the horizontal trace in a more convenient point i, and A_0B_0 in D_0, from D_0 obtain d, then c will lie on id produced.

2. The traces of a plane each make 50° with xy. A point A in the plane is 1″ from the vertical plane, and 2″ above the ground. Determine the projections of a line AB lying in the plane, and inclined at 35° (a) to the horizontal plane, (b) to the vertical plane. What is the greatest inclination which AB may have either to the horizontal or vertical plane?

3. Determine a plane which passes through the point A, Ex. 2, and is parallel to the given plane.

207. PROBLEM.—**Having given the plan of any polygon or plane figure lying in a given oblique plane, to find the elevation and true shape of the figure.**

Draw an edge elevation of the plane as in Prob. 198, then proceed as in Prob. 186.

208. PROBLEM.—**Having given a straight line and an oblique plane, to determine (a) the trace of the line on the plane; (b) the projection of the line on the plane; and (c) the angle between the line and plane.**

Draw an edge elevation of the plane, and a new elevation of the point on the same ground line. The problem is thus reduced to Prob. 190, and is worked in the manner there described.

209. Examples on Problems 198 to 208.

***1.** Draw an edge elevation of the plane, Fig. (*a*), on a ground line taken through *b*. Draw also an edge plan on x_1y_1, taken through a'.

***2.** Fig. (*b*) shows the plan of a prism with equilateral ends, resting on the ground. Draw the elevation, and show the section by the given plane. Draw also a sectional projection of the prism.

***3.** Find the intersection of the line and plane in Fig. (*e*). Also in Fig. (*f*).

***4.** In Fig. (*a*) determine a perpendicular from A to the plane VTH.

***5.** Determine a plane bisecting the line AB, Fig. (*e*), at right angles.

***6.** Find the distance between the parallel planes of Fig. (*d*). Show the projections of any line perpendicular to the planes with one end in each plane.

***7.** In Fig. (*a*) the point B lies in the plane; find its elevation. Show a line containing B, lying in the plane, and making 70° with the trace *ht*.

***8.** Determine a line through B, Fig. (*a*), lying in the plane and inclined at 30° to the ground.

***9.** In Fig. (*a*) determine a line containing B, lying in the plane and making 50° with *vt*. Also one inclined at 50° to the vertical plane.

***10.** In Fig. (*c*) the triangle whose plan *abc* is given lies in the given plane. Determine the elevation and true shape of ABC.

***11.** In Fig. (*e*) determine the trace of the given line on the given plane. Show the projection of the line on the plane. Find the angle between the line and plane.

***12.** Obtain the results of Ex. 11 with reference to Fig. (*f*).

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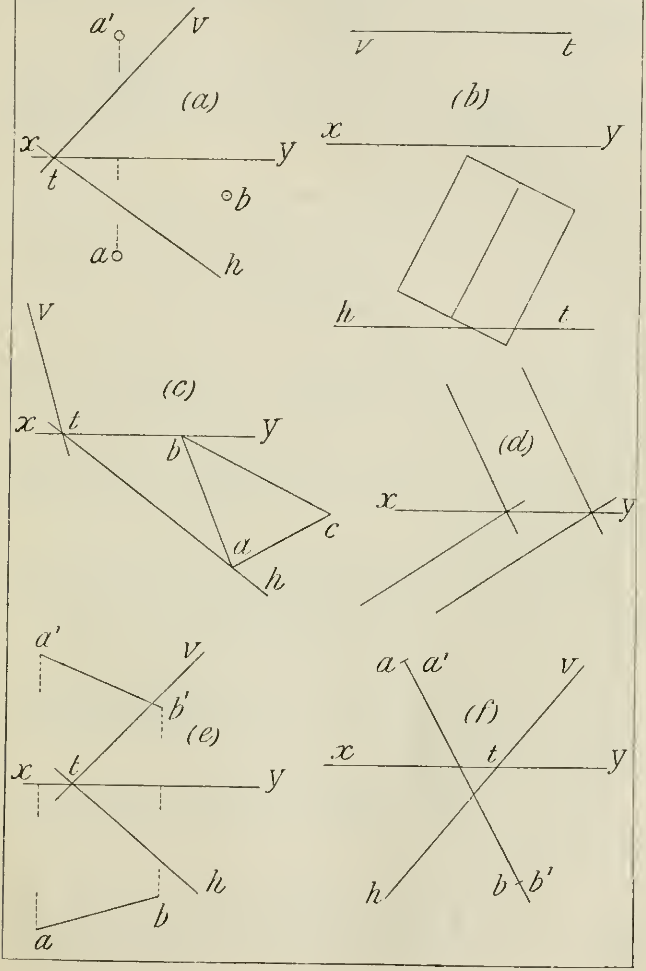


210. PROBLEM.—**Having given an oblique plane VTH, and one projection of a point A which lies in the plane, to determine the other projection.**

The method of solution is to first find the projections of any line drawn through A and lying in the plane. Then to draw a projector from the given plan to intersect the elevation of the line in a'.

(1) *Let the plan a be given.*

In Fig. (a), through a draw bc, the plan of any line which has its lower and upper ends, B and C, respectively in the horizontal and vertical traces of the plane; the elevation of the line is evidently $b'c'$, B and C being respectively in the horizontal and vertical planes of projection. The projector from a, intersecting $b'c'$ in a', gives a' the required elevation.

In Fig. (b) the line through A is horizontal, so that its plan ac is parallel to th, and its elevation $a'c'$ parallel to xy.

In Fig. (c) the line is taken parallel to the vertical plane; therefore its plan ab is parallel to xy, and its elevation $a'b'$ parallel to tv.

(2) *Let the elevation a′ be given.*

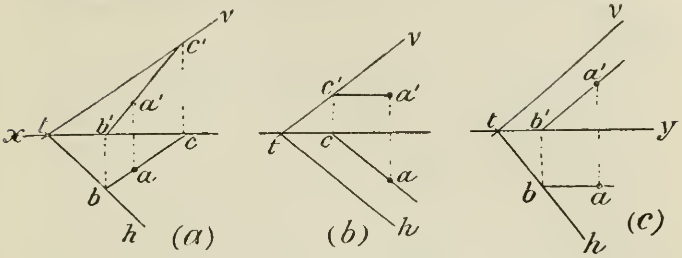
The elevation of the line through A is first drawn; the plan of the line is then determined; and finally the required plan a is found by projecting from a'.

211. PROBLEM.—**Having given the plan and elevation of a point A, and one trace of a plane which contains A, to find the other trace.**

Let a, a', and th be given, Fig. (a).

Draw bc through a; project b'; through c draw the projector to intersect $b'a'$ produced in c'. Then tc' is the required vertical trace.

If the given trace be nearly parallel to xy, so that the point t is not readily accessible, two lines such as BC must be drawn through A, thus determining two points in the other trace. See Fig. 215.



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Examples.—1. Represent a plane by its traces; select a point *a* anywhere except in *xy*. Draw the plan of any line whatever which lies in the plane and passes through *A*. Find the elevation of this line, and by means of it find the elevation of *A*. Find the inclination of the line through *A* which you have represented.

2. Draw a line making 40° with *xy*, and also the projections of any point whatever. Determine the vertical trace of the plane which contains the point, and has the first line for its horizontal trace.

3. Draw a line inclined at 30° to *xy*, and draw the projections of any point not in either of the planes of projection. Determine the horizontal trace of the plane which contains the point and has the first line for its vertical trace.

4. Draw the projections of any point. Take a point $1\frac{1}{2}''$ below *xy*, and through it draw a line inclined at 5° to *xy*. Let this be the horizontal trace of a plane containing the first point. Determine the vertical trace.

Hint.—Draw the plans of *any* two lines through the given point. Regard these as lying in the plane; draw their elevations and their vertical traces, then join the latter. See Fig. 215.

We advise the student to remember this useful construction.

5. A point *A* is $1''$ above the ground and $1.8''$ behind the vertical plane. A second point *B* is $1.5''$ below the ground and $0.5''$ in front of the vertical plane. Determine a plane which contains *A* and *B* and is parallel to the ground line.

6. Represent a plane which contains the ground line and also the point *A* of Ex. 5. Find the angle between this plane and the plane of Ex. 5.

212. PROBLEM.—**To determine a plane which shall pass through a given point A and be parallel to a given plane VTH.**

The student may easily work this problem by drawing an edge elevation. The following is the special method :—

Through *a* draw *ac* parallel to *th*. This is the plan of a horizontal line in the required plane, and *a′c′*, the elevation of the line, is drawn as shown.

A point *C* in the vertical trace of the required plane is thus found. Through *c′* draw *lm* parallel to *tv*, and through *m* draw *mn* parallel to *th*. The required plane is *LMN*.

213. PROBLEM.—**To find the rabatments of a given point A which lies in a given oblique plane VTH.**

First, to find A_0, the rabatment of *A* into the horizontal plane, about *th*.

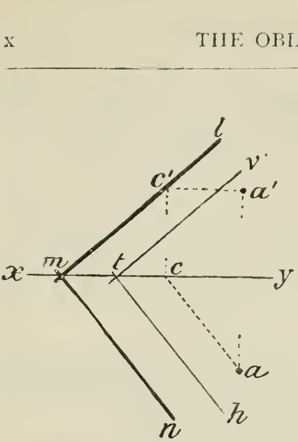
Draw *am* perpendicular to *th*. Then *am* is the plan of a line in the plane, at right angles to the horizontal trace. It is also, when produced, the plan of the circular path traced by *A* during rabatment; so that A_0 must lie on this line.

Obtain *a″am*, the rabatment of the right-angled triangle *MAa* about *am*; observe that $aa'' = la'$. Make $mA_0 = ma''$, the true length of *MA*. Then A_0 is the required rabatment of *A* into the horizontal plane.

It will be noticed that the arc $a''A_0$ is the rabatment about aA_0 of the path traced by *A*. As before stated, aA_0 is the plan of the path.

Next, to find A_1, the rabatment of *A* into the vertical plane, about *tv*. The construction is similar to that just described; *a′n′* is the elevation, and a_1n' the rabatment into the vertical plane, of a line *AN* lying in the plane at right angles to the vertical trace; $n'A_1$ is the rabatment of the same line about *tv*; and A_1 is the required rabatment of the point *A* into the vertical plane.

A model to illustrate this important problem may be made in the following manner :—



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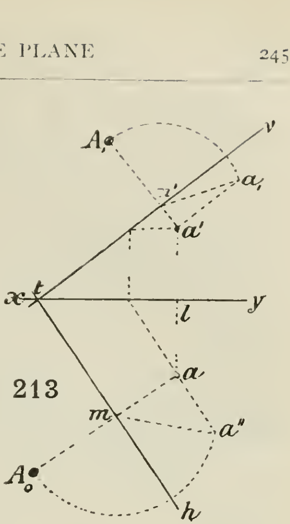


213

Take a model of the oblique plane hinged along *th*, and on it draw the line *AM* perpendicular to *th*. Cut out in paper a right-angled triangle of the shape $a''ma$, with a margin along *am* for attachment to the horizontal plane. The two rabatments about *am* and *th* respectively may then be effected.

This model may be used to illustrate the rabatment of a point into the vertical plane, by turning it so that the horizontal plane becomes the vertical plane, and *vice versa*.

Examples.—1. Draw the traces of any plane *VTH*, and also the projections of a point *A* $1\frac{1}{2}''$ vertically over *ht* and $2''$ from the vertical plane. Determine a plane through *A* parallel to *VTH*.

2. The traces *vt*, *th* of a plane make 50° and 60° with *xy*. Determine the rabatment, into the horizontal plane, of a point *A* in *vt* and $2''$ from *t*.

3. Taking the plane of Ex. 2, determine the rabatment, into the vertical plane, of a point *A* in *ht* $2''$ from *t*.

4. Determine the projections of a point *A* in the plane of Ex. 2. $2''$ from the vertical plane and $1\frac{1}{2}''$ from the horizontal plane. Determine the two rabatments of *A*.

5. A plane at right angles to *xy* contains a point *A* which is $1''$ above the ground and $2''$ from *xy*. Represent the plane and point and determine the two rabatments of the latter.

6. A plane contains *xy* and the point *A* of Ex. 5; set out the two rabatments of *A*.

214. Problem.—**To determine ω, the true angle between the traces of a given oblique plane VTH.**

Take a, a', the projections of any point A in the vertical trace of the given plane.

Obtain A_0, the rabatment of A about th, by drawing aA_0 perpendicular to th, and making $tA_0 = ta'$.

Then tA_0 is the rabatment of the vertical trace into the ground, and the true angle between the traces is that marked ω. Two cases, (*a*) and (*b*), are shown.

215. Problem.—**Having given two intersecting straight lines, to determine the plane containing them; also, to find the true angle between the lines.**

In order that the lines shall intersect, the apparent intersections in plan and elevation must lie on the same projector.

The horizontal and vertical traces of the plane must pass through the corresponding traces of the line.

Let the given lines be as shown in Fig. 215.

As in Prob. 177, determine H, K, the horizontal traces, and V, S, the vertical traces of the given lines.

Then, lines drawn through h, k and v', s' are respectively the horizontal and vertical traces of the required plane.

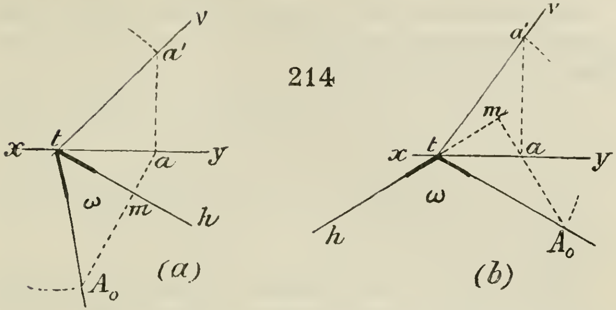
To find the true angle between the lines, obtain A_0, the rabatment of A, as in Prob. 213. Join A_0h, A_0k. Then hA_0k is the rabatment of the triangle HAK, and the angle hA_0k is the true angle between the lines.

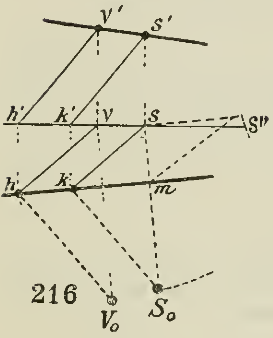
216. Problem.—**Having given two parallel straight lines, to determine the plane containing them; also, to find the true distance between the lines.**

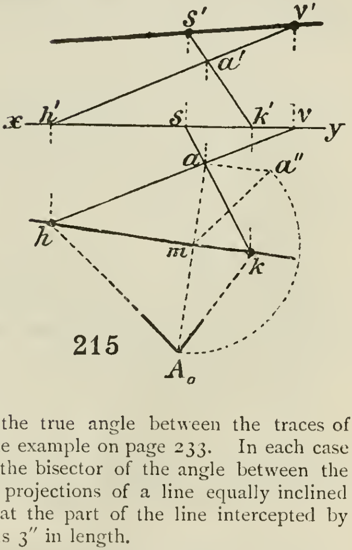
Let the given parallel lines be as shown in Fig. 216. Their plans and elevations are respectively parallel.

Determine the traces of the lines, through which draw the traces of the required plane, as in the last problem.

The true distance between the lines is that between their rabatments, which are shown dotted in the figure, and are obtained as in the last problem.







Examples.—1. Determine the true angle between the traces of each of the planes of the example on page 233. In each case draw the projections of the bisector of the angle between the traces. Also, draw the projections of a line equally inclined to the traces and such that the part of the line intercepted by the planes of projection is 3″ in length.

2. Draw the projections of any two lines *AB*, *AC*. Find (*a*) the traces of the plane containing the lines, (*b*) the inclination of this plane, (*c*) the angle *BAC*, (*d*) the projections of the bisector of the angle *BAC*, (*e*) the projections of the perpendicular from *A* on to *BC*.

3. Draw the projections of any two parallel lines. Determine the plane containing them, and the distance apart of the lines.

217. Problem.—**To determine a plane which shall contain three given points, A, B, C; also, to find the true shape of the triangle ABC.**

Draw lines through any two pairs of the given points, and then find the plane containing these lines, as in Prob. 215.

To find the true shape of the triangle ABC, obtain the rabatments of A, B, and C by the method of Prob. 213, or work by an auxiliary elevation, as in Prob. 207.

218. Problem.—**To determine a plane which shall contain a given straight line and a given point; also, to find the true distance of the point from the line.**

Take any point in the given line and join it to the given point. Then the plane through the two intersecting lines, found as in Prob. 215, is the one required.

To find the true distance between the point and the line, rabat the plane containing them, and draw the perpendicular from the rabatment of the point to the rabatment of the line.

219. Problem.—**To determine a line which shall lie in the plane of two given intersecting straight lines, and shall bisect the angle between them.**

Draw the rabatment of the lines as in Prob. 215.

Then through A_0 (Fig. 215) draw a line bisecting the angle hA_0k, and intersecting hk in p (not shown).

Thus we obtain A_0p, the rabatment of the required bisector, and ap its plan; and since P is on the ground the elevation of AP is known.

220. Problem.—**To determine the angle between two non-intersecting straight lines.**

Through any point in one of the lines draw a line parallel to the other.

The angle between these two intersecting lines, found by Prob. 215, is the angle required (*Def.* 10, *Appendix*).

221. PROBLEM.—**To find the angle between two given planes.**

Take any point *A*, and through *A* draw two lines respectively perpendicular to the two planes, in accordance with the theorem stated in Prob. 201.

Find the angle between these two lines by Prob. 215. Then this angle, subtracted from 180°, will give the angle between the planes.

This solution is based on a proposition of pure solid geometry, which states that the angle between two planes is equal to the supplement of the angle between any two lines respectively perpendicular to the planes. Refer to Prob. 227 for another method of solution.

222. PROBLEM.—**To find the angle between a given straight line and a given oblique plane.**

One method of determining this angle has already been given in Prob. 208. The following is an alternative solution :—

Take any point *A* in the given line, and through *A* draw a line perpendicular to the given plane. Find the angle between these two lines by Prob. 215. Then this angle, subtracted from 90°, gives the angle required.

This solution is based on a proposition of pure solid geometry, which states that the angle between a line and a plane is equal to the complement of the angle between the line and a line perpendicular to the plane.

223. PROBLEM.—**To determine a plane which shall contain a given straight line, and be perpendicular to a given plane.**

Take any point *A* in the given line, and through *A* draw a line perpendicular to the given plane (Art. 201). Then the plane through the two intersecting lines, determined as in Prob. 215, is the one required.

This construction is based on Theorem 6 of the Appendix, and is often required.

224. Problem.—**To determine the plan and elevation of the line of intersection of two given planes.**

Let *VTH* and *LMN* be the given planes; four cases are shown in the figure.

In each case let the horizontal traces intersect at r, and the vertical traces at s', so that *RS* is that part of the intersection intercepted by the planes of projection.

The elevation of R is r', and the plan of S is s, r' and s being in xy. Hence rs is the plan, and $r's'$ the elevation of the required intersection.

In (*b*) the vertical traces intersect below xy, and *RS* is between the planes forming the fourth dihedral angle. In (*d*) the horizontal traces are parallel to each other, and *RS* is a horizontal line parallel to either trace.

225. Problem.—**To determine the plan and elevation of the point of intersection of three given planes.**

The point required is that in which the line of intersection of any two of the planes intersects the third plane. Or it may be determined as follows:—

Find *RS* and *PQ*, the lines of intersection of any two of the three pairs of planes. These lines, produced if necessary, will intersect in the required point.

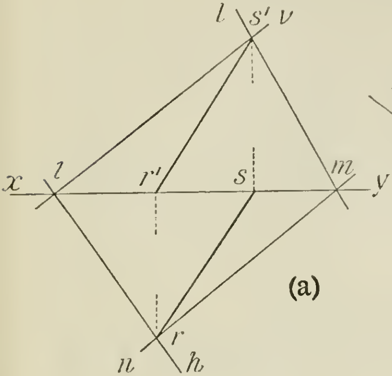
The accuracy of the work may be tested by observing whether the apparent intersections, in plan and elevation, lie on the same projector.

Examples.—**1.** Take two points t, m in xy 3″ apart. Above xy construct the triangle tlm, where tl is $3\frac{1}{4}$″, and lm is 1.8″. Below xy construct the triangle tnm, making tn equal 1.3″, and nm $2\frac{3}{4}$″. Determine the projections of *LN*, the line of intersection of the planes *LTN* and *LMN*.

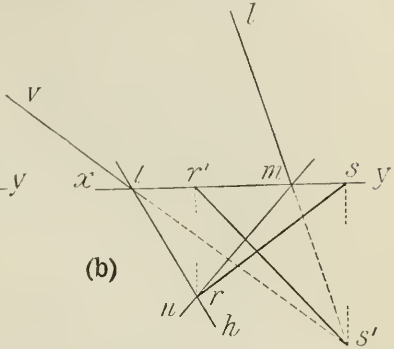
Find the inclinations of *LN* to the planes of projection.

2. Two vertical planes meet xy at points 2″ apart, and the planes make angles of 55° and 40° respectively with xy. Draw the plan and elevation of their intersection.

3. An inclined plane makes 45° with the ground. An oblique plane is parallel to xy and is inclined at 50° to the ground. Determine the projections of the intersection of these two planes.



(a)



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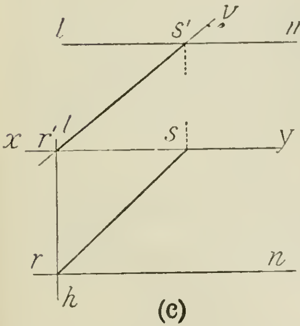

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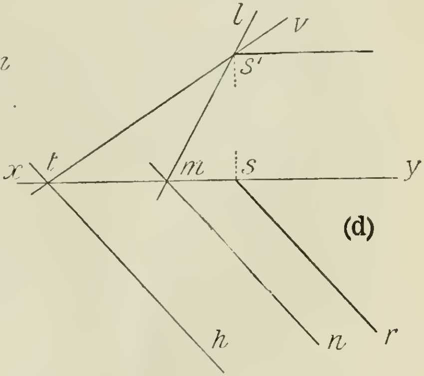

(d)

4. Take any point in the intersection of Ex. 3 and draw the projections of a line passing through this point, lying in the oblique plane, and making an angle of 55° with the intersection.
   Hint.—Rabat the oblique plane about its horizontal trace.
5. Draw the projections of a line *AB* inclined at 35° and 50° respectively to the horizontal and vertical planes. Draw the traces of any pair of planes which intersect in *AB*.
6. Draw the traces of any three planes and determine the projections of their point of intersection.

226. Examples on Problems 210 to 225.

***1.** In Fig. (*a*) the plan of a point *A* lying in the plane *VTH* is given; find the elevation of the point.

***2.** In Fig. (*b*) the projections of a point and the horizontal trace of a plane are given. Find the vertical trace.

***3.** Find the traces of the three planes which pass through *A*, Fig. (*e*), and are parallel to the three given planes.

***4.** In Fig. (*d*) the point whose elevation is given lies in the given plane; draw its plan and obtain the two rabatments of the point. Work the corresponding problems having reference to Figs. (*a*) and (*b*).

***5.** Find the angles between the traces of each of the three planes of Fig. (*e*).

***6.** Find the traces of the plane determined by the two intersecting lines, Fig. (*c*). Find the angle *BAC*.

***7.** Find the positions of *B* and *C* on the given lines, Fig. (*c*), so that *ABC* shall be an isosceles triangle having $AB = AC = 2.5''$.

***8.** Determine the projections of the bisector of the angle *BAC* in Fig. (*c*).

***9.** In Fig. (*e*) find each of the angles which *AB* makes with *xy*, with the trace *LM*, and with the trace *MN*.

***10.** Find the angles between the planes *LMN* and *PQN* in Fig. (*e*). Find also the angles between *LMN* and *VTHT*, and between *PQN* and *VTHT*.

***11.** Find the angle between the line *AB* and the plane *LMN* in Fig. (*e*). Also between *AB* and *PQN*. And between *AB* and *VTHT*.

***12.** Determine the three planes which contain the line *AB*, Fig. (*e*), and are respectively perpendicular to the planes *LMN*, *PQN*, *VTHT*.

***13.** In Fig. (*e*) determine the line of intersection of the planes *LMN* and *PQN*. Also of the planes *LMN*, *VTHT*. And of the planes *PQN*, *VTHT*.

***14.** Find the point common to the three planes of Fig. (*e*).

15. The traces *ht* and *vt* of a plane make 55° and 45° with *xy*. A point *A* is in the plane and $1\frac{1}{2}''$ above the ground. Is this information sufficient to define the position of *A*? If not, what is the locus of *A*? Draw the projections of the locus of *A*.

16. In Ex. 15 determine the positions of *A*, if in addition to the data there given, you are told (1) that the point is $1\frac{1}{2}''$ from the vertical plane, or (2) $2''$ from *xy*, or (3) that the point is $3''$ from the point *t* where the plane cuts the ground line.

17. Determine the intersections of the three planes of Fig. (*e*) with plane which contains *xy* and the point *A*.

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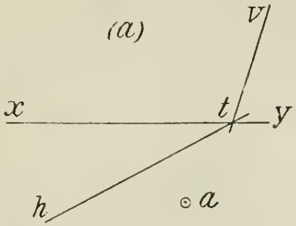
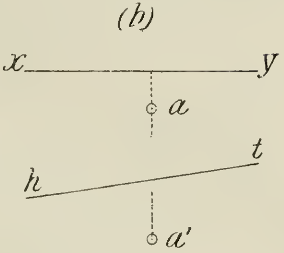
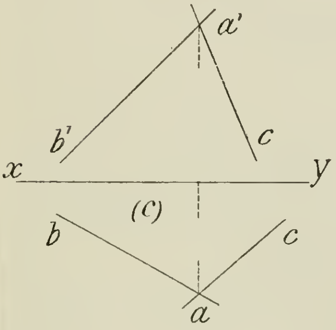
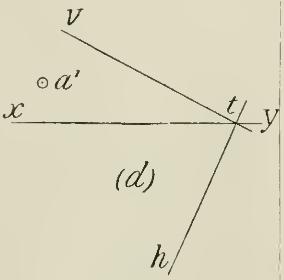
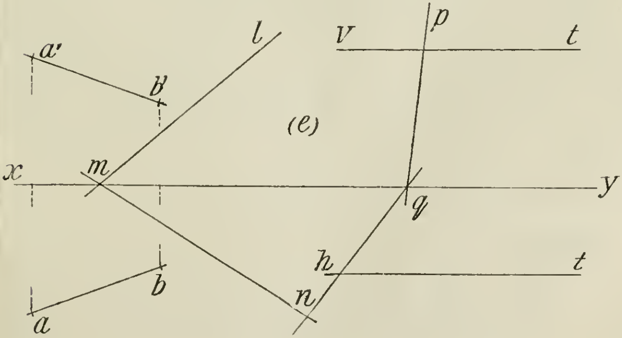










227. PROBLEM.—**A given irregular triangular pyramid rests with one face on the ground; to determine its six dihedral angles, and the true shapes of its faces.**

Let *abcd* be the given plan of the pyramid, the face *ABC* being on the ground, and the height of *D* being as indexed.

First to find the dihedral angle between the faces which intersect in *AD*.

Take *xy* parallel to *ad*, and on *xy* draw $a'd'$, the elevation of *AD*. Draw any line *vt* perpendicular to $a'd'$, and *th* perpendicular to *ad*; these lines are the traces of a plane perpendicular to *AD*. This plane intersects *AD* in *P*, and the two faces *ADB*, *ADC* in *PR*, *PS*, which two lines are perpendicular to *AD* (*Theorem* 16, *Appendix*), and therefore contain an angle which measures the angle between the faces in question (*Definition* 9, *Appendix*).

The true angle between these lines is the angle rP_0s, obtained by a rabatment of the plane *VTH* about *th*.

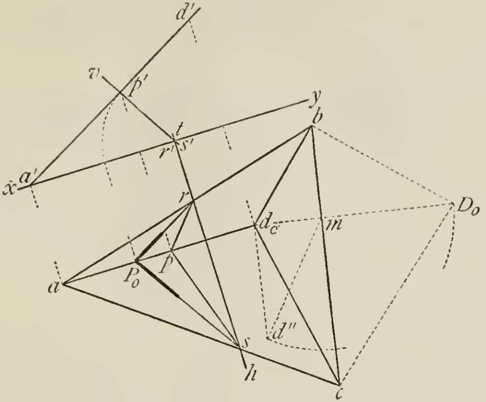
The angles between the other two pairs of sloping faces may be found in a similar manner.

Next to find the angle between the faces which intersect in *BC*, and the true shape of the face *DBC*.

Rabat *DBC* into the ground about *bc*. The construction for this is shown in the figure; it is the same as that explained in Prob. 186. The angle between the faces which intersect in *BC* is measured by the angle d_0md'', and the true shape of the face *DBC* is the triangle D_0bc.

The remaining dihedral angles, and the true shapes of the remaining faces, may be similarly found.

Example.—Draw a triangle *ABC*, making *AB* $3\frac{1}{2}''$, *AC* $2\frac{3}{4}''$, and *CB* $3\frac{1}{4}''$; take a point *D* in the triangle $1\frac{3}{8}''$ from *A* and $1\frac{3}{4}''$ from *C*. Join *AD*, *BD*, *CD*. This is the plan of an irregular pyramid with the face *ABC* on the ground, and the vertex *D* $1\frac{3}{8}''$ above the ground; determine the six dihedral angles and the true shapes of its faces.



228. Some tangential properties of cones.

Place a cone with its base on a horizontal plane, and let a cardboard model of a plane, having one edge on the horizontal plane, rest against, or be tangential to, the surface of the cone. It will be observed:

(1) *That the tangent plane (if sufficiently extended) contains the vertex of the cone;* (2) *that the inclination of the plane is equal to the base angle of the cone; and* (3) *that the trace of the plane touches the circular base, or trace, of the cone.*

If the cone have its base in the vertical plane, similar remarks apply, having reference to the vertical plane.

A plane can in general be found tangential to two cones. (1) *When the cones have their axes parallel, and their vertical angles equal;* (2) *when the cones have a common vertex;* (3) *when the cones circumscribe the same sphere.*

If the student finds difficulty in the application of these principles to any of the remaining problems of this chapter, let him return to the problems after reading Chapters XIII. and XIV., where the principles are illustrated in more detail. See also the theorems in the Appendix.

229. Problem.—**To determine the inclinations θ and ϕ of a given oblique plane to the horizontal and vertical planes of projection.**

Method 1.—Suppose a cone with its base angle equal to θ, to be cut into two halves by a plane containing the axis; then one of these half-cones may be placed with its triangular face in the vertical plane, and the semicircular base on the ground, so that the given plane shall be tangential to it. The following solution consists in drawing the plan and elevation of the half-cone in this position, and so finding the base angle θ.

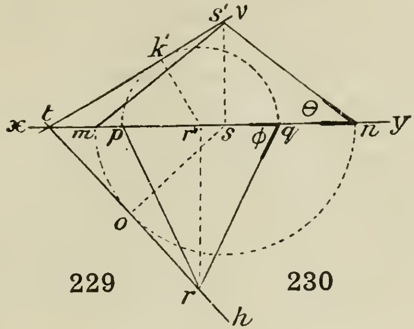
Take for the vertex of the half-cone any point S in the vertical trace, s, s' being its projections. With centre s describe the semicircle *mon* touching *th* at *o*. This is the plan of the half-cone. Join s' to m and n. Then $ms'n$ is the elevation of the half-cone, and the base angle marked θ is the required inclination of the plane to the ground.

To find ϕ, take for the vertex of a half-cone any point R in the horizontal trace, r, r' being its projections. With centre r', describe the semicircle $pk'q$, touching tv at k'. This semicircle is the elevation of a half-cone with its base in the vertical plane, and to which the plane is tangential. The plan of the half-cone is *rpq*, and the base angle, marked ϕ, is the required inclination of the plane to the vertical plane.

Method 2. The inclination θ may also be found by drawing an edge elevation of the plane as in Prob. 198. The inclination ϕ may similarly be obtained by drawing an *edge plan* of the plane on x_1y_1, taken perpendicular to the vertical trace.

The model illustrating the first of these constructions will also serve to illustrate the second, by turning it into a position so as to reverse the planes of projection. And the construction given for the first will also serve to illustrate the second, if it be read with the page upside down, so as to reverse the traces of the plane.

Method 3.—The construction for the rabatments of a point A, in Prob. 213, also gives the inclinations of the plane, the angle θ being equal to the angle ama'' of Fig. 213, and the angle ϕ equal to $a'n'a_1$ of the same figure.



230. Problem.—**Having given one trace of a plane, and also the inclination of the plane to one of the planes of projection, to find the other trace.**

Let the horizontal trace *th* be given, and also θ, the inclination of the plane to the ground.

Take any point *s* in *xy*, and with centre *s* describe the semicircle *mon*, touching the given horizontal trace at *o*. Through *m* and *n* draw lines making θ with *xy*, and intersecting at *s′*. Then the line *ts′* is the required vertical trace.

Examples.—1. Determine the inclinations θ and ϕ of a plane whose horizontal and vertical traces make 30° and 50° with *xy*.

2. A line intersecting *xy* at 50° represents both traces of an oblique plane. Find θ and ϕ, the inclinations of the plane to the horizontal and vertical planes.

3. The real angle between the traces of a plane inclined at 50° is 80°; represent the plane by its traces.

4. A line making 40° with *xy* is the horizontal trace of a plane inclined at 50° to the horizontal plane; determine its vertical trace.

5. A line making 45° with *xy* is the vertical trace of a plane inclined at 60° to the ground. Determine the horizontal trace.

6. A line making 60° with *xy* is the horizontal trace of a plane inclined at 65° to the vertical plane. Determine its vertical trace.

7. A line making 40° with *xy* is the vertical trace of a plane inclined at 60° to the vertical plane. Determine its horizontal trace.

231. Problem.—**To determine the traces of a plane which shall be inclined at given angles θ and ϕ to the horizontal and vertical planes of projection.**

First observe, as stated in Art. 228, that if there be two cones such that the vertex of the second cone does not fall within the first cone, we can conceive a plane which touches the first cone and passes through the vertex of the second. Now this plane will also touch the second cone if the *two cones circumscribe the same sphere.* See the theorems of the Appendix, relating to the sphere, cone, and cylinder.

With any centre c in xy, and any radius, describe a circle. This is both plan and elevation of a sphere with its centre in xy.

Draw tangents $v'r'$, $v's'$ to this circle as shown, each inclined at θ to xy; and with centre c describe the semicircle $r'as'$. These are the projections of an upright half-cone, base angle θ, circumscribing the sphere.

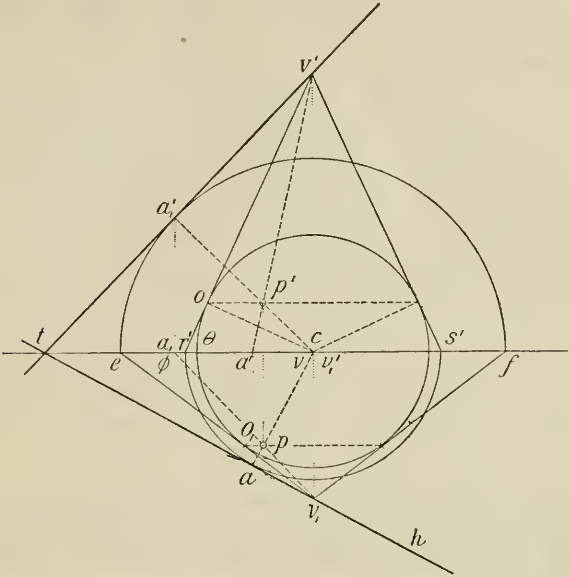
Also draw tangents v_1e, v_1f to the circle, each making ϕ with xy, and draw the semicircle $ea_1'f$ as shown. These are the projections of a second half-cone circumscribing the sphere, with its base in the vertical plane, and base angle equal to ϕ.

Finally, from v' draw the tangent $v'a_1'$ to the semicircle $ea_1'f$; and from v_1 draw the tangent v_1a to the semicircle $r'as'$. These tangents will be found to intersect in a point t in xy, and are the traces of the plane required.

The projections of the two generators AV, A_1V_1, and the point P in which the plane touches the two cones and sphere are shown in the figure.

Limits to the data.—A tangent could not be drawn from v' to the semicircle on ef, if v' were inside this semicircle; therefore, cv' must not be less than ce. Now, observing that cov' and co_1e are right-angled triangles, having the sides co, co_1 equal, it appears that if cv' is greater than ce, the angle $ov'c$ opposite oc must be less than ϕ opposite o_1c. But $\theta + ov'c = 90°$, therefore $\theta + \phi$ cannot be less than 90°. Since neither θ nor ϕ can be greater than 90°, $\theta + \phi$ cannot be greater than 180°.

Thus $\theta + \phi$ must lie between 90° and 180°, both inclusive.



Examples.—1. Determine the traces of a plane inclined at 60° to the ground, and 45° to the vertical plane.

2. A point *A* is $2\frac{1}{4}''$ above the ground, and $1\frac{3}{4}''$ in front of the vertical plane; determine the traces of a plane passing through *A* and inclined at 40° and 55° to the horizontal and vertical planes of projection.

3. Draw the traces of a plane which is inclined at 45° and 50° to the horizontal and vertical planes of projections, the angle between the traces being obtuse.

Hint.—Take the vertical cone in the above figure, with its apex downwards.

4. Determine two planes, one of which is inclined at 40° and 50° to the horizontal and vertical planes of projection, and the other at 90° and 90°. Find the angle between these planes. *Ans.* 90°.

5. Work Prob. 231 when $\theta = 60°$, and $\phi = 45°$, taking the radius of the sphere to be $1''$. Determine the contact of the plane with each cone and with the sphere.

232. PROBLEM.—**To determine a plane which shall be inclined at a given angle θ, and shall contain a given straight line AB.**

Find H and V, the horizontal and vertical traces of the line. These are points in the traces of the required plane.

Next represent a cone with its base on the ground, its vertex at any point in the line, say A, and having a base angle equal to θ. The required plane will be tangential to this cone, and the horizontal trace must touch the plan of the circular base (Art. 228).

The isosceles triangle with its vertex at a', and its base angle equal to θ, is the elevation of this cone; the plan is the circle with centre a.

The horizontal trace of the required plane is the line ht drawn through h to touch this circle, and the vertical trace is the line drawn through t and v', where v' is the elevation of the vertical trace of BA.

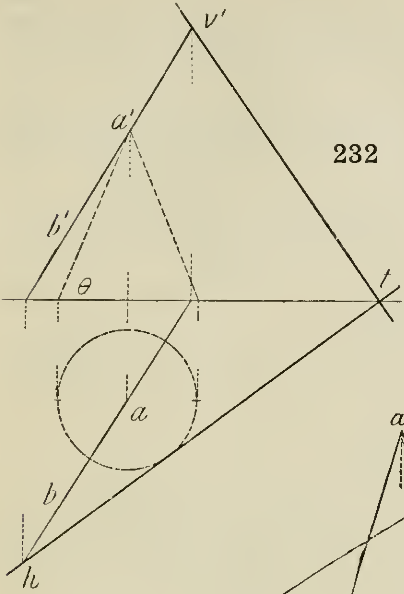
233. PROBLEM.—**To determine the traces of a plane which shall be perpendicular to a given oblique plane VTH, pass through a given point A, and have a given inclination θ.**

The required plane will be perpendicular to the plane VTH, if it contain any line perpendicular to the plane VTH.

Therefore draw ar perpendicular to ht, and $a'r'$ perpendicular to vt; these are the projections of a line AR perpendicular to the plane vth. The problem therefore reduces to the last one.

Therefore draw the projections of an upright cone with vertex A and base angle θ. Obtain n, the horizontal trace of AR; and from n draw a tangent (there are two) to the plan of the base of the cone, thus obtaining mn, the horizontal trace of a plane satisfying the required conditions. The vertical trace ml is readily obtained since A is a point on the required plane.

For examples see page 263.



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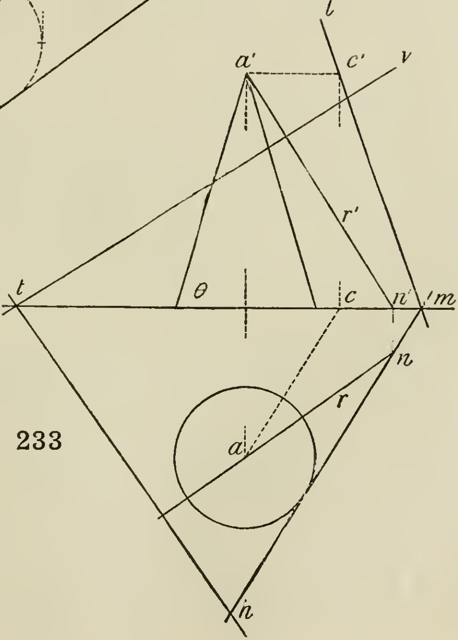

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234. Problem.—**To determine the traces of a plane passing through a given point A, having a given inclination θ, and making an angle α with a given plane VTH.**

If there be two cones having A as a common vertex, their base angles being θ and α respectively, then if these cones have their bases on the ground and the given plane respectively, a plane which touches both cones will satisfy the given conditions.

Convert the given oblique plane VTH into the inclined plane VOH as in Prob. 198, and obtain a'', the new elevation of A on $x'y'$.

Now draw $a''r''s''$ and $a''u''w''$, the elevations (on $x'y'$) of the cones described above.

The horizontal trace of one cone is the circle with centre a, and that of the other is an ellipse whose elevation is $e''f''$.

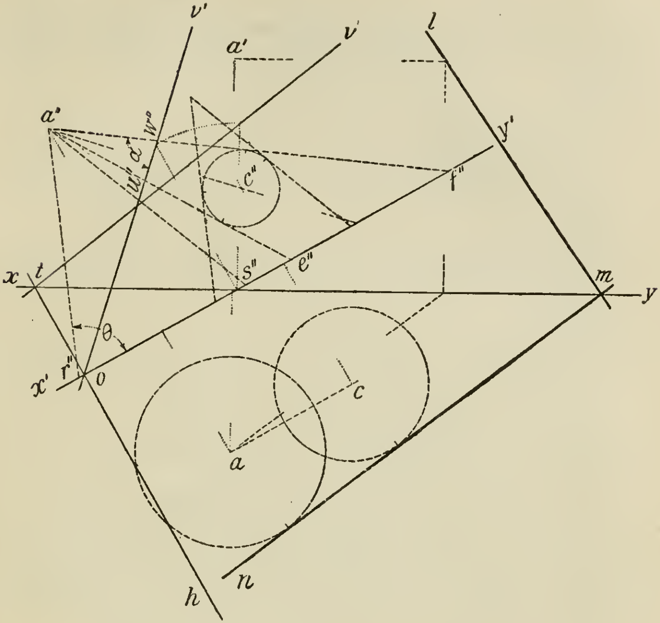
We shall avoid having to draw this ellipse in plan, by the following artifice :—

Any convenient sphere is conceived as inscribed in the cone whose elevation is $a''e''f''$; c'' is the elevation of the centre of such a sphere. This sphere is then conceived as circumscribed by a cone with its base on the ground, and base angle θ ; the plan and elevation of this cone are shown. Observe that the plan c of the centre of the base of this last cone will be in the line through a at right angles to ht.

Draw a common tangent nm to the plans of the two upright cones. This is the horizontal trace of *one* plane satisfying the given conditions, and the vertical trace lm is readily obtained as shown, since A is in the plane.

For, a plane which touches the two vertical cones has the given inclination ; it also passes through the given point A, and touches the sphere centre C, and therefore is tangential to the cone whose elevation is $a''u''w''$.

Note 1.—The trace mn might have been obtained by first drawing the elliptical trace referred to above, and then drawing a common tangent to this ellipse, and the circle centre a.



Examples.—1. A point A is $2''$ in front of the vertical plane, and $1\frac{1}{2}''$ above the ground. A point B is $1\frac{1}{2}''$ to the left of A, $1''$ below it, and $\frac{3}{4}''$ farther from the vertical plane. Determine the traces of two planes each containing AB and inclined at 50° to the ground.

2. The traces vt, th of a plane make angles of 50° and 45° with xy. A point A is $1\frac{1}{2}''$ to the right of t, $2''$ above the ground, and $1\frac{1}{2}''$ in front of the vertical plane. Determine the traces of a plane which passes through A, is inclined at 65° to the ground, and is perpendicular to the given plane.

3. Determine a plane inclined at 65°, passing through the point A, and making an angle of 60° with the plane in Ex. 2.

4. A plane parallel to xy is inclined at 60°. Determine a second plane which shall be inclined at 45° both to the first plane and to the vertical plane.

235. PROBLEM.—**To determine the traces of a plane which shall pass through a given point A, make a given angle α with a given line AB, and have a given inclination θ.**

First observe that the required plane will touch each of two cones, namely:

(1) An upright cone with vertex A, and base angle θ.

(2) A cone with vertex A, axis AB, and semi-vertical angle α.

The projections of the first cone are easily drawn, and those of the latter may be obtained in the following manner:—

Suppose AB to turn about the vertical line Aa until it is parallel to the vertical plane. The plan of AB will now be aB_1, and the elevation $a'B_1'$.

Through a' draw two lines each making an angle α with $a'B_1'$, and describe a circle to touch these lines, the centre being *any* point c_1' on $a'B_1'$.

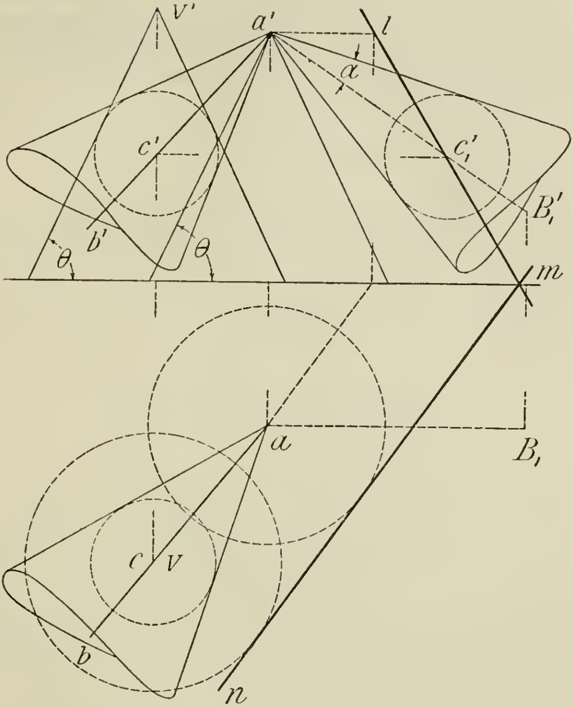
Now describe a circle with the same radius, but with c' as centre, $c_1'c'$ being parallel to xy. Draw tangents to this circle from a', then these form the elevation of a cone of indefinite length, having AB for its axis and a semi-vertical angle α, C being the centre of an inscribed sphere. The outline of the plan of this cone is not required.

Next draw the projections of an upright cone with base angle θ circumscribing the sphere, centre C, its base being on the ground.

Draw nm to touch the plans of the bases of the two vertical cones, then nm is the horizontal trace of the required plane (there are two such), and the vertical trace is readily drawn because A is a point in the plane.

It will be obvious that since the plane LMN touches the two upright cones, it must pass through A and touch the sphere centre C; hence it must touch the inclined cone.

Note.—In connection with this problem the student should study the theorems on the cone and cylinder given in the Appendix.



Examples.—1. A line *AB* is parallel to the vertical plane and distant $1\frac{1}{2}''$ therefrom; its inclination to the ground is 50°. Determine the traces of a plane which makes 30° with *AB* and has an inclination of 60°.

Note.—Observe that there are limits to the data in this problem. Thus the inclinations of the line and plane being 50° and 60°, show that the angle between the line and plane may not be *any* angle, but must be between 0° and 70°.

2. Find the angle between a face of a cube and a diagonal of the solid. Then determine a plane which contains the face and is inclined at 65°, the diagonal of the cube being parallel to the vertical plane and inclined at 45°.

236. PROBLEM.—**The projections of a line AB being given, and the horizontal trace, ht, of a plane which passes through A, it is required to determine the projections of a line AP lying in the plane HT and making a given angle β with the given line.**

Note.—The given angle must not be less than that between the given line and plane.

First determine and set out the true length of AB. Draw AP so that the angle BAP is β, and choose any point C on AP.

We can now find the projections of C because it lies on the surface of

1. A sphere, centre A, radius AC.
2. A sphere, centre B, radius BC.
3. The given plane.

Draw $x'y'$ at right angles to ht; project a'', then oa'' is the edge view of the given plane. Project b''.

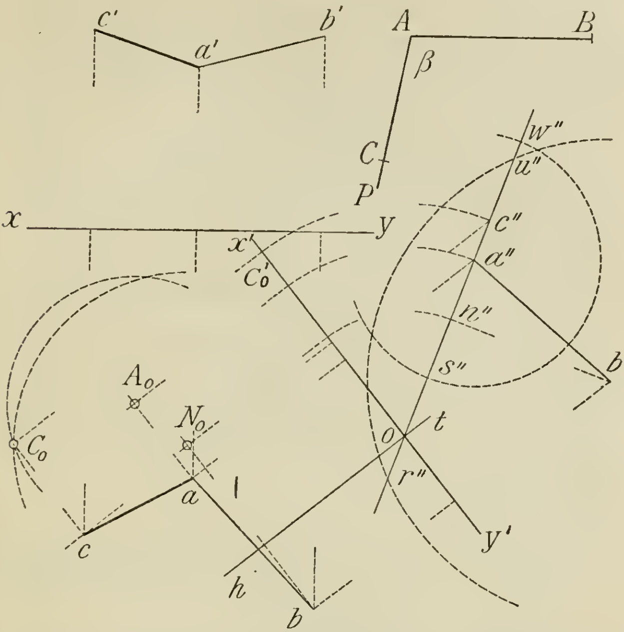
With a'' as centre and AB as radius describe a circle. With b'' as centre and BC as radius describe another circle. These are the elevations, on $x'y'$, of the two spheres mentioned above, and they are intersected by the given plane in circles the elevations of whose diameters are $s''w''$ and $r''u''$.

Rabat the given plane and these two circles into the ground as shown; the rabatments of the circles intersect in C_0 (there are two such points).

From C_0 obtain C_0' and c''; from C_0 and c'' obtain the plan c, then ac is the plane of *one* line fulfilling the required conditions.

The elevation of AC on xy can be drawn since the height of C is known.

We have given the above method because the device of employing spheres to fix the position of a point is one which may be used in solving many problems.



Examples.—1. A line AB is parallel to the vertical plane and inclined at 60° to the ground. An inclined plane passes through A and makes 70° with the ground. Determine the projections of a line AC 2″ long which lies in the given plane and makes 65° with AB.

Would the solution be possible if 45° were substituted for 65°? *Ans.* No.

2. A line AB makes 30° and 50° with horizontal and vertical planes; draw its projection if A is $1\frac{1}{2}''$ from each plane. A plane contains A and makes 45° and 55° with the horizontal and vertical planes; draw its traces. Determine the projections of a line AC 2″ long which lies in the plane and makes 60° with AB.

3. The traces vt, th of a plane make 50° and 35° with xy. Determine a line which passes through T, lies in the plane VTH, and makes an angle of 65° with the ground line xy.

237. PROBLEM.—**An oblique plane VTH is given, and the plan of a straight line CD which lies in this plane. It is required to determine a plane which contains CD and makes an angle α with the given plane.**

Convert the oblique plane *VTH* into the inclined plane *V′OH* by Prob. 198, and project d'' from *d*.

Draw $d''e''$ perpendicular to ov'. Draw the elevation, $d''a''r''$, of a cone which has *DE* for its axis, and semi-vertical angle equal to $90^\circ - \alpha$.

By Prob. 274 determine aa_1, bb_1, f, f_1, the plans of the axes and foci of the ellipse (but not the curve) in which this cone intersects the horizontal plane.

Now the required plane must touch this cone, hence its horizontal trace must touch the ellipse; it must also pass through *c*, the horizontal trace of *CD*.

Therefore by Prob. 94 draw a tangent from *c* to the ellipse; this is *nm*. And *ml* drawn through d' will be the required vertical trace.

Thus *lmn* represents the required plane.

Example. — The traces *vt*, *th* of a plane make angles of 45° and 35° with *xy*. *C* and *D* are points in the traces, each distant 2.5″ from *t*. Determine a plane which contains *CD* and makes an angle of 50° with the plane *vth*.

238. Examples on Problems 225 to 237.

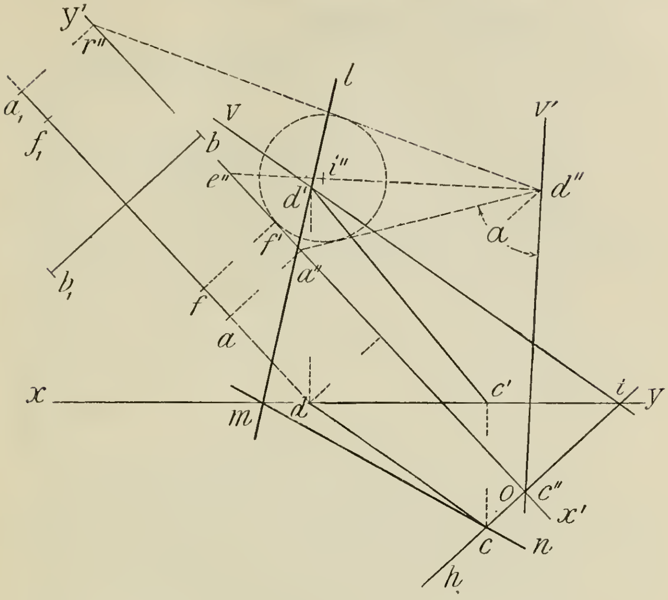
1. A point is $1\frac{1}{4}''$ from *xy*. State and project its locus. *Ans.* A cylinder of indefinite length. The projections are straight lines distant $1\frac{1}{4}''$ from, and parallel to *xy*.

2. A point is $1\frac{1}{2}''$ distant from a point *c* in *xy*. State and project its locus. *Ans.* A sphere 3″ diameter with centre at *c*. The projections are coincident circles.

3. A point is 1″ above the ground. State and project its locus. *Ans.* A horizontal plane 1″ high. The vertical trace is parallel to *xy*.

4. A point is $1\frac{1}{4}''$ from *xy*, and $1\frac{1}{2}''$ distant from the point *c* in *xy*. State and project its locus. *Ans.* The circle in which the cylinder and sphere of Exs. 1 and 2 intersect. Projections, coincident straight lines perpendicular to *xy*.

5. A point is $1\frac{1}{4}''$ from *xy* and 1″ above the ground. State and project its locus. *Ans.* The two horizontal lines in which the cylinder of Ex. 1 is cut by the plane of Ex. 3.



6. A point is distant $1\frac{1}{2}''$ from the point c in xy, and $1''$ above the ground. State and project its locus. *Ans.* The horizontal circle in which the sphere of Ex. 2 is cut by the plane of Ex. 3.

7. A point is $1\frac{1}{4}''$ from xy, $1\frac{1}{4}''$ from a point c in xy, and $1''$ above the ground. Determine its position. *Ans.* Either of the two points common to all the loci of Exs. 1 to 6.

8. The traces of th and tv of a plane make 55° and 45° with xy. A point A is $2''$ distant from t, $1\frac{1}{4}''$ from the plane VTH, and $1\frac{1}{2}''$ from the vertical plane. Find all the positions of A which satisfy these conditions.

9. Three planes are mutually perpendicular. One is inclined at 40°, a second at 60°; find the inclination of the third.

10. Three lines are mutually perpendicular. One is inclined at 40°, a second at 30°; find the inclination of the third.

11. Draw the complete plan of an equilateral triangle ABC of $3''$ edge, having given—the plan ab is $2\frac{1}{2}''$ long and the plan ac makes 30° with ab.

239. Miscellaneous Examples.

***1.** Determine the true angle between the lines *AB* and *BC*.

***2.** Determine the projections of a circle of $1\frac{3}{4}''$ diameter lying in the plane containing the two given lines *AB*, *BC*, produced if necessary, and touching both. (1891)

***3.** The traces of a plane are given and the projections of a point *P*. From *P* draw two lines each $2\frac{1}{4}''$ long, meeting the plane in points on the same horizontal line and $2\frac{1}{2}''$ apart. (1890)

***4.** From the given points *A*, *B*, draw two lines meeting on the given plane and making equal angles with it. (1891)

***5.** p, p' are the projections of a given point *P*; *ab*, *a′b′* those of a given line *AB*. Draw the projections of an equilateral triangle, with one vertex at *P* and the side opposite that vertex on the line *AB*. (1897)

***6.** *a* is the plan of a point *A* lying in the given plane *VTH*. Through *A* draw a line in the given plane *VTH* parallel to the given plane *BOC*. (1888)

***7.** Draw a plane perpendicular to the plane of the triangle *ABC* and bisecting the sides *BC* and *AB*. (1894)

8. Draw the traces of any plane equally inclined to the planes of projection, and determine the true angle between its traces.

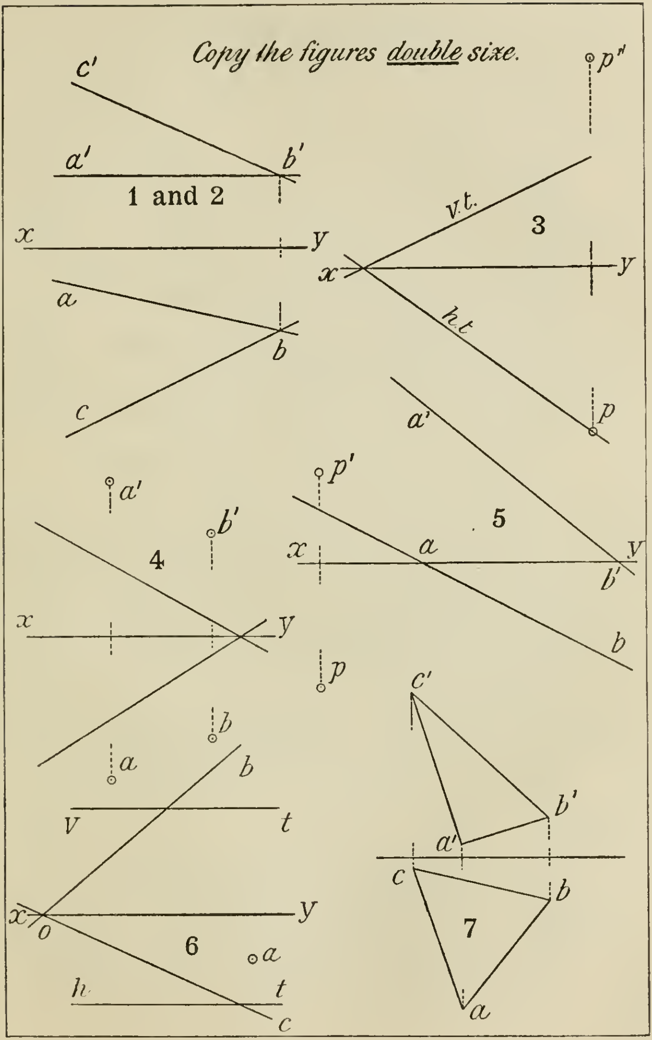
9. A plane is equally inclined to both co-ordinate planes, and the real angle between the traces is 50°. Draw the traces. (1892)

10. The vertical trace of a plane makes 40° with the ground line, and the plane is inclined at 50° to the horizontal plane. Draw its horizontal trace. Determine the point (*P*) in the plane $1''$ in front of the vertical plane and $1\frac{1}{2}''$ above the horizontal plane, and through *P* draw a line in the plane making equal angles with its traces. (1889)

11. Draw a plane inclined at 45° to the horizontal plane, and at 60° to the vertical plane. Draw a line in the plane inclined at 30°, and $2''$ long between its traces. Lastly, draw the plan of a regular hexagon lying in the plane of which the above line is a diagonal. (1897)

12. Draw the traces of *any* plane inclined at 40°. Determine the projections of a line in this plane inclined at 27°. Draw the traces of a plane containing this line, and inclined 63°. Find the angle contained by these two planes. Determine also the perpendicular distance between *xy* and the intersection of the two planes.

13. The traces *vt*, *th* of an oblique plane make 30° and 50° with *xy*. Take a point *a* $1''$ from *ht* and $2''$ from *xy*; this is the plan of a point *A* $2\frac{1}{2}''$ above the ground. Draw *af* $2\frac{3}{4}''$ long making 15° with *xy*. Determine the height of *F*—(1) if *AF* is $3\frac{1}{2}''$ long, and *F* is lower than *A*; (2) if *AF* meets *xy*.



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***14.** *ad* is the plan of a line in a plane whose traces are *ab*, *bc*. One side of an equilateral triangle of 1.5″ side lies along *ad*, one extremity at *a*. The plane of the triangle is inclined at 40° to the horizontal plane. Draw the projections of the triangle. (1895)

***15.** A line *AB* and a plane are given. Through *AB* draw a plane perpendicular to the given plane and determine the line bisecting the angle between *AB* and the line in which the planes intersect. (1892)

***16.** *ABC* is the base of a tetrahedron, and is in the horizontal plane. *D* is at 2.5″ above it. (*a*) Find the values of the dihedral and plane angles round the vertex *D*. (*b*) *P* is a point in the base; find the points where perpendiculars from *P* meet the three other faces of the tetrahedron. (1895)

***17.** Determine the traces of any two planes containing the given line *AB* and including an angle of 60°. (1884)

***18.** *p* is the plan of a point *P* distant $\frac{1}{2}''$ (measured perpendicularly to the surface) from the plane *VTH* and above it. Through *P* draw

(1) a line parallel to the plane *VTH*, and inclined at 45° to the horizontal plane;

(2) a line also parallel to the plane *VTH*, but making an angle of 15° with the line inclined at 45°. (1896)

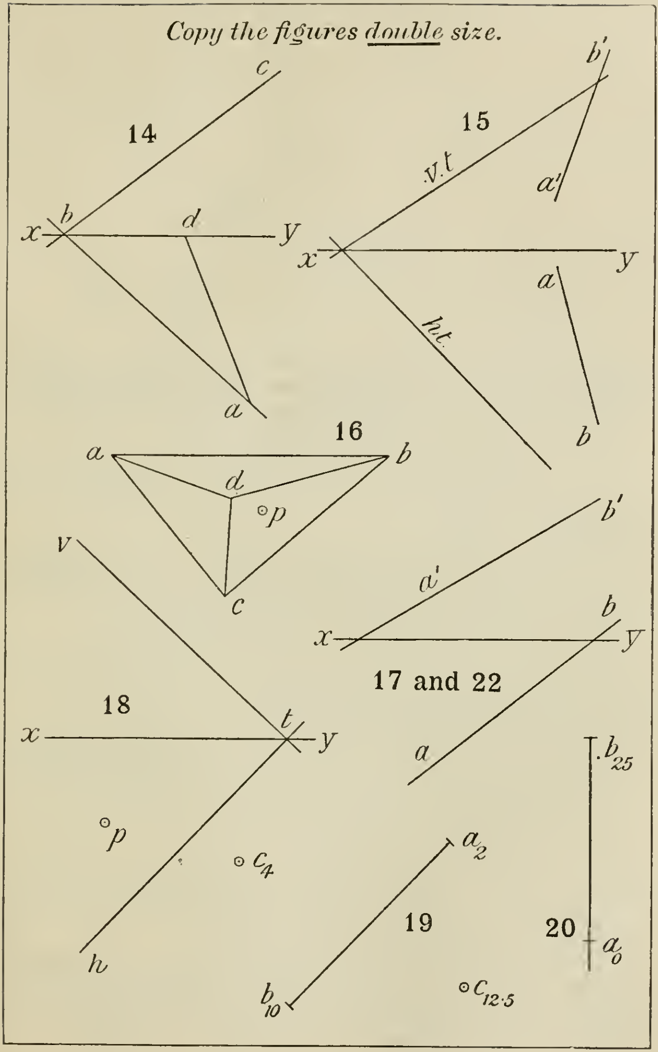
***19.** *AB* is a given line, *C* a given point. Find the scale of slope of the plane of *A*, *B*, and *C*; and draw the plan of a square in that plane, one diagonal to be a perpendicular let fall from *C* on the line *AB*. Unit = 0.1″. (1895)

***20.** Determine a line through the point *C* to meet the line *AB* at a point *D* such that the angle *CDA* shall be equal to 50°. Unit = 0.1″. (1894)

21. Draw an arc of a circle 3″ in diameter with centre *v*. Along the circumference set off adjacent chords *ab*, *bc*, 2″ and $1\frac{1}{2}''$ long. Join *av*, *bv*, *cv*. Assuming the circle to be in the horizontal plane and the height of *V* to be 3″ above this plane, determine the angle made by the plane *ABV*, with the plane *CBV*. (1893)

***22.** Draw (1) a plane (*M*) inclined at 65° to the horizontal plane, and containing *AB*; (2) a plane (*N*) through *B* inclined at 50° to the horizontal plane, and perpendicular to *M*. (1896)

23. The horizontal and vertical traces of a plane make angles of 35° and 40° with *xy*. Draw the plan of *any* line lying in this plane and inclined 30°. Now draw the plan of a line which is parallel to this, and lies in the plane, so that the part of it lying between the traces of the plane is 2″.



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CHAPTER XI

HORIZONTAL PROJECTION, OR FIGURED PLANS

240. PROBLEM.—**Having given the indexed plans of three points A, B, C, to determine the indexed plan of a horizontal line which passes through A and intersects BC.**

Case I.—Let a_8, b_{12}, c_6 be the given plans.

First Method, Fig. (1).—Join $b_{12}c_6$, and draw $b_{12}b'$ and c_6c' perpendicular to $b_{12}c_6$, and equal to 12 and 6 units respectively; join $b'c'$.

Make $b_{12}h$ equal to 8 units, and draw hd', $d'd$ respectively parallel and perpendicular to $b_{12}c_6$; affix the suffix 8 to the point d, then a_8d_8 is the required plan.

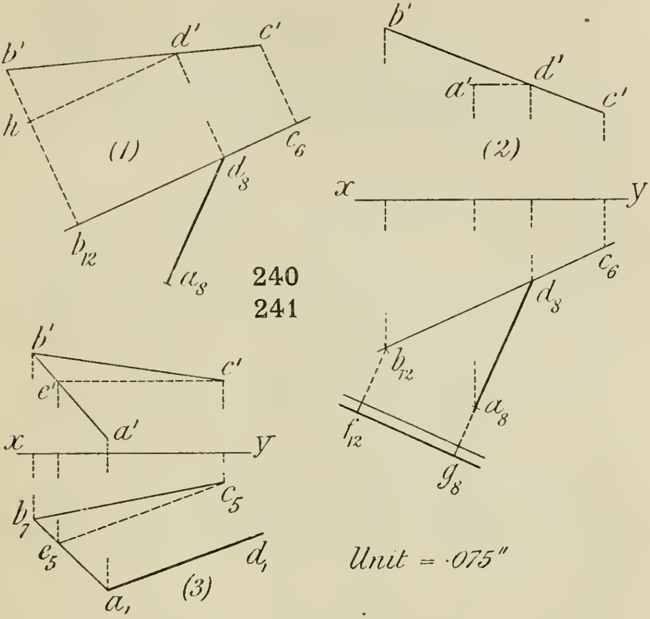
Second Method, Fig. (2).—Draw any xy which is not perpendicular to $b_{12}c_6$, nor so nearly perpendicular as to lead to an ill-conditioned construction, and obtain a', b', c' the elevations of A, B, C.

Join $b'c'$, and draw $a'd'$ parallel to xy; draw the projector $d'd$; affix the suffix 8 to d; then a_8d_8 is the required plan.

Note.—By this method xy may be drawn with the tee-square, and the set-square used for the projectors.

Case II.—Let a_1, b_7, c_5 be the given plans, Fig. (3). The line BC is nearly horizontal, and the point on it at the level of A is supposed to be inaccessible.

Determine the elevations of A, B, C as in the second method of Case I. Join $a'b'$; draw $c'e'$ parallel to xy; project e' to e_5; join e_5c_5, and draw a_1d_1 parallel to e_5c_5. Then a_1d_1 is the required plan.



240
241

Unit = ·075″

241. Problem.—**To determine the scale of slope of a plane containing three points A, B, C, the indexed plans of which are given.**

Let a_8, b_{12}, c_6 be the given indexed points, Fig. (2).

By one of the methods of Prob. 240 determine a_8d_8, the plan of a horizontal line in the plane of *ABC*.

Draw any double line at right angles to d_8a_8. Let d_8a_8 and a line through b_{12}, parallel to d_8a_8, meet this double line in g_8, f_{12}. Then the required scale of slope is g_8f_{12}.

Example.—Draw a triangle $a_{20}b_8c_{15}$, making $a_{20}b_8$ $2\frac{1}{4}''$, $a_{20}c_{15}$ $3''$, and b_8c_{15} $3\frac{1}{4}''$. Draw the plan of a horizontal line *CD* which intersects *AB*. Draw an elevation of *ABC* on a plane at right angles to *CD*. Draw a scale of slope for the plane of *ABC*.

242. Problem.—**Having given a plane by its scale of slope ab, to determine (1) the inclination θ of the plane; (2) a scale of slope of the plane which passes through the given point P and is parallel to the given plane; and (3) a scale of slope of the plane which passes through P, is perpendicular to the given plane, and is inclined at a given angle ϕ.**

(1) Draw xy parallel to ab, and obtain the elevation $a'b'$ by making $na' = 10$ units, and $mb' = 30$ units; join $b'a'$, and produce to meet xy. Then $a'b'$ is an edge view of the plane, and the angle marked θ is the required inclination.

(2) Obtain p', the elevation of P, and draw $p'e'$ parallel to $b'a'$; then $p'e'$ is an edge view of the required plane, and $e'e$ drawn perpendicular to xy is its horizontal trace.

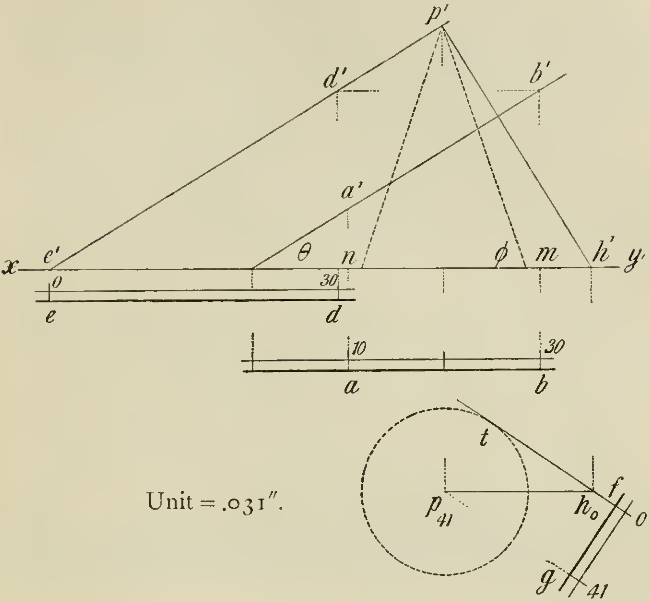
At any point e in this trace draw the double line parallel to ab, and on it mark off ed equal to, say, 30 units measured from the scale of ab; index the two points of the scale 0 and 30 as shown, and the scale of slope of the required plane is completed. Observe that since the planes represented by ab and ed are parallel, equal differences of level of the two planes correspond to equal lengths on their scales of slope.

(3) Every plane passing through P at right angles to the given plane ab must contain the line through P at right angles to that plane.

Draw $p'h'$ perpendicular to $a'b'$, and $p_{41}h_0$ parallel to ab; then these are the projections of the line through P perpendicular to the given plane, and H is the horizontal trace; hence the horizontal trace of the required plane will pass through h_0.

Draw the projections of a cone with its base on the ground, its base angle ϕ, and its vertex at P; the required plane will touch this cone, and the horizontal trace will touch the plan of the base of the cone.

The plan of the cone is the circle, centre p_{41}; hence the horizontal trace of the required plane is the tangent h_0t drawn from h_0 to this circle.



Unit = .031″.

Take any point f in h_0t; draw the double line fg perpendicular to h_0t, and draw pg parallel to h_0t; then ft is the horizontal trace of the required plane, and $p_{41}g$ is the plan of a horizontal line at the level 41; hence the scale of slope fg can be indexed as shown.

Examples.—1. Draw a double line ab 2″ long, and attach indices of 10 and 30 to a and b. Regard this as the scale of slope of a plane. A point c_{25} is 3″ from a, and $2\frac{3}{4}$ from b. Determine the scale of slope of a plane (a) through C parallel to the given plane; (b) through C perpendicular to the plane and inclined at 65° to the ground. Unit = 0.1″.

2. Determine the plan of a line in the plane (a) Ex. 1, at a level 15, and of one at the same level in the plane (b); show the plan of the intersection of these lines, and index it.

243. PROBLEM.—**The figured plan $c_{25}d_{13}$ of a line, and the scale of slope ab of a plane are given; to determine (1) the figured plan of the point of intersection of the line and plane; (2) the angle between the line and plane; and (3) the bisector of this angle. Unit 0·1″.**

(1) Take *ab* as a ground line, and obtain *ab′* the edge view of the given plane, and *c′d′* the elevation of the given line *CD*; these intersect in *i′*. Draw the projector *i′i*, then *i* is the plan of the point of intersection of the given line and plane, and its index may be obtained by measuring *i′m*.

(2) Determine *c′n′*, $c_{25}n$, the projections of *CN*, the perpendicular from *C* to the plane. Then obtain I_0N_0 by rabatment of the plane about its horizontal trace; and next find C_0 by rabatment of the triangle *INC* about I_0N_0.

Then $N_0I_0C_0$ is the required angle between the line and plane.

(3) Bisect the angle $N_0I_0C_0$ by I_0H_0. This is the rabatment of the required bisector.

Set off *n′h′* equal to N_0H_0 and project *h* from *h′*. Then *ih* is the plan of the bisector. The indices for *i* and *h* may be found by measuring the heights of *i′* and *h′* above *ab*.

Example.—Draw a triangle *abc*, making *ab* = 2″, *ac* = 3″, and *bc* = 2.5″. Take *d* 2″ from *b* and 1¾″ from *c*, and outside the triangle *abc*. Index the points *a* and *d*, 8 and 25; this is the plan of *AD*. Regard *bc* as the scale of slope of a plane, *B* and *C* being at levels 10 and 30. Unit 0.1″.

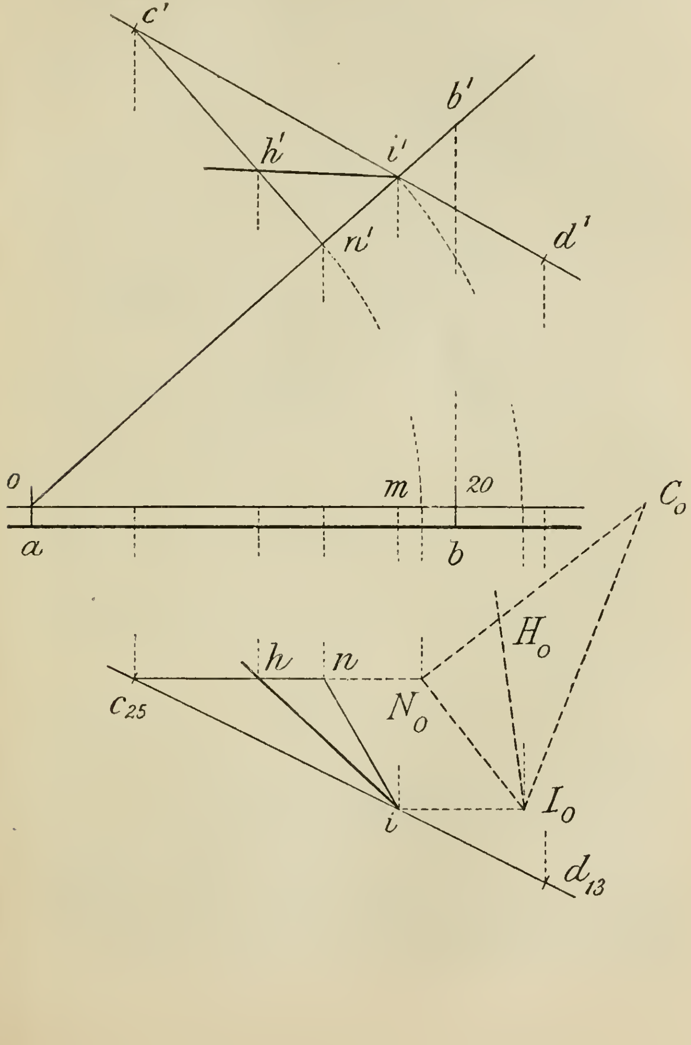
(*a*) Determine the indexed plan of the point of intersection of the given line and plane.

(*b*) Find also the angle between the line and plane.

(*c*) Draw the indexed plan of the bisector of the angle (*b*).

(*d*) Draw the scale of slope of a plane which bisects the line *AD* at right angles.

(*e*) Determine a scale of slope of the plane which contains *AD* and is parallel to *BC*, and of the plane which contains *BC* and is parallel to *AD*. Find the distance between these parallel planes.


c'
b'
h'
i'
n'
d'
0
m
20
C0
a
b
H0
h
n
c25
N0
I0
i
d13

244. PROBLEM.—**To determine the line of intersection of two planes given by their scales of slope, ab and cd.**

Case I.—The method here employed depends upon the fact that two horizontal lines at the same level, and one on each plane, must intersect each other at a point which lies in both planes, that is, on their intersection.

Through the division 0 on each scale draw lines *aj*, *cj* respectively perpendicular to the scales of slope; these are the plans of two horizontal lines at a level 0, one on each plane, hence j_0 is the indexed plan of a point on the line of intersection. Similarly, *bi* and *di* drawn through the divisions 10 determine i_{10}, the plan of another point on the line of intersection; hence *IJ* is the required intersection.

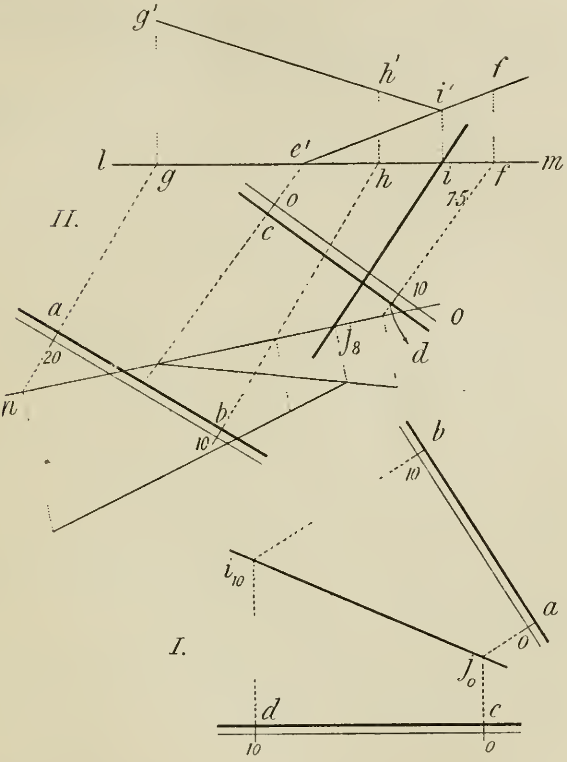
Case II.—Let the scales of slope be nearly parallel; the above method will be inconvenient.

Conceive both planes to be cut by any vertical plane the plan of which is, say, *lm*. Draw the plans of the horizontal lines through *a* and *b* on one of the given planes, and through *c* and *d* on the other, intersecting *lm* in *g*, *h*, *e*, *f* respectively. Obtain $g'h'$ and $e'f'$, the elevations of *GH* and *EF*, on *lm* as ground line; they intersect in i', from which the plan *i*, in *lm*, is determined by projecting from i'. Now the point *I* is in the plane *ab*, since it is in the line *GH* contained by this plane; and *I* is in the plane *cd*, since it is in *FE*; therefore *I* is in both planes, or is one point in their required intersection. On measuring ii' the index for *i* is seen to be 7.5.

By taking any other vertical plane, say *no*, the point j_8 is obtained in a similar manner; hence $i_{7\cdot5}j_8$ is the required indexed plan.

This method is applicable to any case where the line of intersection is within the limits of the paper.

Case III. (no figure).—Let the scales of slope be parallel to each other. In this case the horizontal lines on the two planes are parallel to each other, and the line of intersection of the planes is horizontal. We may proceed as in II., or as follows:—



Take *xy* parallel to *ab* or *cd*, and obtain *a′b′*, *c′d′*, the edge views of the two planes; let these intersect in *i′*. Then *i′* is the end view of the line of intersection, and the plan *ij* may be obtained by a projector from *i′*. The distance of *i′* from *xy* determines the indices of *i* and *j*.

Example.—Draw a quadrilateral *abcd*, making $ab=bc=2''$, $ad=2\frac{1}{2}''$, $abc=70°$, $bad=60°$. Regard $a_{30}d_0$ and $b_{25}c_{10}$ as the scales of slope of two planes; determine the indexed plan of their line of intersection. Unit 0.1″.

245. PROBLEM.—**To determine the scale of slope of the plane which bisects at right angles the angle between two lines AB, AC, whose indexed plans are given.**

The required plane must contain the line which bisects the angle *BAC*. It must also contain the line through *A* at right angles to the plane of *BAC*. These two lines define the plane, and their projections are first found.

On $b_{30}a_{10}$ produced determine n_4, the plan of a point *N* on the same level as *c*, by the method of Prob. 240; join c_4n_4. Then *CN* is a horizontal line in the plane of *ABC*.

Draw *xy* at right angles to c_4n_4, and obtain $c'a'b'$, the edge elevation of the triangle *CAB*. Now conceive that the triangle revolves about *CN* until it is horizontal; its plan while in this position may be obtained thus:—

Through c' draw $w'z'$ parallel to *xy*, and with c' as centre describe the arcs $b'B_0'$ and $a'A_0'$; draw bB_0 and and aA_0 parallel to *xy* to meet projectors from B_0' and A_0' in B_0 and A_0. Then $A_0B_0c_4$ is the required plan.

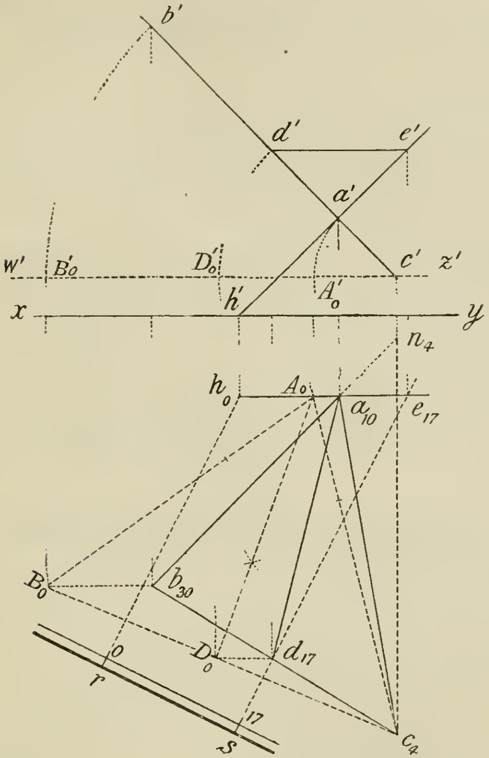
Bisect the angle $B_0A_0c_4$ by the line A_0D_0. Find d', the elevation of *D* when the triangle *ABC* is brought back to its original position; also, from d' project d_{17} on $b_{30}c_4$, measuring the index 17 from the elevation.

Through a' draw $a'h'$ perpendicular to $c'a'b'$; this is the elevation of the line at right angles to the plane of the triangle, and $a_{10}h_0$ drawn at right angles to n_4c_4 is its plan.

The required plane contains *AD* and *AH*, and its horizontal trace could be determined by obtaining the horizontal traces of *AD* and *AH*, and then joining these; the scale of slope would be at right angles to the joining line, and could be indexed, since the height of a point *A* in the plane is known.

In the figure the horizontal trace of *AD* is not found. We avail ourselves of the fact that a line which intersects *AD* and *AH* must lie in the required plane.

Draw $d'e'$ parallel to *xy*; this is the elevation of a *horizontal* line intersecting *AD* and *AH*; its plan $d_{17}e_{17}$ is



obtained by projecting from *d'* and *e'* on to $b_{30}c_4$ and h_0a_{10}, and measuring and indexing the level 17.

Draw the scale of slope at right angles to $e_{17}d_{17}$, meeting the latter in *s*, and draw h_0r parallel to $e_{17}d_{17}$ to meet the scale of slope in *r*. Then *rs*, with the indices 0 and 17, is the required scale of slope.

Example.—Determine the scale of slope of the plane which bisects at right angles the angle *BAC* in the example page 275. Also that for the angle *ABC*.

246. Problem.—**To determine the angle between two planes given by their scales of slope ab and cd.**

Determine $i_{20}j_0$ the plan of the intersection of the planes, Prob. 244.

Let a plane be taken at right angles to the intersection, cutting the given planes in two lines and the horizontal plane in a third line; these lines form a triangle with its base on the ground, the vertical angle being the one required.

Obtain $i'j_0$ the elevation of the line of intersection, on $i_{20}j_0$ as ground line. Draw vt, th the traces of an inclined plane at right angles to IJ. Let vt cut j_0i' in r'. Draw j_0a, j_0c perpendicular to ab, cd, and produce these lines to meet h_0t in h_0, f_0.

Now rabat the triangle FRH about its base; that is, with centre t, draw the arc $r'R_0$, and join h_0R_0, f_0R_0. We thus obtain $f_0R_0h_0$, the required angle between the planes.

247. Problem.—**To determine the scale of slope of the plane which bisects the angle between two planes given by their scales of slope ab, cd.**

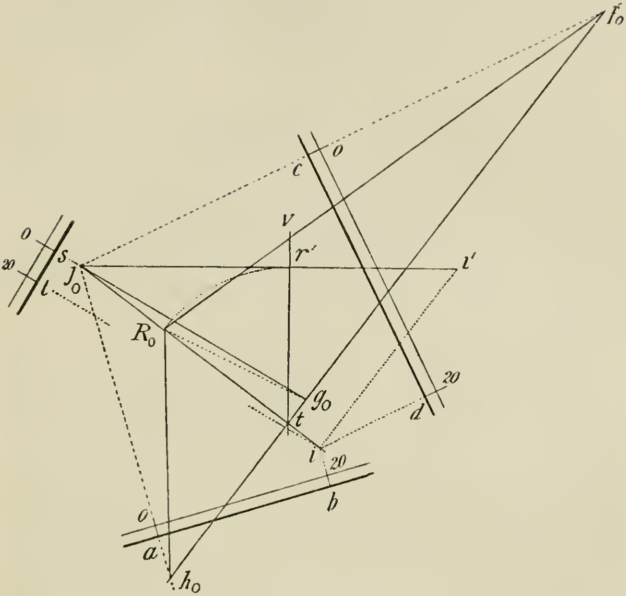
Determine $f_0R_0h_0$, the angle between the given planes in the manner explained in Prob. 246, and draw R_0g_0 bisecting the angle $f_0R_0h_0$, and meeting f_0h_0 in g_0.

Now the triangle $f_0R_0h_0$ is the rabatment of the triangle FRH about FH, and R_0g_0 is the rabatment of the bisector RG. The required plane will bisect the plane angle FRH, and will therefore contain RG.

But since G is on FH it will not move during the rotation of the triangle FRH, hence the required plane contains g_0, which coincides with G. It also contains JI, the intersection of the given planes.

Join j_0g_0. Then since j_0 is the horizontal trace of JI, j_0g_0 will be the horizontal trace of the required plane.

Draw the double-lined scale of slope at right angles to g_0j_0 and meeting it in s; draw $i_{20}l$ parallel to gs; index s and l, 0 and 20 respectively, then sl is the required scale of slope.



246 and 247

Examples.—**1.** Draw a quadrilateral *abcd* having given—sides $ab = 4.5''$, $bc = 4.0''$, $cd = 2.5''$, $da = 2.0''$; diagonal $ac = 4.0''$. Let the indices of the points *a*, *b*, *c* be 10, 30, 5. Unit = 0.1″. If this figure is the projection of a *plane* quadrilateral *ABCD*, index the plan *d*.

2. Let the index of *d* be 40. (*a*) Find the angle between the two planes of which $a_{10}b_{40}$, c_5b_{30} are the scales of slope. (*b*) Find the angle between the planes *ABC*, *DBC* of the gauche quadrilateral *ABCD*.

3. Determine the scale of the slope of the plane which bisects the angle between the planes $a_{10}b_{40}$, c_5b_{30} of Ex. 2. Also work this problem for the planes *ABC*, *DBC* of the same example.

248. PROBLEM.—**The indexed plans a_{10}, b_{20} of two points A, B, and the scale of slope c_0d_{20} of a plane are given; it is required to determine a point P in the given plane which shall be distant 9 units from A and 12 units from B. Unit = ·05″.**

Since the required point *P* is 10 units from *A* it must be situated on the surface of a sphere with *A* as centre and radius 10 units. Similarly *P* must be situated on the surface of a second sphere, with *B* as centre and 12 units as radius. It is also contained by the given plane.

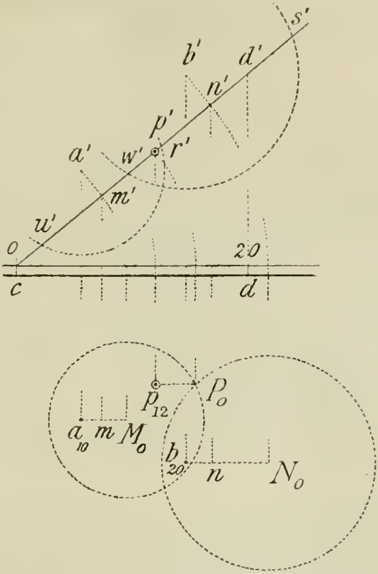
Hence the required point is either of the two points where the circles in which the plane intersects the spheres cut one another. These points are determined in the following manner:—

Take *cd* as a ground line and project *cd′* the edge elevation of the given plane, and also *a′*, *b′* the elevations of *A* and *B*. With *a′* as centre and radius 9 units describe a circle intersecting *cd′* in *u′* and *r′*. With *b′* as centre and radius 12 units describe another circle intersecting *cd′* in *w′* and *s′*. These circles are the elevations of the spheres referred to above, and *u′r′*, *w′s′* are the elevations of the circles in which the given plane intersects these spheres.

Draw *a′m′* and *b′n′* at right angles to *cd′*, then *M* and *N* are the centres of the circles.

These circles intersect each other in two points (*P* and *Q*), each of which satisfies the required conditions; the projections of *P* will now be determined.

Conceive the plane with the two circles to be turned into the ground about its horizontal trace. We thus obtain the rabatments of the two circles, viz. the circles with centres M_0 and N_0; these intersect each other in two points, one of which is P_0. Obtain *p′* by a process the reverse of that by which M_0 and N_0 were obtained from *m′* and *n′*; from *p′* project the plan p_{12}, the index being obtained by measuring the height of *p′*. Thus the figured plan of the required point *P* is found.



Note.—This problem is of considerable importance in so far that many other problems may be reduced to it.

The point P was determined by regarding it as one of the two points of intersection of three surfaces—two spheres and a plane. Now there are two methods of determining such a point. We may determine the intersection with each other of any two of the surfaces, and then obtain the points in which this intersection meets the third surface; or we may determine the intersections of one of the surfaces with each of the other two, and afterwards obtain the points in which these two intersections meet each other. The latter method has been adopted in the above solution.

Examples.—1. Draw a quadrilateral $a_0b_{30}c_{28}d_{15}$ making $a_0b_{30} = 20$; $b_{30}c_{28} = 10$; $a_0d_{15} = 7$; $a_0c_{28} = 20$; $b_{30}d_{15} = 17$. Regard a_0b_{30} as the scale of slope of a plane, and determine the indexed plan of a point on the plane, distant 10 units from C, and 15 units from D. Unit = 0.1″.

2. In Ex. 1 determine the indexed plan of a line lying in the given plane and making 60° with the given line.

249. PROBLEM.—**To determine the shortest line MN between two given lines, AB, CD.**

Let $a_4\ b_{12}$, $c_{10}d_0$ be the given indexed plans of the lines *AB*, *CD*.

Suppose that from any point in one of the lines, say *B* in *AB*, a line *BE* be drawn parallel to the other *CD*. Then *ABE* determines a plane through *AB* parallel to *CD*. Let now *CD* be projected on this plane, and let the projection cut *AB* in *M*. At *M* erect a perpendicular to the plane. This perpendicular will meet *CD* in the point we have called *N*. Then *MN* is the shortest line required. *It is perpendicular to both AB and CD.*

Through b_{12} draw $b_{12}e_2$ parallel and equal to $c_{10}d_0$, the index 2 of *e* being determined so that the difference of the indices of *b* and *e* is the same as that of *c* and *d*, viz. 10. Then *BE* is parallel to *CD*.

Determine f_0 and g_0, the traces of *BA* and *BE*; or by Prob. 240 determine the plan of any other horizontal line in the plane of *ABE*.

Draw *xy* perpendicular to the horizontal line; and on *xy* project the elevations f', a', b', c', and d'. Then $f'a'b'$ is the edge elevation of the plane *ABE*, and $c'd'$ will be parallel to $f'a'b'$.

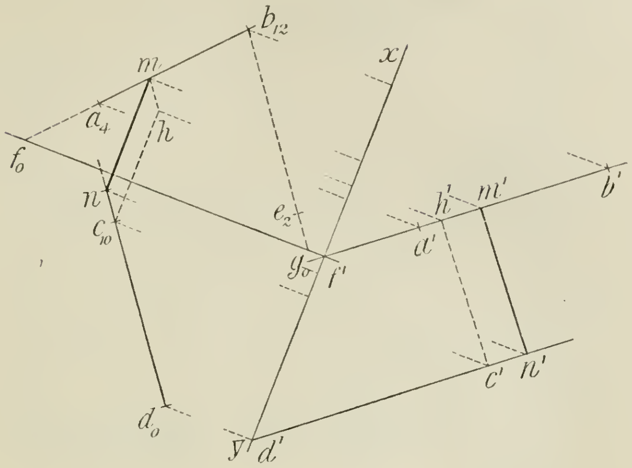
Select any point in $c'd'$, say c', and draw $c'h'$ perpendicular to $a'b'$. Project from h' to h, where $c_{10}h$ is perpendicular to f_0g_0. Through h draw hm parallel to d_0c_{10}. Draw mn perpendicular to f_0g_0.

Then *MN* is the shortest line required. The indices of m and n may be found by projecting m', n' as shown, and then measuring the heights of these points above *xy*.

Examples.—**1.** Two lines a_0b_{18}, c_6d_{28}, each 2.5″ long, which bisect one another at 60°, are the indexed plans of two lines *AB*, *CD*. Determine the indexed plan of *MN*, the shortest line between them.

2. Measure the angle between *AB* and *CD*.

3. Determine a line *PQ* which meets *AB* and *CD* each at an angle of 65°.



4. Draw a triangle *aoc*, having $ac = 2\frac{1}{4}''$, $ao = 1\frac{3}{4}''$, $oc = 1\frac{1}{2}''$. Regard *ac* as the plan of one edge of a regular tetrahedron; the plan of an adjacent edge *AB* coincides, in direction only, with *ao*; complete the plan of the tetrahedron, the indices of *a* and *c* being 12 and 25 respectively. Unit = 0.1″.

Hint.—See Prob. 248. Find the true length of an edge *AC* of the tetrahedron; then the point *B* is fixed in the following way. (1) It lies on a sphere, centre *A*, radius *AC*; (2) it lies on a sphere, centre *C*, radius *AC*; (3) it lies on a vertical plane whose plan is given by the line *ao*. Therefore draw the projections of these spheres, and find (by rabatting the vertical plane) the circular sections of the spheres by the vertical plane; the circles intersect in two points either of which is the rabatment of *B*. The plan *b* may then be readily determined.

5. A line a_0b_{20}, $2\frac{1}{2}''$ long, is the indexed plan of a square *ABCD*; unit = 0.1″. The plan *ad* of an adjacent side makes 45° with *ab*. Complete the indexed plan of the square.

6. Suppose *ABCD* of Ex. 5 to be the base of a right pyramid 3″ long, vertex *V*; draw the indexed plan of the pyramid.

7. Draw the indexed plan of an equilateral triangle *ABC*, having given $a_0b_{10} = 2''$, $a_0b_{10}c = 30°$.

250. Miscellaneous Examples.

***1.** Determine a plane, bisecting the angle between the two given planes. (1886)

***2.** *ab* is a given line, *CD* the scale of slope of a given plane. Determine the projection of the line *ab* on that plane. Unit = 0.1″. (1897)

***3.** Determine the intersection of the three given planes *A*, *B*, and *cod.* Unit 0.1″. (1888)

***4.** *cab* is the horizontal trace of a plane inclined at 40° to the horizontal plane. *cd* is the plan of a line *CD* in that inclined plane. Draw the traces of a plane inclined to the first plane at 40°, the intersection of the two planes to be the line *CD.* (1897)

***5.** The figured plan of a triangle *ABC* is given; *ef* is the plan of a line which is bisected by the plane of the triangle *ABC.* Obtain the index of *f.* (1893)

***6.** Find the common perpendicular to the two lines *AB*, *CD*, and give its length. Unit = 0.1″. (1897)

***7.** Two non-intersecting lines *AB* and *CD* are given (see figure for Ex. 6). Determine the traces of a plane containing *CD*, and parallel to *AB*, and determine the projection of *AB* on this plane.

8. Two lines *ab*, *cd*, at right angles, are the plans of the centre lines of two horizontal shafts, *AB*, *CD*, one of which is two feet above the other. They are connected by a third shaft, intersecting them in *P* and *Q*, such that *PQ* makes angles of 60° and 45° respectively with *AB* and *CD.* Draw the plan of *PQ.* Scale $\frac{1}{2}$ inch to the foot.

***9.** Determine the angle between *AB* (Ex. 6) and a line *AE* which intersects *CD* and lies in the same vertical plane as *AB.*

***10.** *AB* and *CD* (take the figure of Ex. 6, but alter the index of *b* to 16), two non-intersecting straight lines, are given by their figured plans. From *A* draw a line making an angle of 45° with *AB* and intersecting *CD.* Unit 0.1″.

Hint.—The required line must lie on the surface of a cone, vertex *A*, axis *AB*, semi-vertical angle 45°; it must also be in the plane *ACD.* Draw the plan of the cone, and take any vertical plane perpendicular to *AB* cutting the cone in a circle; draw the elevation (on this vertical plane) of the circle. Take *any* two points on *CD*, say *C* and *D.* Find the points *E* and *F* in which *AC* and *AD* meet the vertical plane, then either of the points, in which the line *EF* meets the circle, when joined to *A* will give a line satisfying the required conditions.

Observe that *EF* is the intersection of the plane *ACD* and the vertical plane cutting the cone.

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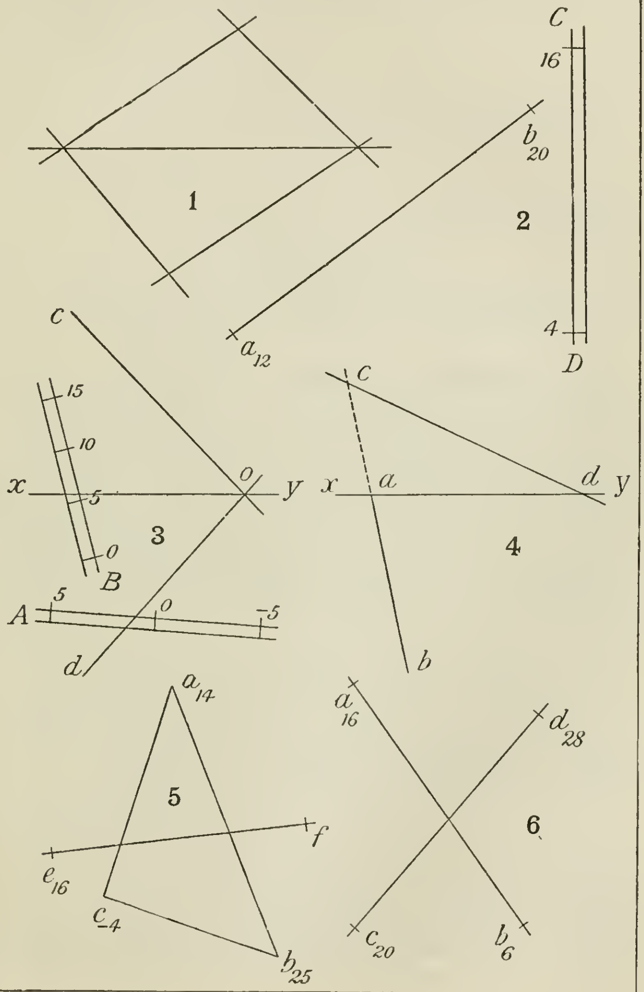

CHAPTER XII

PLANE AND SOLID FIGURES IN GIVEN POSITIONS

251. How position is defined.—Let the student take a square cut out in paper, and place it so as to lie on a plane surface which is inclined at a given angle, one side of the square making another given angle with horizontal trace of the plane; then he will observe that the *shape* of the *plan* is always the same no matter what position the square may occupy on the plane, so long as the two angles remain unaltered. The shape of the plan in this case is therefore made definite by two angles being given. If we suppose the square to be one face of a cube, then the shape of the plan of the cube is also definite.

The *angular* position relatively to the horizontal plane may be fixed in other ways; for example, by having given the inclination of two sides; *or*, what amounts to the same thing, the differences in the heights of three corners of the square. The shape of the plan is again definite.

The shape of the *elevation* is *not* completely defined by the conditions just stated, but depends further on the position which the object occupies relatively to the vertical plane of projection. If the shape of the elevation is to be made definite as well as that of the plan, one more angle must be given; *e.g.* the angle which one side makes with the vertical plane.

Reasoning thus, it is seen that the shape of the *plan* of a polyhedron is definite when any one of the following sets of conditions are given :—

(*a*) *The inclination of a face and that of a line in the plane of the face.*

(*b*) *The inclinations of two lines*, or *the heights of two points above a third, all connected with the solid.*

(*c*) *The inclinations of two faces.*

(*d*) *The inclination of a face, and that of a line connected with the solid, not in the plane of the face.*

The shape of the *elevation* is also definite if, in addition, one of the following conditions be given :—

(*e*) *The inclination of a line to the vertical plane*, or *the difference of the distances of two points from the vertical plane.*

(*f*) *The inclination of a face of the solid to the vertical plane.*

It should be observed that the conditions stated in the preceding article do not define completely the position of a figure in space relatively to the surrounding objects, but only its angular position. Complete definition of position may be obtained as follows:—

Let three planes mutually perpendicular be taken as planes of reference, then the position of a point in space is defined if we know

(*g*) *The distance of the point from each of the three planes of reference.*

The position of a finite line is definite if we know

(*h*) *The position of one point as in* (*g*), *and the inclinations of the line to two of the planes of projection.*

The position of a polygon or polyhedron in space is defined if we know

(*i*) *The position of one point as in* (*g*) *and also the angular position relatively to the planes of reference. To define the latter three angles are necessary, which may be those of* (*a*) *and* (*e*) *in the preceding article, or other combinations.*

252. Problem.—**To draw the plan of an equilateral triangle, $1\frac{1}{2}''$ side, the plane of which is inclined at θ, and one side at α.**

Draw a plane inclined at θ, and in it draw any line AB inclined at α, and obtain bA_0 the rabatment of the line as described in Prob. 191 and here repeated.

Draw an equilateral triangle $C_0D_0E_0$ of the given size, with one side parallel to, or in, bA_0; let this be the rabatment of the triangle about *th*. The plane of rabatment is now to be turned back into its original position *vth*, carrying the triangle along with it; the projections of the triangle will then be *cde*, $c'd'e'$, obtained by the construction of Prob. 186 reversed. The circular arcs with *t* as centre are the elevations of the paths of C, D, E, the plans of these paths being C_0c, D_0d, E_0e, perpendicular to *th*.

The problem is thus solved. The plane of the triangle is inclined at θ, and the side CD at α to the ground.

253. Problem.—**Draw the plan of a cube of given edge, with one face inclined at θ, and a diagonal of that face at α.**

Begin, as in the last problem, by finding L_0M_0, the rabatment of a line inclined at α, lying in a plane inclined at θ. The construction lines for this are not shown.

Draw the square $A_0B_0C_0D_0$, of the given side, with the diagonal A_0C_0 parallel to L_0M_0. Then $A_0B_0C_0D_0$ may be taken as the rabatment of the face of the cube, and also as the plan of the cube in the rabatted position.

The plane of rabatment must now be turned back into the original position *vth*, and the elevation of the face $ABCD$ in the plane obtained as in the last problem.

The elevation of the cube is completed by drawing the lines $a'a'$, $b'b'$, $c'c'$, $d'd'$ perpendicular to *tv*, each equal to the edge of the cube, and then joining $a'c'$.

The plan of the cube is now found by projecting the points in elevation on to the lines through corresponding points of the rabatment, drawn at right angles to *th*.

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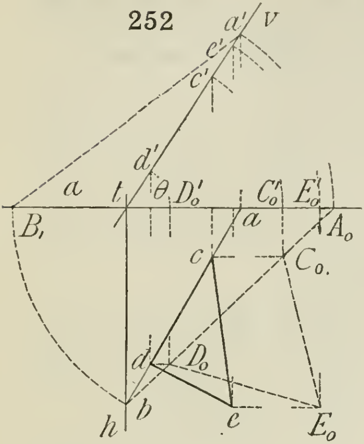

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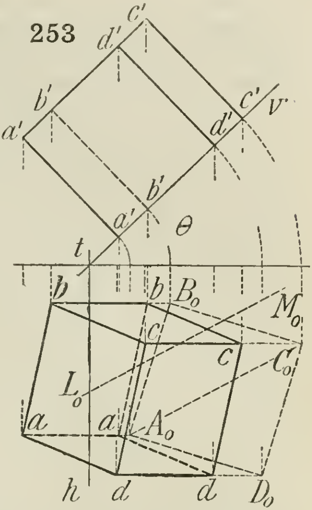


Examples.—1. Draw the plan of a square, 2″ side, the plane of which is inclined at 50°, one side being inclined at 35°.

2. Draw the plan of a square *ABCD*, 2″ side, when the line joining *A* to the middle point of *BC* is inclined at 35° and the plane of the square at 50°, the point *A* being on the ground.

Determine the inclinations of the diagonals of the square.

3. Draw the plan of a regular hexagon of $1\frac{1}{8}$″ side in any position such that its plane is neither horizontal nor vertical.

4. A regular hexagon of $1\frac{1}{4}$″ side has one side in the horizontal plane. The plane of the hexagon is vertical, and inclined at 43° to the vertical plane of projection. Draw the elevation of the hexagon.

5. Draw the plan of a cube of $2\frac{1}{2}$″ edge, one face being inclined at 50° and one side of that face at 35°.

6. An isosceles triangle, base 2.5″, sides 3″, has its base inclined at 35°, and its plane at 50°; this is the end of a right prism $2\frac{1}{2}$″ long. Draw the plan and elevation of the solid.

7. Draw the plan of a square pyramid, side of base 2″, height 3″, when the base rests on a plane inclined at 60°, one diagonal of the base making 50° with the horizontal trace of the plane of the base.

254. PROBLEM.—**An octahedron, 2″ edge, has one face resting on a plane inclined at 30°, one side of the face making an angle of 70° with the horizontal trace of the plane. Draw the plan and an elevation of the solid.**

Let VTH be the plane inclined at 30°. Draw any line L_0M_0 making an angle of 70° with the horizontal trace, and construct an equilateral triangle $A_0B_0C_0$, having one side in the line L_0M_0. Let $A_0B_0C_0$ be the rabatment of that face ABC of the octahedron which rests on the plane at 30°, AB being the side of the face which makes an angle of 70° with the horizontal trace. Complete the plan of the solid in its rabatted position as follows:—

Inscribe a circle in $A_0B_0C_0$, and draw f_1d_1, d_1e_1, e_1f_1, tangential to the circle, respectively parallel to A_0B_0, C_0A_0, B_0C_0. Then $d_1e_1f_1$ is an equilateral triangle, and is the plan of the upper face of the octahedron. The plan of the solid is completed by drawing the outline shown.

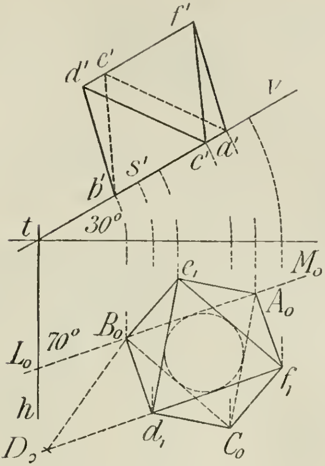
Now let the plane of rabatment be turned back to the original position vth; the elevation $a'b'c'$ of the face ABC is readily found as in previous problems, and the construction is evident from the figure.

To determine the elevation of the face DEF, first find the distance between the parallel faces ABC, DEF. This is done in the figure by finding B_0D_0, the rabatment of BD about its plan B_0d_1. That is, d_1D_0 is drawn at right angles to the plan, and B_0D_0 made equal to BD, $1\frac{1}{2}''$. Then d_1D_0 is the distance between the faces.

To find the elevation of D determine the point s', as shown, s' being the elevation of the foot of the perpendicular from D to the plane of the face ABC, and draw $s'd'$ perpendicular to tv and equal to d_1D_0. In a similar manner e' and f' may be found. The elevation of the octahedron is then completed by drawing the lines representing its edges.

The determination of the plan is left as an exercise.

Example.—An octahedron 2″ edge has a face inclined at 60°, and one side of that face at 40°. Draw its plan.



Examples on Problems 255 to 258.

1. The sides of a square are inclined respectively at 30° and 45°; draw the plan and determine the inclinations of the diagonals.
2. Draw plan of a square pyramid, side of base $1\frac{1}{2}''$, height $1\frac{3}{4}''$, when two sides of the base are inclined at 30° and 40° respectively. Determine the inclinations of the sloping edges.
3. Determine the plan of a hexagon of 1″ side, when two adjacent sides are inclined 30° and 40° respectively.
4. Draw the plan of a regular hexagon *ABCDEF*, $1\frac{1}{4}''$ side: (*a*) when two diagonals are inclined at 30° and 50° respectively; (*b*) when two alternate sides are inclined at 30° and 50° respectively; (*c*) when *AE* and *CF* are inclined at 30° and 50° respectively.
5. Draw the plan of a regular tetrahedron, 2″ edge, when a face and an edge are inclined respectively at 55° and 40°. Show an elevation of the solid on a vertical plane which makes an angle of 30° with the edge which is inclined at 40°.
6. Obtain the projections of an equilateral triangle, 2″ side, when two sides are inclined at 30° and 50° to the horizontal plane, and the third side at 40° to the vertical plane.
7. Two lines meet at an angle of 60°. The plane containing them is inclined at 40°, and one of the lines is inclined at 50°. Find the inclination of the other.

255. PROBLEM.—**To draw the plan of a square, having given the inclinations α_1 and α_2 of two of its sides.**

Let AB, AD be the sides of the square which are inclined respectively at α_1 and α_2, and suppose the point A to be on the ground.

Commence the solution by drawing a square $A_0B_0C_0D_0$; let this be the rabatment of $ABCD$ from the required position, into the ground, about the horizontal trace of the plane containing the square.

The horizontal trace must now be found; it is a line through A_0 such that if the rabatted square be turned back about this trace, until AB be at the inclination α_1, then *at the same time* AD shall be at the inclination α_2.

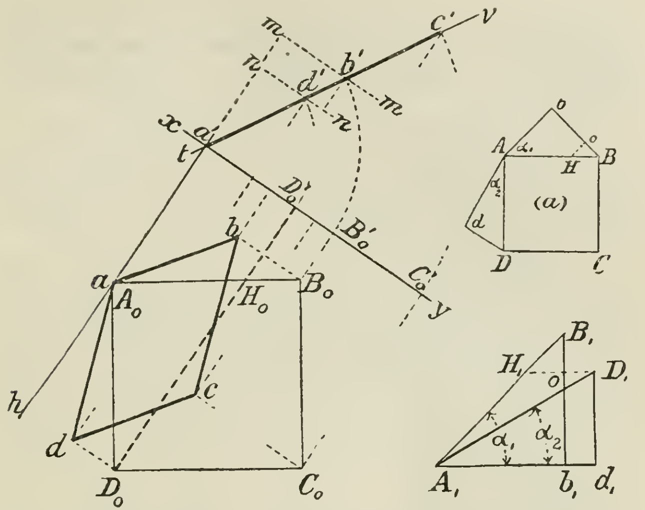
Draw the right-angled triangles $A_1B_1b_1$, $A_1D_1d_1$, having the base angles respectively equal to α_1 and α_2, and the sides A_1B_1, A_1D_1 each equal to the side of the square; draw D_1H_1 parallel to the base. Then, as in Art. 179, B_1b_1, D_1d_1 are the *heights* of B and D above A, *i.e.* above the ground; and A_1b_1, A_1d_1 are the *lengths* of the plans of AB, AD. Also H_1 in A_1B_1 determines the position of a point H in AB, which is at the same height as D.

Make $A_0H_0 = A_1H_1$, and join D_0H_0. Then D_0H_0 is the rabatment of a *horizontal line* in the plane of the square. The horizontal trace of the plane of rabatment is therefore a line through A_0 parallel to D_0H_0; it is denoted by *th* in the figure.

Finally, the required plan may be obtained in two ways.

First Method.—With centre A_0, radii A_1b_1, A_1d_1, the lengths of the plans of AB, AD, describe arcs intersecting the lines dawn through B_0, D_0, perpendicular to *th*, in *b* and *d* respectively; join *ab*, *ad*, and draw *dc*, *bc* parallel to *ab*, *ad*. Then *abcd* is the required plan of the square. B_0b, D_0d are the plans of the circular paths of B and D traced during the rabatment.

Second Method.—Take an *xy* perpendicular to D_0H_0, or *th*; draw the two horizontal lines *mm*, *nn*, at heights above *xy* equal to B_1b_1, D_1d_1 respectively. With centre *t*, describe



arcs through B_0', D_0' (projected from B_0, D_0), intersecting the horizontal lines *mm*, *nn* in b' and d' respectively. Then b', d' are the elevations of *B*, *D*, a' is the elevation of *A*, and a', d', b' will lie in one straight line, which is the edge elevation of the plane of the square. The arc $C_0'c'$ determines c'; and the plan of the square is obtained by drawing projectors from the points in elevation, to intersect lines from the corresponding points of the rabatment, drawn at right angles to *th*, the horizontal axis of rotation.

A model may be made to illustrate this problem.

Draw a square *ABCD*, Fig. (*a*), and draw the lines *Ab*, *Ad*, making angles of α_1, α_2 with *AB*, *AD*. Draw *Bb*, *Dd* perpendicular to *Ab*, *Ad*. Make *bo* equal to *Dd*, and draw *oH* parallel to *bA*. Cut out this figure in paper, then indent and fold the triangles *ABb*, *ADd* about *AB*, *AD* to such a position that when *Ab*, *Ad* rest on the ground, *b* and *d* are the plans of *B* and *D*.

By studying this model, the reasons for drawing the various construction lines should be quite evident.

256. PROBLEM.—**To draw the plan of a regular hexagon of given side: (a) when the inclinations of two diagonals are given; (b) when the inclinations of two alternate sides are given.**

Draw $A_0B_0C_0D_0E_0F_0$, the rabatment of the hexagon.

(a) The rabatment C_0H_0 of a *horizontal line* is found with the aid of Fig. (*a*), as in Prob. 255. The horizontal trace of the plane of rabatment is parallel to B_0H_0, and may be taken through O_0.

The plan is then determined as in Prob. 255, either by the first or second method. If the first method be adopted, and the plan of OAB be thus found, the plan of the hexagon can be completed, without drawing an elevation, by making use of the following proposition of pure solid geometry:—"*If a number of straight lines are parallel to each other and of equal lengths, then their projections on any plane are also parallel to each other and of equal lengths.*"

Further, the elevation on any xy can be obtained without first drawing the edge elevation, by making use of the properties of the perpendiculars A_1a_1, B_1b_1, as explained in Art. 179.

(b) Let the sides AF, DE be those having the given inclinations. Produce them to meet in P.

Then by the aid of Fig. (*b*) the rabatment D_0K_0 of a *horizontal line* in the plane of the hexagon is determined.

The horizontal trace of the plane of rabatment is parallel to D_0K_0, and may be taken through P_0.

The plan can be obtained by either of the methods of Prob. 255, making use of the proposition stated above if the first method be the one adopted.

257 PROBLEM.—**An equilateral triangle has two of its sides AB, AC inclined at α_1 and α_2 to the horizontal plane, and the third side BC inclined at β to the vertical plane; determine its plan and elevation.**

Obtain the plan as in Prob. 255, by first drawing the

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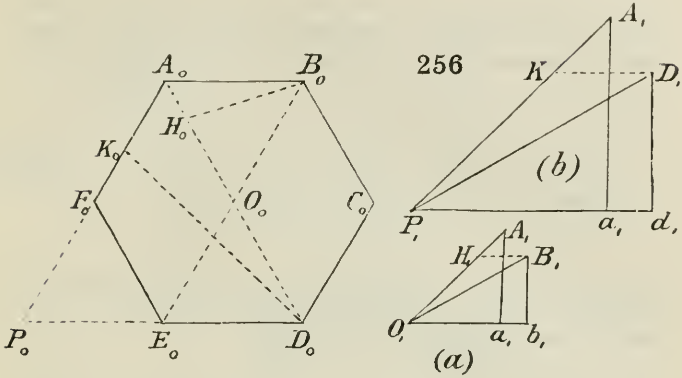

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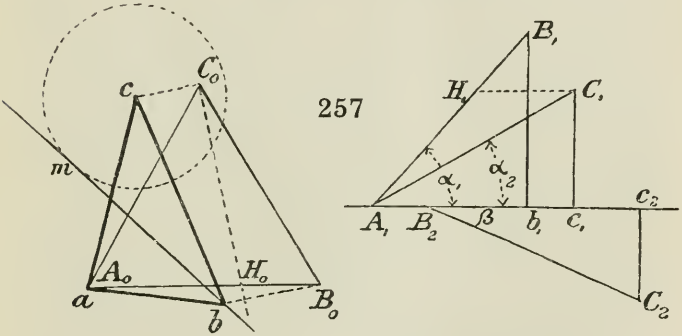

rabatment $A_0B_0C_0$; then by the aid of the triangles $A_1B_1b_1$, $A_1C_1c_1$ find C_0H_0 the rabatment of a *horizontal line*; and finally determine the plan *abc* by the first method.

To find the elevation, first draw a right-angled triangle $B_2C_2c_2$, where $B_2C_2 = BC$, and the angle $C_2B_2c_2 = \beta$. With centre c, radius C_2c_2, the *difference* in the distances of B and C from the vertical plane (Art. 179), describe the circle shown, and draw the tangent *bm*.

Then the plan *abc* and an elevation on an *xy* parallel to *bm* will satisfy the conditions. Compare with Prob. 180.

258. PROBLEM.—**Two lines meet at an angle of 65°. The plane containing them is inclined at 50°, and one of the lines is inclined at 40°; find the inclination of the other.**

Represent a plane inclined at 50°, in which place a line AB inclined at 40°, and find its rabatment A_0B_0.

Draw a line A_0C_0 making 65° with A_0B_0.

Determine the plan and elevation of AC when the plane of rabatment has returned to its original position.

Then find the true inclination of AC.

259. PROBLEM.—**Draw the plan of a regular tetrahedron, $1\frac{1}{2}''$ edge, three of its corners being at heights of $\frac{1}{4}''$, $1\frac{1}{4}''$, and $1\frac{1}{2}''$ respectively above the ground.**

Draw $A_0B_0C_0d_1$ the plan of the tetrahedron, with the face ABC on the ground, and consider this the plan of the solid in its rabatted position.

In the two triangles shown to the right, make $A_1B_1 = A_1C_1 = 1\frac{1}{2}''$, the length of edge of the solid, and B_1b_1, C_1c_1 respectively equal to $1\frac{1}{4}''$ and $1''$, the heights of B and C above A. Draw C_1H_1 parallel to c_1A_1, and make $A_0H_0 = A_1H_1$; then C_0H_0 is the rabatment of a *horizontal line* in the face ABC.

Take xy perpendicular to C_0H_0, and determine $a'b'c'$ the elevation of ABC, as in Prob. 255, second method, assuming the point A in the ground, and drawing the horizontal lines mm, nn, at distances equal to B_1b_1, C_1c_1, above xy.

To determine the elevation of D, first find d_1D_0, the distance of D from the face ABC, by the rabatment of CD about its plan as in the figure, or by any other method. Obtain the point s' as shown, and draw $s'd'$ perpendicular to $a'b'$ and equal to d_1D_0. Then d' is the elevation of D, and the elevation of the tetrahedron is completed by drawing the lines representing the edges.

The plan is to be obtained as in previous problems, and is left as an exercise for the student.

Finally, if an xy be drawn parallel to the one shown, and $\frac{1}{4}''$ below it, all the conditions of the problem are satisfied.

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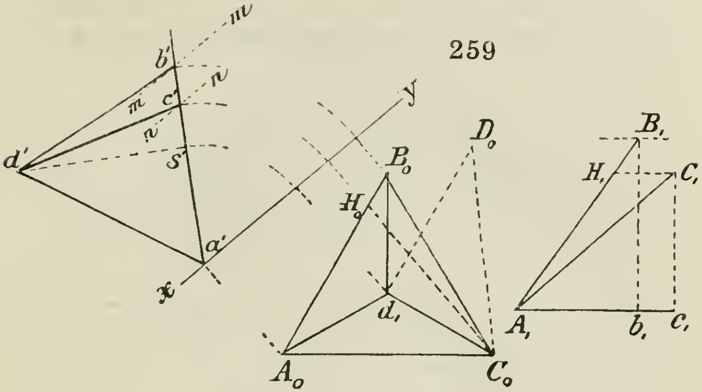


Examples.—1. Determine the plan of a hexagon of $\frac{3}{4}''$ side, when three alternate angular points are $1''$, $1\frac{1}{4}''$, and $1\frac{3}{4}''$ high respectively.

2. Draw the plan of a square $2\frac{1}{4}''$ side, when the heights of its centre and two corners, not opposite each other, are $1''$, $1\frac{3}{4}''$, and $2\frac{1}{2}''$ respectively.

3. Draw the plan of an isosceles triangle, base $2\frac{1}{2}''$, sides $2''$, when the extremities of the base and the middle point of one of the sides are at heights of $\frac{3}{4}''$, $1\frac{1}{2}''$, and $2''$ respectively.

4. Draw the plan of an equilateral triangle, $3''$ side, when the three middle points of the sides are at heights of $1''$, $1\frac{1}{4}''$, $1\frac{3}{4}''$ respectively.

5. Draw the plan of a regular tetrahedron, $2\frac{1}{4}''$ edge, the heights of three of its corners above the horizontal plane being respectively $\frac{1}{2}''$, $1\frac{1}{2}''$, $2''$.

6. Draw the plan of a cube, $2\frac{1}{2}''$ edge, when three of its angular points are at heights of $1''$, $1.25''$, and $2''$ above the horizontal plane. Make an elevation on a plane parallel to one of the diagonals of the solid.

7. An isosceles triangle is the plan of an equilateral triangle. Find the inclination of the plane of the triangle (1) when the equal sides are each three-fourths of the base; (2) when the base is three-fourths of each of the equal sides.

8. Draw the plan of an octahedron of $2''$ edge, when two diagonals of the solid are inclined at 26° and 36° respectively.

9. Three corners of a square, $2''$ side, are $1''$, $1.4''$, and $2.1''$ high; find the height of the centre.

260. PROBLEM.—**To draw the plan of a cube having given the length of edge and the inclinations θ and ϕ of two faces.**

Draw the traces *vth* of a plane inclined at θ.

By Prob. 233 determine a plane at right angles to *VTH* and inclined at ϕ. That is, choose any point *A* in the plane and draw the projections of a cone, vertex *A*, base on ground, base angle ϕ. Find also the horizontal trace *N* of a line through *A* perpendicular to the plane *VTH*.

Then the tangent *nlm* is the horizontal trace of the plane at ϕ.

The line *ias* is the plan of the intersection of the planes. If now one edge of the cube coincide with *IA*, and two faces with the two planes, the conditions will be satisfied.

Rabat the plane *VTH*, and thus obtain iS_0.

Draw the square with one side in iS_0; this is the plan of the cube while on the ground. Now let the plane *VTH* be raised to its proper position, and draw the corresponding plan and elevation of the cube.

261. PROBLEM. — **To draw the plan of a tetrahedron, having given the length of edge and the inclinations θ and ϕ of two faces.**

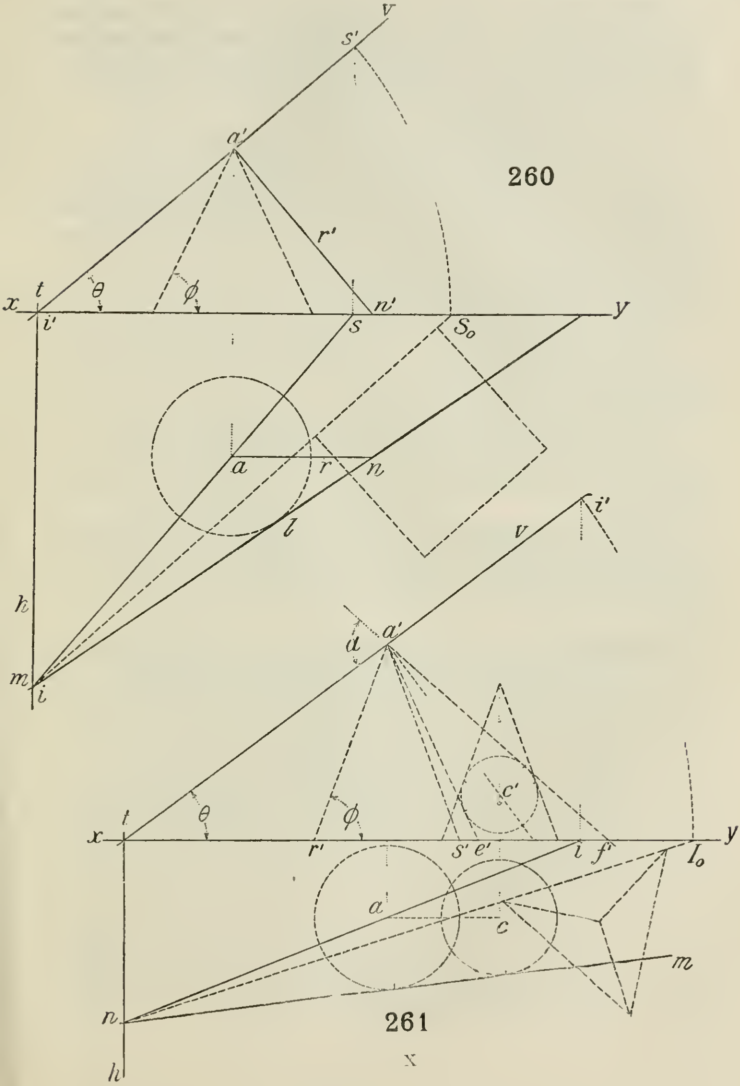
Draw the traces *vth* of a plane inclined at θ.

Then by Prob. 234 determine the traces of a plane which is inclined at ϕ, and makes with the plane *VTH* an angle α equal to that between the base and one of the equal faces of the tetrahedron.

Determine the intersection of these two planes and rabat it, along with the inclined plane, into the ground.

Proceed now exactly as in the last problem, by drawing an equilateral triangle having one side (equal to the given edge) in the rabatment of the intersection.

While the inclined plane is in the ground complete the plan of the tetrahedron. Finally suppose the inclined plane to be raised to its proper position, and thus complete the required plan and elevation of the tetrahedron.



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262. PROBLEM.—**To draw the plan of a cube, having given the length of edge, the inclination θ of one face, and the inclination ϕ of a diagonal of the solid.** (No figure.)

There are three preliminary problems to be worked.

1. Find the length of the diagonal.
2. Find the angle α between a diagonal and a face.
3. Find the angle β between a diagonal and an edge.

Having solved these, draw the projections of a diagonal inclined at ϕ. Since a plan only is required, the diagonal may be parallel to the vertical plane.

Then by Prob. 235 determine a plane inclined at θ, and making an angle α with the diagonal.

Next obtain the plan of an edge which meets the diagonal and lies in the plane. This is done by Prob. 236, the edge making β with the diagonal.

Now rabat the plane with the edge into the ground.

On the rabatment of the edge construct a square. This will be the plan of the cube while one of its faces is on the ground.

Now turn the plane back into its original position, carrying the cube with it, and thus complete the plan of the solid by well-known methods.

Examples.—1. Draw the plan of a cube, $2\frac{1}{2}''$ edge, when two of its adjacent faces are inclined at 45° and 75° respectively.

2. A building brick $9'' \times 4\frac{1}{2}'' \times 3''$ has one end inclined at 40°, and a long face at 60°. Draw its plan. *Scale* $\frac{1}{3}$.

3. A tetrahedron, $2\frac{1}{2}''$ edge, has two of its faces inclined at 40° and 70°. Draw the plan and elevation.

4. A square pyramid, base $2''$ side, axis $3''$ long, has its base inclined 40°, and one of its faces at 60°. Draw its plan, and a sectional elevation on a vertical plane which bisects any two of its long edges.

5. A cube, $2\frac{1}{2}''$ edge, has a diagonal inclined at 60°, and a face at 65°; determine its plan.

6. Draw the plan of a regular tetrahedron of $2''$ edge, when the line joining a vertex to the centre of the opposite face is horizontal, one of the other faces being inclined at 60°. What is the least angle which may be substituted for the 60°, so as not to make the solution impossible? *Ans.* $19\frac{1}{2}°$.

263. Miscellaneous Examples.

1. Draw the plan of a hexagon of 1.5″ side, the heights of three successive adjacent corners to be 1″, 1.5″, and 0.75″. (1878)

2. An isosceles triangle (sides 2½″, base 1¾″) has its base in the horizontal plane and one side in the vertical plane. The base makes an angle of 35° with *xy*. Determine the plan and elevation of the triangle. (1893)

3. A square of 3″ side lies on a plane inclined at 50°. One side of the square makes 40° with the horizontal trace of the plane. Draw its plan. (1885)

4. Draw the plan of a cube of 1½″ edge, one face inclined at 50° and a second face at 60°. (1890)

5. Draw the plan of an octahedron of 2″ edge, one edge being inclined at 30°, and another (on the same face) at 20°. (1882)

6. Draw the complete plan of a cube of 1.5″ edge, two faces inclined respectively at 60° and 70° to the horizontal plane. (1895)

7. A right pyramid has for its base a regular pentagon of which the diagonals measure 2.5″. The vertex is 2″ above the base. Draw the plan and elevation of the pyramid, with its base in a plane inclined at 55° to the vertical plane, and at 60° to the horizontal plane; one diagonal inclined at 30°, and one end of that diagonal in the vertical plane. (1894)

8. Draw a plane inclined at 45° to the horizontal plane, and at 60° to the vertical plane. Draw a line in the plane inclined at 30°, and 2″ long between its traces. Lastly, draw the plan of a regular hexagon lying in the plane of which the above line is a diagonal. (1897)

9. An octahedron of 2¼″ edge has the plane containing two of its diagonals inclined at 30°, and that containing one of these, and the other diagonal inclined at 70°. Draw its plan.

10. Determine the projections of a cube on three planes mutually perpendicular, having given the inclinations of three adjacent edges, one to each plane.

11. Determine the projections of an equilateral triangle on three planes mutually perpendicular, having given the inclinations of its sides, one to each plane.

12. Draw the projections of a cube on three planes mutually perpendicular, having given the inclinations of three adjacent faces, one to each plane.

CHAPTER XIII

THE PROJECTION OF CURVES AND CURVED SURFACES

264. General method of procedure.—In the problems hitherto considered, the figures represented in projection have been made up of points, straight lines, and planes. The projection of curves, and of solids bounded by curved surfaces, is now to be considered.

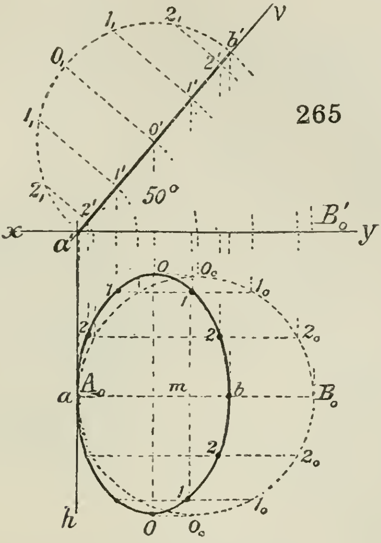
The general method of projecting a curve is to first find the projections of a number of isolated points in it; then to draw a curve by freehand through the points thus determined.

In projecting a solid, the projections of the straight or curved edges are first found; then if necessary the outline of the projection is completed, as determined by projectors which *touch* the surface of the solid. This will be more particularly explained in the problems which follow.

265. PROBLEM.—**Draw the plan of a circle, $1\frac{1}{4}''$ diameter, when its plane is inclined at 50°.**

First Method.—Begin with the rabatment, a circle A_0B_0, $1\frac{1}{4}''$ diameter. Take twelve equidistant points on the circumference of this circle, as shown. Find the plans and elevations of these points when the plane of rabatment is turned back into its original position *vth*, inclined at 50°.

The line $a'b'$ is the edge elevation of the circle; and a fair curve carefully drawn through the plans of the points is the plan of the circle. The plan is an *ellipse*.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Second Method.—Begin by drawing an edge elevation of the circle, the line $a'b'$, $1\frac{1}{4}''$ long, inclined at 50°. On $a'b'$ draw a semicircle, which divide into six equal arcs as shown. From the points of division draw lines perpendicular to $a'b'$, intersecting the latter in $0'$, $1'$, $2'$.

Conceive $0'$, $1'$, $2'$ as the elevations of chords of the circle perpendicular to the vertical plane. Then the dotted perpendiculars represent the halves of these chords, when half the circle is turned about AB into a position parallel to the vertical plane.

To obtain the plan of the circle, draw projectors from the points in $a'b'$. Consider the projector from $1'$: let m be its intersection with ab; make $m1$, $m1$ each equal to $1'1_1$; repeat this construction for the other points. In this way the points 0, 1, 2 on the plan of the circle are found, and the ellipse is drawn through them.

266. PROBLEM.—**A given sphere resting on the ground is cut by a given vertical plane. Draw the plan and elevation of the trace of the section plane on the surface, and show a sectional elevation of the sphere.**

The projections of the sphere are circles, equal in diameter to the sphere, drawn with centres c, c', the plan and elevation of the centre of the sphere. Since the sphere rests on the ground, the elevation touches xy.

Let VTH be the given vertical section plane. The section is a circle of which ab is the plan. The elevation may be found as follows:—

On ab as diameter draw a semicircle, and divide its circumference into six equal arcs as shown. From the points of division draw lines perpendicular to ab, intersecting the latter in 0, 1, 2.

Let 0, 1, 2 be the plans of vertical chords of the circle. Then the dotted perpendiculars represent the halves of these chords, when half the circle is turned about AB into a horizontal position.

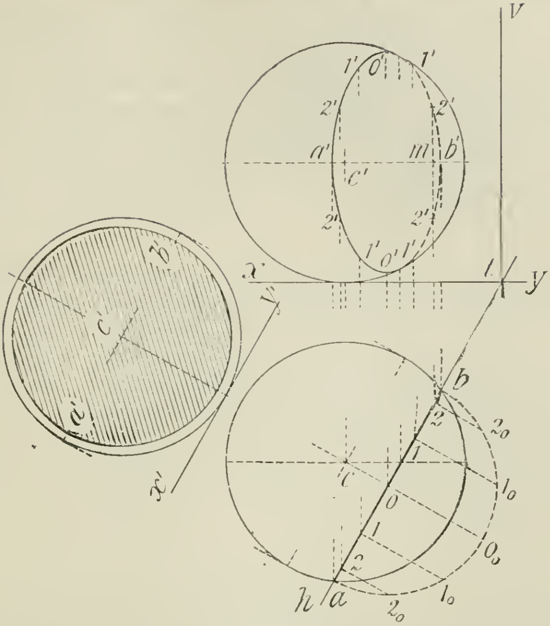
Draw projectors from the points on ab. Consider the projector from 2: let m be its intersection with the horizontal line through c'; make $m2'$, $m2'$ each equal to $2_0 2$; repeat this construction for the other points, and through the points in elevation thus found draw a curve, which is an ellipse, and is the elevation of the required trace of the section plane on the surface of the sphere.

The sectional elevation is that on $x'y'$, taken parallel to ab. The method of obtaining this needs no further explanation.

Examples.—**1.** Draw the plan of a circle $2\frac{1}{2}''$ diameter, when its plane is inclined at 50°.

2. Draw the elevation of a circle $2\frac{1}{2}''$ diameter whose plane is vertical and inclined at 60° to the vertical plane.

3. A hemisphere, $2\frac{1}{2}''$ diameter, rests with its flat face on the ground. It is cut by a vertical plane $\frac{1}{2}''$ distant from its centre, which makes an angle of 50° with xy. Draw the elevation, showing the curve of intersection.



4. A sphere 3″ diameter is cut into four equal parts by two perpendicular planes through its centre. Draw the plan of one of these parts when resting with a flat face on the ground; and give a sectional elevation on a vertical plane which makes 45° with the straight edge of the solid, and is distant 1″ from the centre.

5. A sphere, 4″ diameter, is cut into eight equal parts by three planes mutually perpendicular. Draw the plan of one of these parts when resting on a flat face. Also draw an elevation on a vertical plane, which makes an angle of 30° with a horizontal edge of the solid, so as to show the flat faces.

6. Draw the plan of the solid of Ex. 5 when two of its edges are inclined at 30° and 40°. And add an elevation on a vertical plane, the real angle between which and the third edge is 25°.

7. Draw the plan of the solid of Ex. 5 when its curved surface rests on the ground, two faces being inclined each at 70°.

267. Problem.—**A cone, diameter of base $1\frac{1}{2}''$, length of axis $2''$, rests with its base on the ground, and is cut by a plane inclined at 45°, bisecting the axis. (a) Draw the plan and elevation of the cone, showing the trace of the section plane on its surface; (b) find the true shape of the section; (c) obtain a development of the surface of the cone, showing the trace of the cutting plane on the surface.**

(a) The plan of the cone is the circle *ab*, with *s*, the plan of the vertex, as centre. The elevation is the triangle $s'a'b'$.

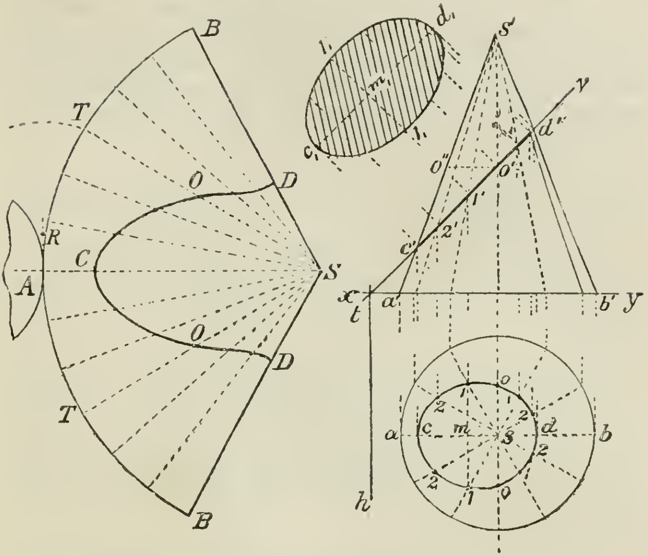
To find the section, take twelve equidistant points on the circumference of the base, join these to the vertex, and draw the plan and elevation of the twelve generating lines thus formed. The intersection of the cutting plane with these generators gives twelve points on the curve of section required. The elevations of the points are in $c'd'$, and the plans are found by projecting from the points in elevation.

A special construction is required to determine the plans *o*, *o*; in this case it is evident that *so*, *so* are each equal to $o'o''$, where $o'o''$ is parallel to *xy*. The curve through the plans of the points is an ellipse.

(b) The true shape of the section may be found by a rabatment of the cutting plane about its vertical trace, which is equivalent to an auxiliary plan on *tv*. The rabatment of *CD* is c_1d_1, and to obtain the points on the curve, make $m_1 1_1$, m, 1, each equal to $m1$; and repeat this construction for the other points. The true shape of the section is an ellipse.

(c) To obtain the development. With *S* as centre, describe an arc with radius equal to $s'a'$, the length of a generator.

Take any point *A* on this arc, and by Rankine's construction, Prob. 113, make arc *AT* equal to $\frac{1}{4}$ circumference of base of cone; and set off $TB = AT$. Subdivide each of these arcs into three equal parts. The development of the curved surface of the cone, with that of the twelve equidistant generators, is thus obtained.



Make $SO = s'o''$ = true distance of the vertex from the point O; and repeat this construction for the other points. A curve through these points is the development of the trace of the cutting plane on the surface of the cone.

Examples.—1. Determine the true shape of the section of a cone $2\frac{1}{2}''$ base, $3''$ axis, by a plane parallel to its axis, the distance between the plane and the axis being $\frac{1}{4}''$. Develop the cone and show the curve of section.

2. Determine the shape of the section of a cone $2''$ base, $3''$ axis, by a plane parallel to a tangent plane, and distant $\frac{1}{4}''$ from the latter.

3. A cone of which the altitude = radius of base = $1\frac{1}{2}''$ is cut in two by a plane containing the vertex, the trace of the plane on the base bisecting a radius at right angles. Draw the plan of the smaller part when resting with its triangular face on the ground; and obtain an elevation on a ground line which makes an angle of 40° with the chord of the segmental base.

268. PROBLEM.—**A cone, diameter of base $1\frac{1}{2}''$, length of axis $2''$, rests with a generator on the ground, the axis being parallel to the vertical plane. Draw the plan and elevation of the cone; and add a sectional elevation on a vertical plane which bisects the plan of the axis at an angle of 45°.**

The elevation of the cone is the triangle $s'a'b'$, copied from the triangle $s'a'b'$ of Fig. 278, but with the generator SA on the ground.

The plan of the circular base is an ellipse determined as in Prob. 265, second method; and the plan of the cone is completed by drawing the two tangents to the ellipse from s.

Each of these tangents is the line generated by the foot of a vertical projector, which projector moves so as always to touch the curved surface of the cone. It may be shown by pure solid geometry that the moving projector touches the surface along two generators, one on each side of the cone; the two tangents from s to the ellipse are the plans of these generators.

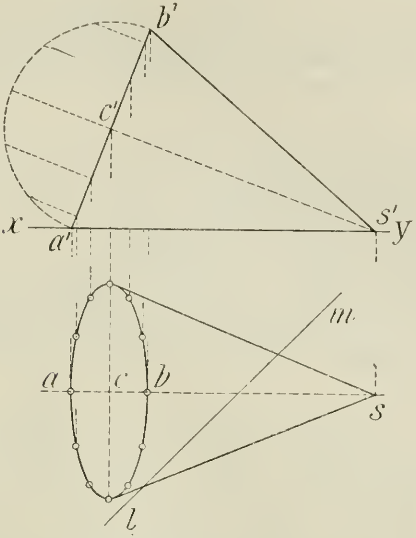
Let lm, bisecting sc at 45°, be the plan of the vertical section plane. To obtain the sectional elevation, take a new ground line $x'y'$ parallel to lm, and find the elevation on $x'y'$ of the vertex and the points on the base according to the rules of Art. 170.

Draw the twelve generating lines in both plan and sectional elevation, whence twelve points on the section are at once found by projection from the plan.

The drawing of this view is left as an exercise for the student.

Examples.—1. A cone, $2''$ base, $3''$ axis, rests with a generator on the ground; draw its plan, and a sectional elevation on a vertical plane which bisects the plan of the axis at 45°.

2. Draw the plan and elevation of a cone $2''$ base, $3''$ axis, when the axis is inclined at 30° to the horizontal plane and 45° to the vertical plane. Show the section by a horizontal plane which bisects the axis.



Examples on Problems 270 to 272.

1. A cylinder, $1\frac{1}{2}''$ diameter, $2''$ long, has a cone $2\frac{3}{4}''$ base, $1\frac{1}{2}''$ axis, placed centrally on one of its ends. Draw the plan of the solid when the bases of both cone and cylinder touch the ground. Draw a sectional plan on a plane which contains the vertex and two points on the circumference of the base of the cone $2\frac{1}{4}''$ apart.

2. Draw a semicircle $5''$ diameter, in which inscribe the largest possible circle. The semicircle is the development of a cone, and the circle that of a line on its surface. Draw the plan and an elevation of the cone when resting with its base on the ground, showing the line on its surface.

3. Draw a sector of a circle $3''$ radius, containing an angle of 120°. This sector is the development of the surface of a cone. Determine the plan and elevation of the cone when resting with its base on the ground.

Suppose a fly, starting from a point in the circumference of the base, to walk round the surface and return to the same point. Show the plan and elevation of the path, when its length is the least possible.

269. PROBLEM.—**A cylinder, 2″ diameter, is 3″ long. Draw the plan and elevation: (a) when the axis is parallel to the vertical plane, and inclined at 45° to the horizontal plane; (b) when the axis is inclined to both planes of projection.**

(a) Draw the rectangle $a'a'b'b'$ having $b'b' = 3''$, $b'a' = 2''$, and $b'b'$ making an angle of 45° with xy. Draw $c'c'$ bisecting the short sides of the rectangle. Then $a'a'b'b'$ is the elevation of the cylinder, and $c'c'$ that of the axis; $a'b'$, $a'b'$ are the elevations of the two circular ends.

The plan is obtained by projecting the two circular ends from the elevation, in the manner of Prob. 265, second method. The two ellipses thus found are joined by two common tangents as shown in the figure, and the plan is complete.

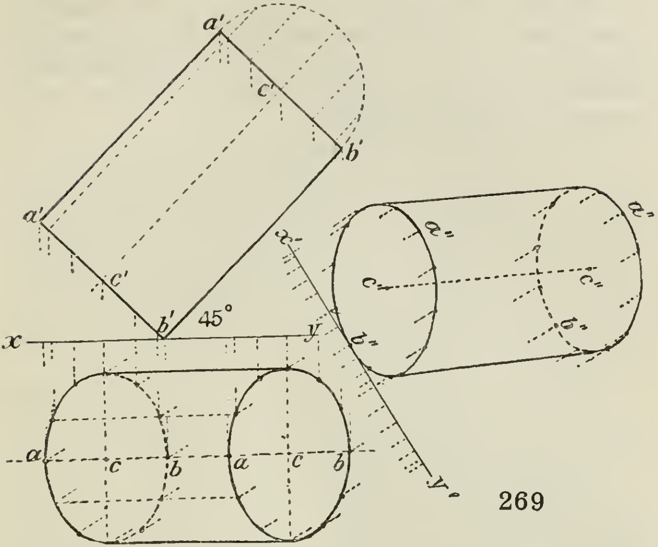
(b) An elevation on any line $x'y'$, not parallel to cc, determined according to the principles of Art. 170, will satisfy the requirements of (*b*). The two ends project into ellipses, common tangents to which complete the elevation.

The common tangents to the ellipses, and the lines $a'a'$, $b'b'$, are the projections of the generators along which the projectors touch the curved surface of the cylinder in each case.

Examples.—**1.** A cylinder, 2″ diameter, is $2\frac{3}{4}''$ long, draw the plan and elevation: (*a*) when the axis is parallel to the vertical plane, and inclined 40° to the horizontal plane; (*b*) when the axis is inclined at 50° and 40° to the vertical and horizontal planes of projection.

2. A cylinder, $2\frac{1}{2}''$ base, $3\frac{1}{2}''$ long, is cut into two parts by a plane containing the axis. Draw the plan of one of these parts, when resting with a rectangular face on the ground, and draw a sectional elevation on a vertical plane bisecting the axis at an angle of 45°.

3. A cylindrical cheese is 16″ diameter, and 5″ thick. A wedge-shaped piece is cut out by two planes containing the axis and including an angle of 50°. Draw the plan of the piece when resting with one of its rectangular faces on the horizontal plane, and draw an elevation on a plane making 25° with the short sides of that face. Scale $\frac{1}{4}$.



269

270. Problem.—**To find the projections of the shortest line on a given developable surface, connecting two given points on the surface.**

First read Prob. 271. The line in question develops into a *straight* line, joining the developments of the points. For the length of the line on the surface is equal to the length of its development, and when the development is straight it is the shortest line possible between the points.

Therefore, first find a development of the given surface and of the given points in it. Draw the straight line between the developed points. Then working backwards from the development, determine the projections of the line. See Fig. 271.

Note.—The shortest line between two points on a cylinder is a helix of uniform pitch ; for the development of this curve is a straight line. See Probs. 369 and 130.

271. Problem.—**The given sector of a circle SBAB is a development of the curved surface of a cone; and HK that of a line on the surface. Draw the plan and an elevation of the cone, when resting with its base on the ground, showing the line HK.**

The diameter of the base of the cone must first be found. Bisect the arc *BB* in *A*, and the arc *AB* in *T*. Draw the tangent *AR* and make *AR* equal to the chord of $\frac{1}{4}$ the arc *AT*. With centre *R* describe an arc through *T*, intersecting in *t* the line *At* drawn at 45° to *SA* produced. Draw *tC* perpendicular to *SC*. Then the circle with centre *C*, radius *CA*, is the development of the base of the cone. For by Rankine's construction, Prob. 113, arc *At* = arc *AT* $= \frac{1}{4}$ circumference of base.

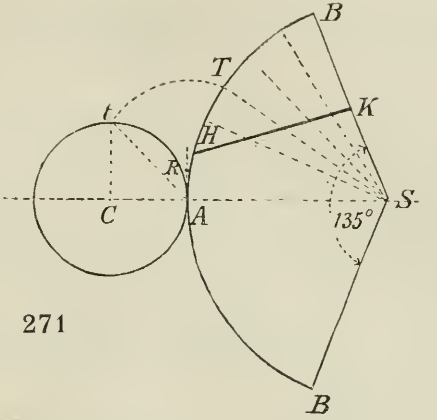
The plan of the cone is thus drawn; and the altitude and the elevation may be found, since the diameter of the base, and the length of a generator, viz. *SB*, are known.

To obtain the plan of *HK*, take a series of generators intersecting the line. In the figure, four of twelve equidistant generators are shown. The construction is then equivalent to that of Prob. 267 (*c*), worked the reverse way, and the completion is left as an exercise.

272. Problem.—**The plan and elevation of a solid are shown in the figure at (a). Draw the plan when the solid is in the position shown in elevation at (b).**

The plan of the cylindrical portion of the solid is found as in Prob. 269.

The curves on the four side faces of the solid are equal and similarly placed circular arcs, being the intersections of the faces with the spherical surface. The points *A*, *B*, *C* on these arcs are similarly and symmetrically taken on the four faces. Hence the distance between *b*, *b* in plan is the same as between *b′*, *b′* on the front face in elevation. This consideration enables the plans of the arcs to be found. The plan of the solid is completed by projecting from the elevation the edges which are straight.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t
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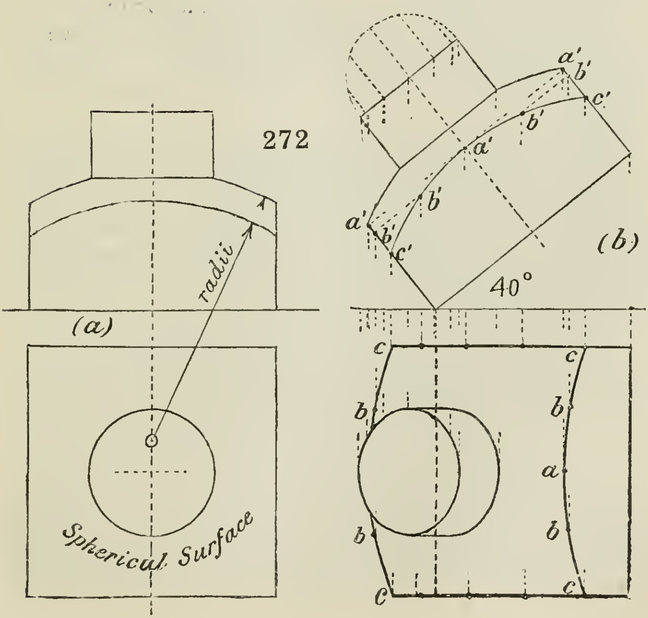
271


radii
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b'
c'
b'
a'
b'
a'
b'
c'
(b)
40°
c
c
b
b
a
b
b
c
c

272

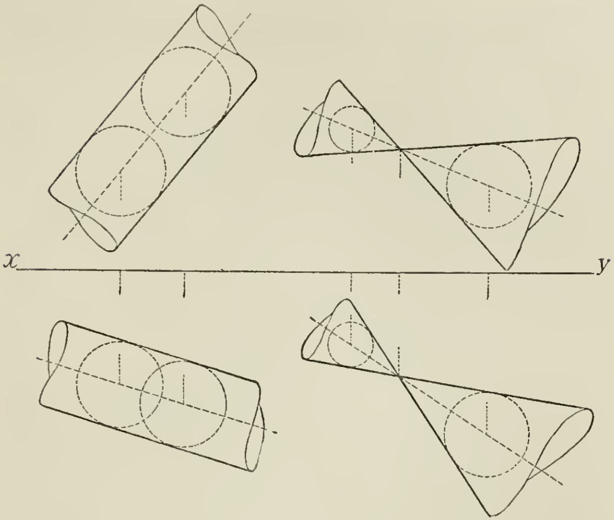
273. Some properties of the cone and cylinder.—Before proceeding with the remaining problems of this chapter we wish to amplify some of the theorems which are given in the Appendix.

Sphere inscribed in cone or cylinder.—Take two intersecting lines of indefinite length and draw the bisector of the angle between them; with any point on this bisector as centre draw the circle touching the lines. Suppose now these lines and the circle to rotate about the bisector as axis, then it is evident that the lines will generate a cone, and the circle a sphere inscribed in the cone, and the sphere will touch the cone in a circle perpendicular to the axis. It is readily seen that an indefinite number of spheres of varying sizes can be inscribed in a cone, all having their centres in the axis.

A cylinder, being the limiting case of a cone where the vertex is at an infinite distance away, may also have an unlimited number of spheres inscribed in it. All these spheres will have a diameter equal to the diameter of the cylinder.

Two cones and a common inscribed sphere.—It can be shown that if two cones circumscribe the same sphere, their surfaces intersect each other in *two ellipses*. Also, since a cylinder is a particular case of a cone, a similar remark applies to two cylinders, and to a cylinder and cone. This property may be frequently taken advantage of to facilitate the working in certain problems.

Projection of a cone or cylinder of indefinite length.—A cone of indefinite length may evidently be defined by any two spheres inscribed in it, for if these two spheres are given, then the line of the axis, the position of the vertex, and angle of the cone are known. One of the spheres may be at the vertex and so become a point; thus a cone of indefinite length is defined by its vertex and an inscribed sphere. The projection of the cone on any plane consists of the two tangents drawn to the two circles which are the projections of the two inscribed spheres. It must be



specified whether the two external or two internal tangents are to be taken. As before, one of these circles may be a point, *i.e.* be the vertex of the cone. Similar remarks apply to a cylinder of indefinite length, but now the two inscribed spheres are of equal size. A cylinder and cone projected on the horizontal and vertical planes are illustrated in the figure. These indefinite cones and cylinders may be drawn broken at the ends as shown.

These statements must be qualified as follows. If the axis of the cylinder be perpendicular to the plane of projection, the projection of the cylinder is a circle which is the edge view of the surface.

If the projection of the vertex of the cone fall within the projection of an inscribed sphere, the common tangents cannot be drawn; in this case the indefinite cone has no definite form of projection, but it may still be represented in projection by the circle and point or by two circles.

274. Problem.—**To determine the horizontal trace of a given cone of indefinite length, whose axis EV is parallel to the vertical plane.**

The cone may be given by its vertex V, vertical angle α, and inclination of axis θ; *or* by its vertex and an inscribed sphere. In either case the projections in outline are readily drawn; let them be as shown in the figure. The horizontal trace is a conic section, in the present case an ellipse.

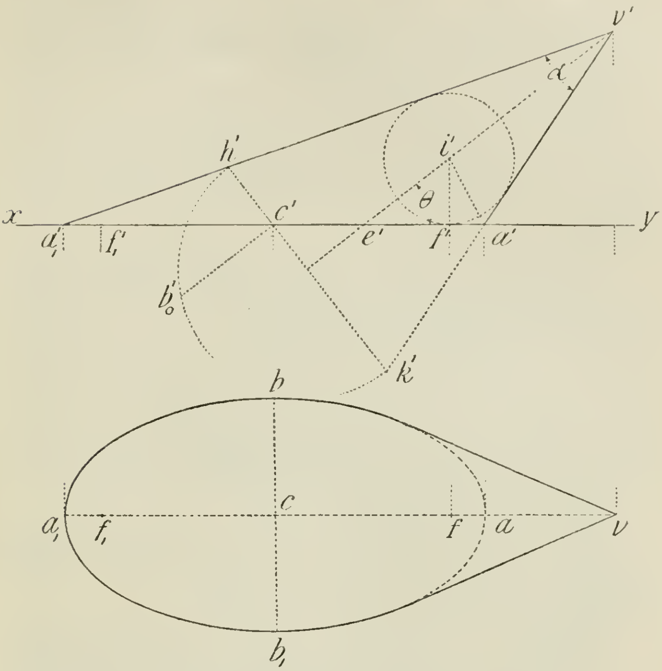
To find the major axis of the ellipse.—Let the outline in elevation intersect xy in a', a_1'; bisect $a'a_1'$ in c'. Project a', a_1' and c' on to the plan of the axis at a, a_1, and c. The horizontal trace of the cone consists of an ellipse of which C in the centre, and AA_1 the major axis.

To set out the ellipse. First Method.—Bisect the angle $v'a'a_1'$ by the line $a'i'$; draw $i'f'$ perpendicular to xy, and with i' as centre and $i'f'$ as radius draw the circle shown; this circle is the elevation of a focal sphere of the cone with respect to the ground as section plane (see Art. 76). Therefore the point of contact f' is the elevation of one focus of the ellipse, and f, projected from f', is its plan. Set off a_1f_1 equal to af_1, then f_1 is the second focus. The minor axis of the ellipse may now be found and the ellipse drawn, as explained in Chapter IV.

Second Method.—Through c', the middle point of $a'a_1'$, draw $h'k'$ perpendicular to $v'e'$. On $h'k'$ describe a semicircle, and draw $c'b_0'$ perpendicular to $h'k'$. Then $c'b_0'$ is the length of the semi-minor axis, and the horizontal trace of the cone can now be completed as before.

To explain this construction, observe that c', being the mid-point of $a_1'a'$, must be the elevation of the minor axis, the minor axis itself being a chord of that circular section of the cone which has $h'k'$ for its elevation. One-half of this circle is turned about HK until it is parallel to the vertical plane; its elevation then being the semicircle on $h'k'$; the semi-minor axis appears as $c'b_0'$.

Third Method.—Repeat the construction of the second method for a number of points in $a'a_1'$ in addition to the



mid-point c'. The lines corresponding to $c'b_0'$ thus obtained will be ordinates of the elliptic trace, and may be set off on each side of va_1, on the projectors from the respective points in $a'a_1'$. This method is useful when the cone is so situated that its trace is a parabola or hyperbola; or an ellipse so elongated that the end A_1 is inaccessible.

Fourth Method.—Draw the plan of the focal sphere or any other sphere inscribed in the cone, and from v draw tangents to this circle. These tangents give the plan of the cone in outline and must touch the trace (see Art. 284). The trace can therefore be drawn, since it is an ellipse, of which the major axis and a tangent are known (see Prob. 98).

275. Problem. — **To determine the horizontal trace of a given cylinder of indefinite length, whose axis ED is parallel to the vertical plane.**

The cylinder may be given by its diameter, the position of a point D in its axis, and the inclination θ of the axis; *or* by two inscribed spheres as explained in Art. 273. In either case the plan and elevation of the *outline* of the cylinder are readily determined. Let them be as shown in the figure.

The horizontal trace of the cylinder, being a plane section, must be an ellipse. To determine the plan of the ellipse we may proceed as follows:—

Let the outline in elevation meet xy in $a'a_1'$; bisect $a'a_1'$ in c'. Project a', a'_1, c' on to the plan of the axis at a, a_1, c, and let the projector from c' intersect the plan of the outline in b, b_1. Then c is the centre, and aa_1, bb_1 the major and minor axes of the elliptic trace. The ellipse can be drawn by any of the methods described in Chapter IV.

The elevation of the trace is $a'a_1'$.

The foci of the ellipse are shown in the figure; these might have been determined by a focal sphere, as was explained for the cone in the last problem.

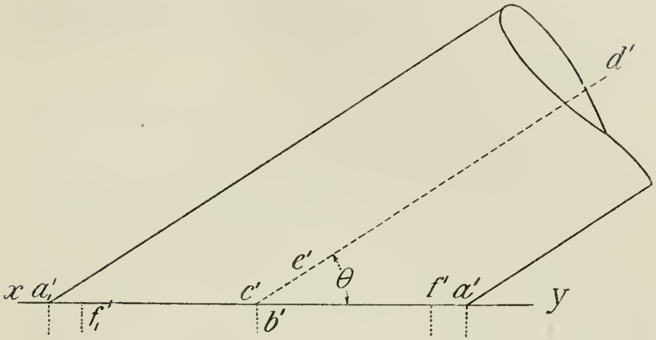
Examples.—1. A circle $\frac{7}{8}''$ diameter, centre s, is the plan of a sphere which rests on the ground. A point v, $1''$ from s, is the plan of the vertex of a cone of indefinite length which circumscribes the sphere, the height of V being $1\frac{1}{2}''$. Draw the plan of the cone and an elevation on a vertical plane parallel to VS. Set out the horizontal trace of the cone.

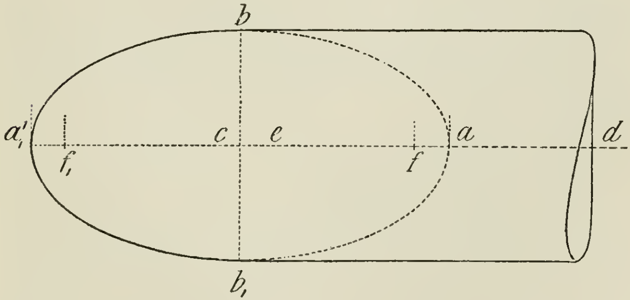
2. A cylinder, $2''$ diameter, has its axis parallel to the vertical plane and inclined 50° to the ground. Determine its horizontal trace.

Show the elevation of the section made by a vertical plane parallel to, and distant $\frac{1}{2}''$ from the axis.

3. Draw a triangle $a'b'c'$ with $b'c'$ in xy; make $b'c'$ $3''$, $a'b'$ $4''$, and $a'c'$ $2\frac{1}{2}''$. This is the elevation of a cone, the plan of whose axis makes 40° with xy. Determine the horizontal trace of the cone. See Prob. 276.

4. A cylinder $2''$ diameter has its axis inclined at 45° to the ground and 30° to the vertical plane. Determine its horizontal trace; also its vertical trace. See Prob. 277.





5. Determine the trace of the cone of Ex. 2, p. 330, on a horizontal plane $\frac{3}{4}''$ above the ground.
6. Determine the section of the cone of Ex. 3, p. 330, by a horizontal plane which touches the smaller sphere at its highest point.
7. A cone, vertical angle 60°, has its axis horizontal and 2″ above the ground. Determine the section by a vertical plane (*a*) which makes 45° with the axis and is distant 2″ from the vertex; (*b*) which makes 20° with the axis, and is distant $\frac{1}{2}''$ from the vertex.
8. Determine the vertical trace of a cylinder, $1\frac{1}{2}''$ diameter, whose axis makes 40° with each plane of projection.

276. PROBLEM.—**Having given m′v′n′, the elevation of a cone, and also the projections of the axis EV, to determine the horizontal trace of the cone.**

Let it be observed that $m'n'$ is *not* the true length of the major axis, nor is it the length of the elevation of the major axis of the required elliptical trace, for the axis of the cone is not parallel to the vertical plane as in Prob. 274. None of the methods there explained can be immediately used.

First Method.—Obtain an auxiliary elevation of the cone as seen when looking in the direction shown by the arrow in plan, that is on $x'y'$ taken parallel to ev. In determining this elevation in outline use may be made of an inscribed sphere, as explained in Art. 273. Then any of the methods in Prob. 274 may be employed.

Second Method.—Bisect the angle $m'n'v'$ by the line $n'i'$ and draw $i'f'$ perpendicular to xy; then I is the centre of one focal sphere, and F is the corresponding focus. Bisect $m'n'$ in c', and by projection from c' and f' obtain c and f on ve, produced if necessary. The point C is the centre of the ellipse.

Draw the projector $n'r$; then it is evident that $n'r$ is a tangent to the required ellipse.

Draw fh perpendicular to $n'r$, and with c as centre and ch as radius describe the arc ha to meet ev in a; then a is one extremity of the major axis of the required ellipse, and a_1 is the other, where ca_1 is made equal to ca. The minor axis can now be found as in Prob. 79, and the ellipse constructed. The plan of the cone is completed by drawing tangents from v to the ellipse.

277. PROBLEM.—**Having given the projections of cylinder of indefinite length, to determine the horizontal trace of the cylinder.** (No figure.)

As in the last problem, the axis of the solid is to be taken inclined to both planes of projection, and the outline in plan and elevation may be drawn as tangents to the projections of two inscribed spheres.

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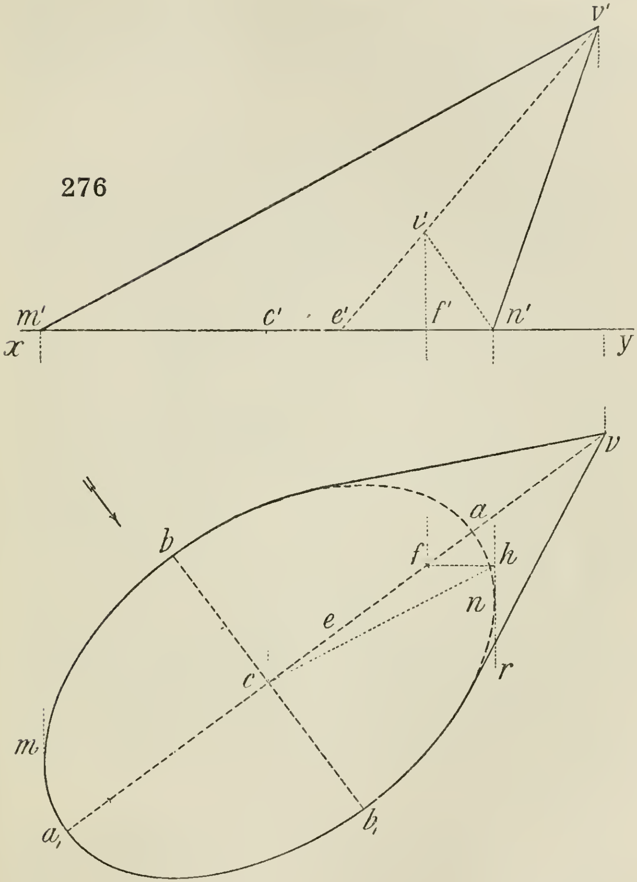


The construction to find the trace is exactly like that for the cone; but in the case of the cylinder may be somewhat simplified, because the minor axis of the ellipse is known, being equal to the diameter of the cylinder.

278. PROBLEM.—**The plan of a cone of indefinite length is given, the plan of the axis being indexed; determine the index of p, the given plan of a point on the upper portion of the cone.**

First Method.—Let *mvn* be the plan of the cone, and *ev*, with the indices attached, the plan of the axis.

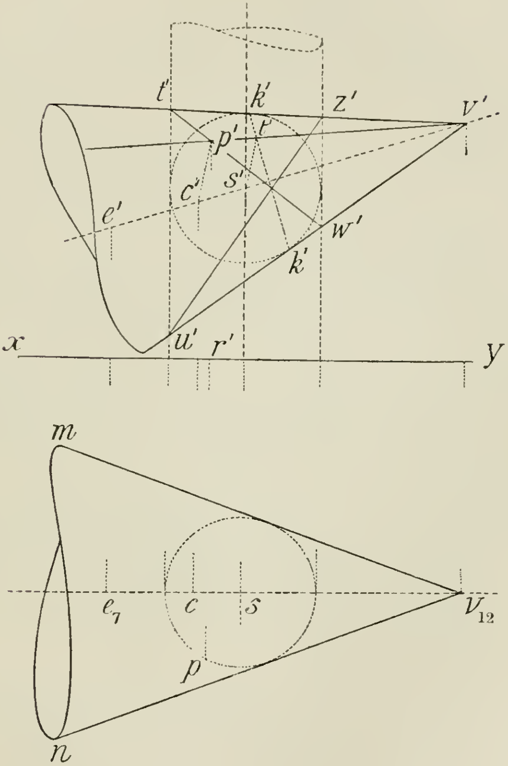
Draw *xy* parallel to *ev*, and project e', v'. Describe a circle to pass through *p* and touch *vm*, *vn* (Prob. 61); let its centre be *s*. First regard this circle as the plan of an inscribed sphere and determine its elevation, that is, the circle with s' as centrè. Complete the elevation of the cone in outline by drawing the tangents $v'k'$, $v'k'$.

Now regard the circle, centre *s*, as the plan of a vertical cylinder, the elevation of which is indicated. Thus, the cone and cylinder circumscribe the same sphere, and hence (see Art. 273) their curve of intersection consists of two ellipses. But it is evident, from considerations of symmetry, that we shall obtain an *edge view* of these ellipses when looking horizontally in a direction at right angles to both axes; and since t', w', z', u' are clearly the elevations of points which are on both surfaces, it follows that $t'w'$ and $u'z'$ are the elevations of these ellipses.

Now *P* is on the surface of the cone, and it has been arranged that it is also on the cylinder, therefore it is on one or the other of the two ellipses; but from the condition that *P* is on the upper half of the cone, it will be seen that *P* is on the ellipse *TW*, and hence p' is on $t'w'$. The required height of *P* is therefore $p'r'$, from which the index of *P* may be measured.

See over leaf for the *second method.*

Note 1.—It should be observed that *P* is *not* on the sphere, centre *S*, although *p* is on the plan of this sphere. Draw $k'k'$ joining the points of contact of the tangents from v' to the circle, centre s'; join $p'v'$ intersecting $k'k'$ in t', then *T* is the point of contact of the generator *PV* with the sphere, centre *S*. Hence if $p'c'$ be drawn parallel to $t's'$, then *C* will be the centre of that inscribed sphere which contains *P*; *c* is the plan of the centre of this sphere. This will be required in some future problems.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v_{12}
p
n

Second Method.—Let *mvn* be the given plan of the cone, e_3v_{14} being the indexed plan of its axis.

Draw any circle, centre *s*, to touch *mv* and *nv*; this is the plan of an inscribed sphere.

Draw *vp*, and regard it, not only as the plan of the generator through *P*, but also as the plan of the vertical plane containing this generator. Such a plane cuts the sphere in a circle, diameter *ab*, to which the generator *PV* is a tangent.

An elevation of this circle and tangent are shown on *xy*, taken parallel to *pv*. In determining this elevation, the heights of e' and v' are known from the given indices; s' is the elevation of the centre of the sphere, and also of the centre of the circle diameter *ab*.

Draw the tangent $v't'$. Then a projector from *p* to meet $v't'$ produced in p', will determine $r'p'$ the height of *P*, and the index of *p* is thus known.

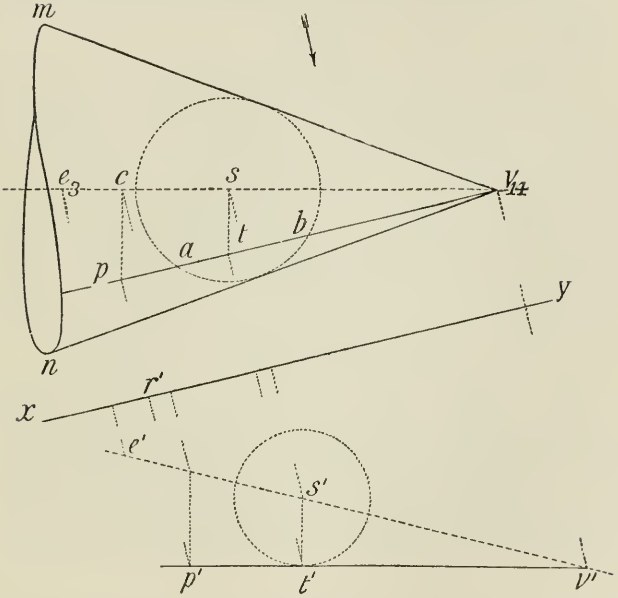
Note 2.—It will be useful for the student to observe that the generator *VP* touches the sphere, centre *S*, at *T*, and by obtaining *t* from t', joining *st* and drawing *pc* parallel to *st*, we obtain the plan of the centre of that inscribed sphere which contains *P*. This sphere will be required in some future problems.

Examples.—1. Describe a circle 1½″ diameter, centre *s*; take a point *v* 1¾″ from *s*; these are the plans of an inscribed sphere and vertex of a cone whose axis is inclined at 25° to the ground. Take a point *p* on the given circle, distant 2¼″ from *v*, and regard this as the plan of a point on the surface of the cone. Determine the height of *P* above the ground, if that of the vertex be 2½″.

Determine the plan of that inscribed sphere of the cone which passes through *P*.

2. A cone whose vertical angle is 90°, rests with a generator on the ground; draw its plan. Determine a point on the surface of the cone which is 1½″ high and 2″ distant from the vertex.

3. Two circles 2″ and 1¾″ diameter, which have their centres 1½″ apart, are the projections of two spheres, the heights of the centres being respectively 1½″ and 1″ above the ground. Regard the points of intersection of the circles as the plans of two points on the upper and lower surface of the cone which circumscribes the spheres. Required the indices of the points; unit = 0.1″. Also determine the inscribed spheres on the surfaces of which the points lie.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279. PROBLEM.—**The plan of a cylinder of indefinite length is given, the plan of the axis being indexed; determine the index of p, the given plan of a point on the lower half of the cylinder.**

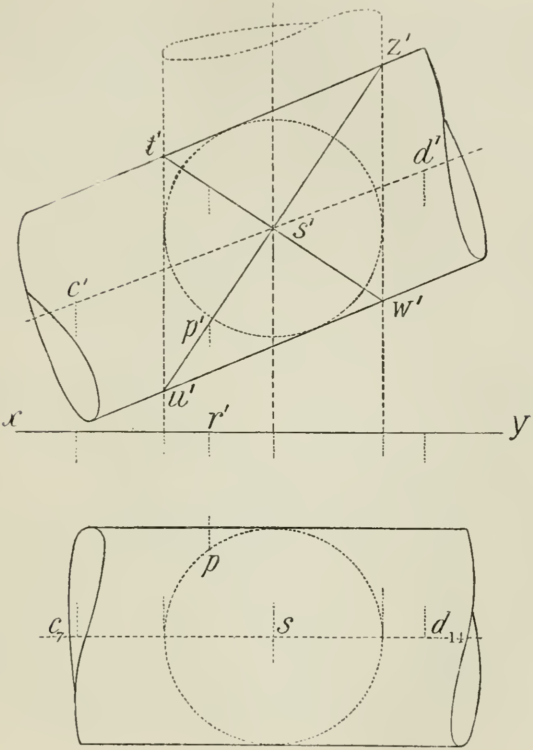
First Method. — Let *cd*, with the indices attached, be the plan of the axis of the cylinder. Take *xy* parallel to *cd*, project *c′d′*, and draw the outline elevation of the cylinder.

Describe a circle to pass through *p* and touch the outline of the given plan of the cylinder. First regard this circle as the plan of a sphere inscribed in the given cylinder, and then regard it as the plan of a vertical cylinder, the elevation of which may be at once drawn. In this way it is ensured that the two cylinders circumscribe the same sphere, and therefore intersect each other in two ellipses. From what was said in the last problem, it will readily be seen that *t′w′* and *u′z′* are the *edge elevations* of these ellipses, and that *p′* will be on *u′z′*, since *P* is on the *lower* half of the cylinder. Measure *p′r′*, and index *p* accordingly.

Note 1.—Since any number of points may be taken on the plan of the vertical cylinder in either this problem or the last, and the corresponding elevations determined, it follows that the projections of any number of generators of the given cone or cylinder may be determined; and by finding their horizontal traces the horizontal trace of the cone or cylinder may be determined, as the fair curve through these points. This constitutes a method of setting out the trace of a cone or cylinder, additional to the methods given in Probs. 274 to 277.

Second Method. — By regarding a cylinder as the particular case of a cone when the latter has its vertical angle diminished indefinitely, it will be seen that the second method of the previous problem (p. 330) will apply to the case of a cylinder, the generator through *P* being now parallel to the axis.

Note 2.—The sphere inscribed in the cylinder and containing *P* may be determined as in Note 2 of the last problem.



Examples.— **1.** Draw a line *ab* $1\frac{1}{2}''$ long; $a_{10}b_{22}$ is the indexed plan of the axis of a cylinder 2″ diameter. Select any point *p* $\frac{3}{4}''$ from *ab*; this is the plan of a point on the surface of the cylinder. Required the index of the point *p*. Unit = 0.1″. Also determine the indexed plan of the inscribed sphere which passes through *P*.

2. Determine a sectional elevation of the cylinder of Ex. 1 on a vertical plane whose horizontal trace makes 60° with *ab*.

3. Copy the above figure double size; then draw the elevation of the ellipses *TW*, *UZ* on a vertical plane which makes 45° with *xy*.

280. Problem.—**The plan of a cylinder of indefinite length is given, gd, the plan of the axis, being indexed. Determine the true shape of the section made by the vertical plane, the horizontal trace of which is lm.**

Bisect the angle nqr by the line qi_1; draw i_1f_1 at right angles to lm, then I_1 is the centre of a focal sphere with regard to the section plane LM, and F_1 is the corresponding focus. From symmetry F is the other focus where $rf = qf_1$.

Draw xy parallel to lm and obtain g', d' by projection from g, d, setting off the heights given by the indices.

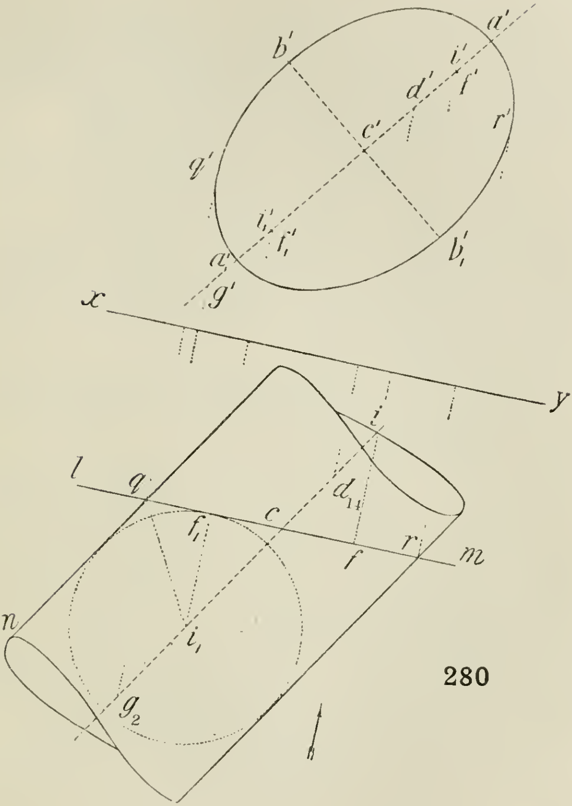
Now since LM is a vertical plane, the radius I_1F_1 of the sphere is horizontal; and I_1 is on the axis of the cylinder. Hence i_1' and f_1' coincide, and each is on $g'd'$.

Project f_1' and f' from f_1 and f. Bisect $f_1'f'$ at right angles by the line $b'b_1'$, and make $c'b' = c'b_1' =$ radius of cylinder; then $b'b_1'$ is the minor axis. The major axis $a'a_1'$ may now be determined and the ellipse may be completed by any of the methods explained in Chapter IV.

281. **Projection of any surface of revolution.**—Reasoning as in Art. 273, it is readily seen that a series of spheres can be inscribed in any surface of revolution. The surface itself may be conceived as generated by, or the envelope of, its inscribed spheres. It is shown in pure geometry that the outline projection of a surface of revolution on any plane is the envelope of the projections of its inscribed spheres; that is, the projection may be generated by a circle of variable diameter moving with its centre on the projection of the axis of the solid.

In illustration, refer to the figure on page 337. The axis CD of the surface of revolution is parallel to the vertical plane, and the elevation consists of two circular arcs, having $c'd'$ as a common chord. The plan of the surface might be determined thus:

First draw a series of circles inscribed in the elevation; then draw the plans of the spheres represented by these circles; then draw the envelope, or the curve which touches all these plans.



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The method of drawing the plan which is adopted in the next problem is fuller and more instructive. The curve *HPRG* on the surface which projects into the outline in plan is determined, and a number of *points* in the plan are obtained. This gives more definiteness than would be the case if the method of envelopes alone were used.

282. PROBLEM.—**A surface of revolution is generated by a given circular arc CTD, which revolves about its chord CD. It is required to determine the projections of the surface when the axis CD is parallel to the vertical plane and inclined at θ to the horizontal plane.**

For the elevation draw $c'd'$ making an angle θ with xy, and complete the elevation by describing the two arcs on $c'd'$. Let o' be the centre of one of these arcs.

To obtain the plan. Suppose a vertical projector to move tangentially round the surface; then the foot of this perpendicular will trace the *outline* of the plan. The projector during its circuit will touch the surface along a curve, the plan of which will be identical with the plan of the outline. We shall determine this curve, employing inscribed spheres for the purpose.

From the centre o' draw any radius $o'b'$, intersecting $c'd$ in s'; with centre s' draw the circle through b', draw $b'a'$ perpendicular to $c'd'$, and $e's'f'$ parallel to xy. The lines $a'b'$ and $e'f'$ intersect in p'.

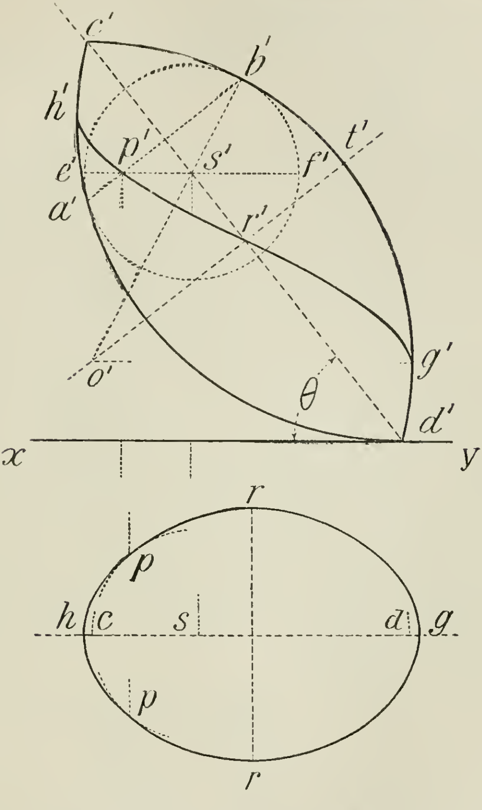
Project from $c'd'$ and thus obtain the plan cd, which is parallel to xy. Project s from s'. With centre s, radius $s'f'$, describe a circle, and draw the projector from p' to cut this circle in p, p. Then p, p are points on the required plan, and p' is the elevation of the point where the projector *touches* the surface.

For let S be the centre of an inscribed sphere. Then $a'b'$ is the elevation of its circle of contact. Let $e'f'$ represent a circle on the sphere. These circles intersect in P. Now the vertical projector through P *touches* the surface at P, because it evidently touches the sphere, and in the immediate neighbourhood of P the two surfaces coincide.

Repeat the above construction for other inscribed spheres. In particular, draw the radii $o't'$ and $o'g'$, the latter being parallel to xy, and obtain the important points R and G, and by symmetry H.

If a curve be drawn as shown through h', p', r', g', and other similar points, it will be the elevation of the curve on the given surface of revolution, such that the outline plan of the surface is identical with the plan of the curve.

Examples.—**1.** Work Prob. 282 when $c'd' = 3\frac{1}{2}''$, $o't' = 2\frac{1}{4}''$, $\theta = 50°$. Add an auxiliary elevation of the surface on a vertical plane which makes 45° with xy.



2. A surface generated by the revolution of a circular arc about an external line is 3″ long, $1\frac{3}{4}''$ diameter at each end, and $\frac{5}{8}''$ diameter in the middle. Draw the plan when its axis is inclined at 45°. Draw also the elevation on a vertical plane making 30° with the plan of the axis.

3. An *annulus* is generated by a sphere $1\frac{1}{4}$ diameter, whose centre moves round a circle $2\frac{1}{2}''$ diameter. Draw the plan of the surface when the annulus rests on a plane inclined at 45°.

Hint.—The plan is similar in shape to Fig. 123.

283. Miscellaneous Examples.

1. The plane of a circle of 2″ diameter is inclined at 50°. A diameter of this circle is inclined at 30°. Draw the plan of the circle and also an elevation on a plane parallel to the inclined diameter. (1893)

***2.** Fig. (*a*). The side elevation of a circular tin canister with the lid (hinged at *A*) partly open is given. Draw its plan, and an elevation on a vertical plane, having *xy* as ground line. (1885)

3. *VTH* is an oblique plane, the traces *vt* and *th* making angles of 61° and 71° with *xy*. A point *c*, 2.2″ from *t* and 1.7 from *xy*, is the plan of the centre of the base of a right circular cone which has a generator in the horizontal plane and its base in the given plane *VTH*. Determine the plan of the cone. (1889)

4. A sphere of radius 1.5″ resting on the ground is cut by a plane inclined at 60°, whose horizontal trace touches the plan of the sphere. Draw the plan of the section. (1877)

5. A vertical cylinder 2″ in diameter is cut by a plane inclined at 50°, and having a horizontal trace which touches the plan of the cylinder and makes 35° with the ground line. Draw the elevation and development of the curve of section. (1877)

6. Fig. (*b*). A horizontal right cylinder lies on a truncated right cone as shown, the axis of the cylinder passing through that of the cone. Draw the figure full size, and make a sectional elevation of the two solids on a vertical plane distant $\frac{1}{2}''$ from the axis of the cone, and making 50° with the axis of the cylinder. (1887)

***7.** Fig. (*e*). $v'a'b'$ is the elevation of a right cone. A circle, whose centre is o', is drawn on the elevation, touching $v'b'$; and two tangents to the circle, $c'd'$, $c'e'$, are drawn. Develop the cone, opening it along the line *VB*, and determine the developments of the circle and lines. (1895)

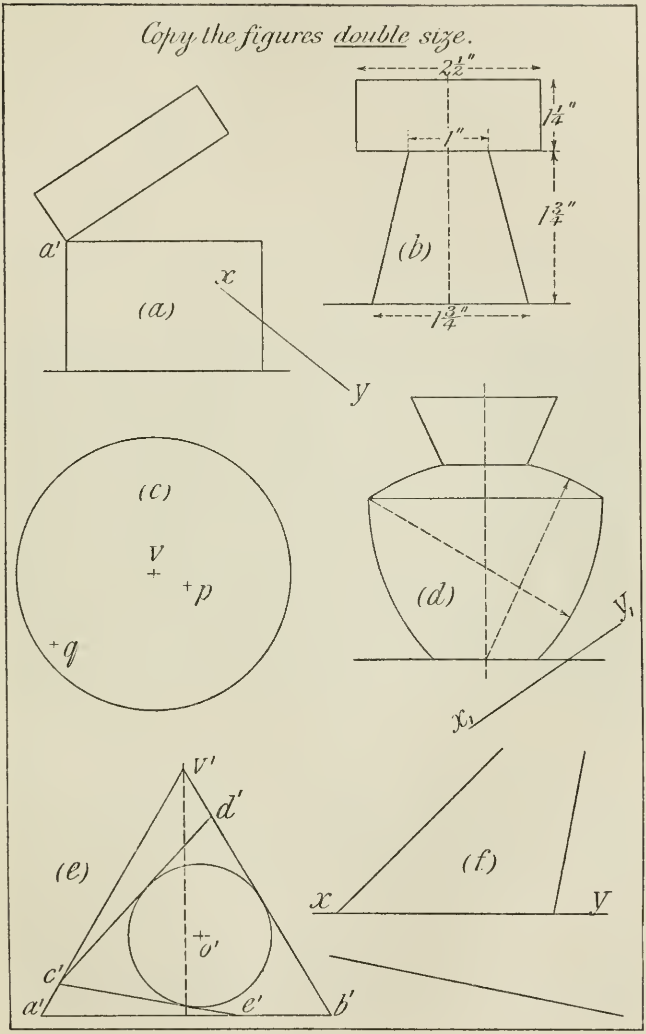
***8.** Fig. (*c*). Find the real length of the shortest line that can be drawn on the right cone, whose plan is given, joining the points whose plans are *p*, *q*. Height of cone $3\frac{1}{2}''$. (1878)

9. The axis of a cone is inclined at 60° to the H.P., the apex pointing upwards and to the right; the generatrix makes an angle of 25° with the axis. Determine a section of the cone, the plan of which will be a circle of 3″ diameter. (1897)

***10.** Fig. (*f*). The elevation of a right cone; and the plan of the axis are given; draw the plan of the cone. (Honours, 1880)

***11.** Fig. (*d*). A surface of revolution is shown in elevation; draw a plan on x', y'.

12. A cone, base 2.70″ diameter, height 2.35″, has its axis inclined at 40°. A curve is traced on the cone, which, in development, would be a circle of 1″ radius touching the base of the cone. Draw the plan of the cone, and of the curve traced on it, touching the base of the cone at its highest point. (1894)



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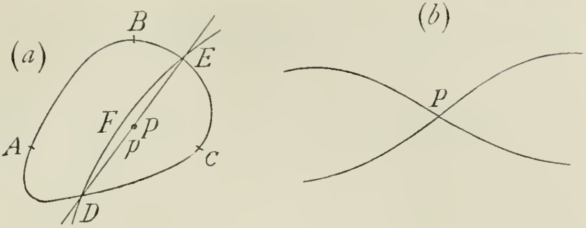
CHAPTER XIV

TANGENT PLANES TO SURFACES

284. Nature of the contact of a tangent plane.—Let any three non-collinear points, A, B, C, Fig. (a), near to one another, be taken on the curved surface of a solid figure, and suppose a section of the figure to be made by the plane through A, B, C. If the surface in the neighbourhood be wholly convex (or wholly concave), like that of a sphere or the rounded portion of a vase, the section plane will intersect the surface in a closed curve of some kind passing through the three points; this is illustrated in Fig. (a). Consider now any point P on the surface and within this curve, and suppose the points A, B, and C to approach indefinitely near to P. Then the plane ABC has a *definite limiting position*, and is called the *tangent plane* to the surface at P. The tangent plane is said to *touch* the surface at P.

Let P be projected on the section plane at p. These two points are too near together to be distinguished separately in the figure, and ultimately coincide. Now any line in the plane ABC through p will intersect the curve ABC, and therefore the surface, in two points, shown at D and E in the figure. Again suppose A, B, C to approach indefinitely near to P, so that in the limit DE becomes a line through P in the tangent plane; hence this line meets the surface in two consecutive points, that is, N *touches* the surface or is *tangential* to it. We thus see that *any line* in a tangent



plane which passes through the point of contact *touches* the surface at the point. Thus we have a series of *tangent lines* at any point P on a surface, which all lie in the tangent plane at P. The tangent plane at P may in fact be looked on as having been *generated by a tangent line which is rotated about P so as to always touch the surface at P*. The axis of rotation is the *normal* to the surface at P.

Again, suppose *any plane* through p, inclined to the plane ABC, to cut the latter in the *line* DE, and the surface in the *curve* DFE. In the limit this line and curve will both pass through P, and *the line will touch the curve at P*. It is so important that this theorem be well understood that we present it in several different forms.

Thus, *if any plane section of a surface be taken through a point P on the surface, then the line in which the tangent plane at P is cut by the section plane is a tangent at P to the curve of the section.*

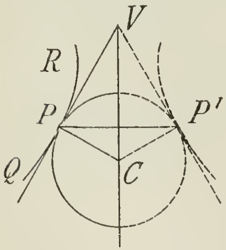
Or, *if a plane touch a surface at P, then the traces of the plane and the surface on any second plane through P also touch one another at P.*

This theorem may be generalised and stated in the form : *If two surfaces touch one another at P, then the traces of the surfaces on any other surface through P will also touch one another at P.*

As a simple example, suppose a cone to stand with its circular base on the ground, then *the horizontal trace of any tangent plane will touch the base.*

The properties of tangent planes and lines above stated have been illustrated for the case of a surface wholly convex or wholly concave in the neighbourhood of the point of contact P. They are equally true for a surface which at P is *convex in some directions* and *concave in others*, like a horse's saddle, or the hollow neck of a vase; but in this case the surface around P lies partly on one side of the tangent plane and partly on the other, and the tangent plane at P cuts the surface in a curve which has a node at P, Fig. (b), the two branches having points of inflexion where they cross. The surface near P is divided into four regions by the two branches of the curve of intersection. Two opposite regions lie on one side of the tangent plane, and the other two on the other side.

285. Surface of revolution with inscribed cone and sphere.—Let QR be any plane curve, and VC any line in its plane. Let PV and PC be the tangent and normal at P. With centre C, describe the circle through P.



Now let the figure rotate about VC as axis. Then the cone and sphere described by PV and the circle are tangential to the surface generated by QR, the contact extending along the circle traced by P, and represented in the figure by PP' perpendicular to the axis. The cone and sphere are said to be inscribed in (or they may circumscribe) the surface. The tangent plane to one of the surfaces, at any point in the circle of contact PP', also touches the other two.

The surface of a cone or cylinder is generated by the motion of a *straight* line, and the tangent plane at any point of either touches the surface along the line or generator through the point, and hence touches all inscribed spheres.

See Theorems 23 to 48, Appendix II.

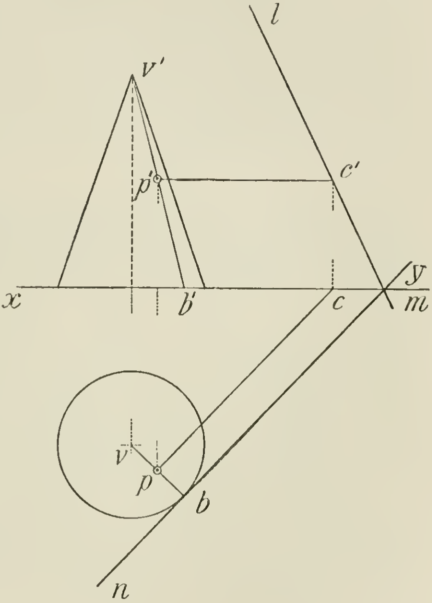
286. PROBLEM.—**To determine the traces of the plane which shall be tangential to a given cone, axis vertical, vertex V, and shall pass through a given point P on its surface.**

The projections of the cone are shown in the figure.

Draw *vb*, *v′b′*, the projections of the generator through *P*, and at *b* draw the tangent *nm* perpendicular to *vb*; this is the horizontal trace of the required tangent plane.

Through *p* draw *pc* parallel to *nm*, and *p′c′* parallel to *xy*. These are the projections of a horizontal line passing through *P*, lying in the tangent plane, and terminated at *C* by the vertical plane.

Hence *lm* drawn through *c′* and *m* is the required vertical trace.



Examples.—**1.** Draw the plan and elevation of a cone with its base on the ground, diameter of base $1\frac{3}{4}''$, height $2\frac{1}{2}''$. Draw the traces of a plane which touches the cone along a generator whose plan makes 45° with *xy*.

2. Draw the plan and elevation of a line inclined at 45° to touch the cone in Ex. 1 at the middle point of the given generator.

3. Determine the traces of a tangent plane as in Ex. 1, but with the cone inverted so that its vertex rests on the ground.

287. PROBLEM.—**A given cone, vertex V, rests with its base on the horizontal plane, determine the traces of a plane which shall pass through a given external point P and be tangential to the cone.**

The required plane must contain the line *VP*. Therefore obtain *h*, the plan of the horizontal trace of *VP*.

Draw *nhm* tangential to the plan of the base of the cone. Then *nm* is the horizontal trace of the required plane.

Determine *r'* the elevation of the vertical trace of *VP*. Then *r'lm* is the required vertical trace.

Note.—Since two tangents may be drawn from *h* to the plan of the base of the cone, there are two tangent planes satisfying the given conditions.

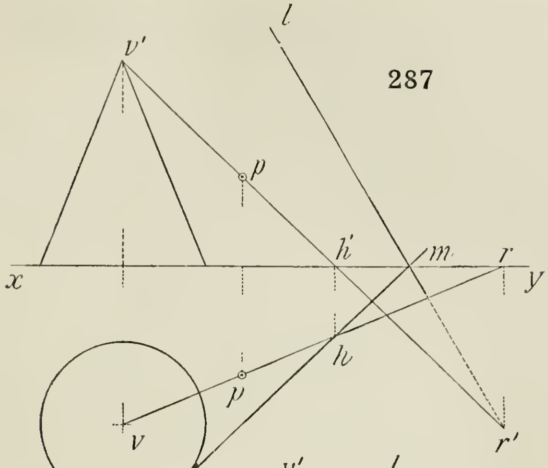
288. PROBLEM.—**To determine the traces of the plane which shall be tangential to a given sphere, centre S, at a point on its upper surface, the plan p of the point being given.**

First Method.—With centre *s* describe a circle to pass through *p*, and obtain *a'b'* the elevation of this circle; a projector from *p* to meet *a'b'* will determine *p'*. At *a'* and *b'* draw tangents to the circle, meeting each other in *v'* and *xy* in *e'*, *f'*; then *v'e'f'* is the elevation of a cone which touches the given sphere along the horizontal circle through *P*. The plan of the cone is easily obtained.

Now determine by Prob. 286 the plane *LMN* to touch the cone at *P*; this will be the required tangent plane.

Second Method.—The tangent plane at *P* is perpendicular to *SP*, hence its traces are perpendicular to *sp* and *s'p'* respectively.

To determine these traces draw *pc* perpendicular to *sp* and *p'c'* parallel to *xy*; then *PC* is a horizontal line passing through *P*, contained by the tangent plane and having *C* in its vertical trace. Hence *lm* drawn through *c'* perpendicular to *s'p'* will be the vertical trace, and *mn* perpendicular to *sp* will be the horizontal trace of the required tangent plane.



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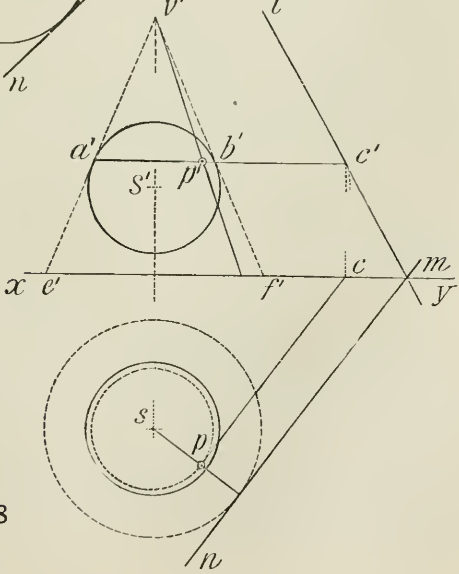

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289. PROBLEM. — **A given cylinder, axis AB, lies with a generator on the horizontal plane. It is required to determine the plane which shall touch the cylinder at a point whose plan p is given.**

Let the rectangle *cdef* be the given plan of the cylinder.

First Method.—Conceive the tangent plane in position; it will touch the cylinder along the generator through *P*. To a person looking horizontally in the direction of the arrow in plan, the cylinder will appear as a circle, and the tangent plane as a line touching this circle. This view will now be drawn.

Take $x'y'$ perpendicular to *ab*, and draw the elevation of the cylinder, that is, the circle with centre a''. Obtain p'' by projection from *p*, and at p'' draw the tangent $v''o$ perpendicular to $a''p''$. This is the edge elevation of the tangent plane, and *on* drawn through *o* parallel to *ba* is the horizontal trace. The vertical trace *lm* may be found as in previous problems, since the horizontal trace and the projections of a point *P* in the plane are known.

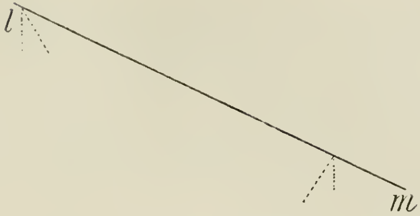
Second Method.—Draw the projections of a sphere inscribed in the cylinder and passing through *P*; then determine the tangent plane to the sphere at *P* (Prob. 288). This will be the required tangent plane (see Art. 285).

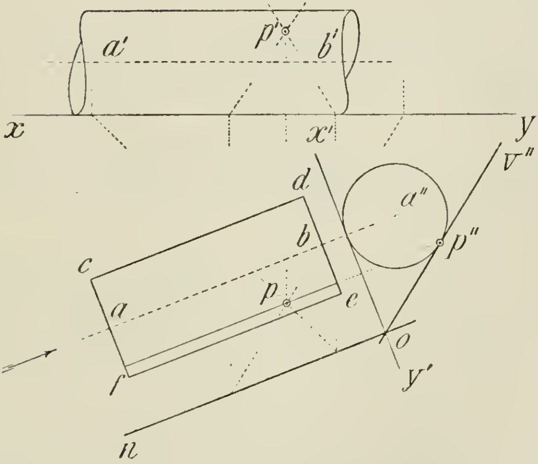
Note.—If *on* be nearly parallel to *xy*, the method of Prob. 211 may be used to determine the vertical trace, as here indicated.

Examples.—**1.** Draw the plan and elevation of a cone with its base resting on the ground, diameter of base $1\frac{3}{4}''$, height $2\frac{1}{2}''$, axis $2''$ from the vertical plane. A point *P* is $2''$ to the right of the axis of the cone, $1\frac{1}{4}''$ above the ground, and $1\frac{1}{2}''$ from the vertical plane. Draw the traces of the two planes which pass through *P* and touch the cone.

2. Draw the projections of the line from *P* to touch the cone, so that the length from *P* to the point of contact is the least possible.

3. A sphere $1\frac{1}{4}''$ diameter has its centre $\frac{3}{4}''$ above the ground, and $2\frac{1}{4}''$ in front of the vertical plane. A point *P* on the upper half of the surface is $1\frac{1}{8}''$ above the ground, and $2\frac{5}{8}''$ in front of the vertical plane. Draw the traces of the tangent plane to the sphere at the point *P*.





4. Draw the projections of the tangent line to the sphere at the point P in Ex. 3, such that it is inclined 30° to the ground.
5. A cylinder $1\frac{1}{2}''$ diameter touches the ground along a generator which makes 30° with the vertical plane; determine the traces of a plane which touches the cylinder at a point P, $1\frac{1}{4}''$ above the ground.
6. Draw a line through the plan of P in Ex. 6, making an angle of 60° with xy and 30° with the plan of the axis. If this is the plan of a line which touches the cylinder at P, draw the elevation.

290. Problem.—**To determine the traces of a plane which shall touch a given cone, vertex V, resting with its base on the ground, and a given sphere, centre S.**

Circumscribe the given sphere by a cone, vertex *A*, so that the two cones are similar and similarly placed. The projections of this circumscribing cone are shown in the figure.

Draw *mn* an *external* common tangent to the plans of the bases of the two cones. This is the horizontal trace of a required tangent plane.

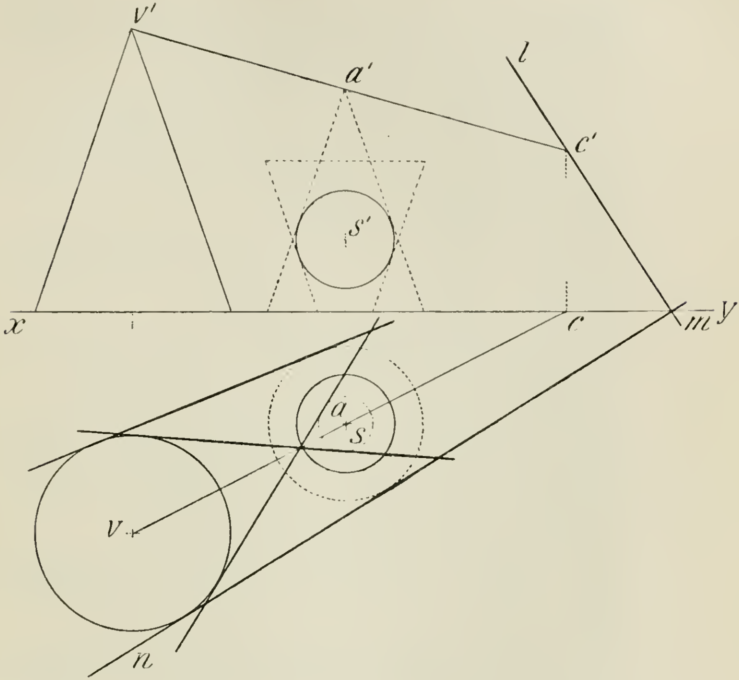
Determine the projections of *C*, the vertical trace of the line *VA*. Then *lm* drawn through *m* and *c′* is the required vertical trace.

For, any plane which touches the two cones must touch the given cone and sphere; and since the plane contains the two vertices its vertical trace passes through the vertical trace of *VA*.

Note.—It will be observed that the plane *LMN* is such that the given sphere and cone lie on the same side of it. There is another such plane the horizontal trace of which is the other external tangent to the circles *v* and *a* as shown.

There is a second pair of planes each of which passes *between* the given sphere and cone. The manner in which the traces are obtained is exhibited in the same figure, in which an *inverted* cone is taken to circumscribe the given sphere, the base angles of this cone and the given cone being equal. The trace of the inverted cone on the horizontal plane is the small circle shown in plan. The *internal* common tangents to this circle and the one with centre *v* will be the horizontal traces of the two tangent planes, the vertical traces being found as in Prob. 211.

Examples.—1. Draw the plan and elevation of a cone with its base on the ground, axis 3″ from the vertical plane, diameter of base 2″, height 3″. A sphere $1\frac{1}{4}''$ diameter has its centre $2\frac{1}{2}''$ to the right of the axis of the cone, $1\frac{1}{4}''$ from the vertical plane, and 1″ above the ground. Determine the traces of one tangent plane to the sphere and cone, and draw the horizontal traces of all the tangent planes.



2. Draw the projections of any line which touches the cone and sphere in Ex. 1.
3. Draw the traces of a plane which shall touch the cone in Ex. 1, page 346, and be $\frac{3}{4}''$ from the given point P.
4. Two points A and B are respectively $2''$ and $1''$ from each of the planes of projection. AB is $2\frac{3}{4}''$ long. Determine the traces of a plane through A inclined at 55° to the ground, and distant $\frac{3}{4}''$ from B.
5. Determine a sphere which shall contain the point A, Ex. 4, be distant $\frac{3}{4}''$ from B, and shall make an angle of 55° with the vertical plane.
6. Determine a plane which shall touch a sphere, $2''$ diameter whose centre is in xy, and shall make 30° with xy.

291. PROBLEM.—**To determine the traces of a plane which shall be tangential to two given spheres, centres S and C, and shall have a given inclination θ.**

Let the two spheres be each circumscribed by a vertical cone with base angle equal to the given inclination θ.

These cones may be either upright or inverted, and by taking the four possible combinations *eight* common tangent planes may in general be obtained.

In the figures the horizontal traces only are shown; the vertical traces may be found as in the preceding problems.

In (*a*) two upright cones are taken.

In (*b*) one cone is upright and the other inverted.

In (*c*) both cones are inverted.

In (*d*) one cone is inverted and the other upright.

In determining the horizontal traces of the tangent planes care must be taken to draw the proper common tangents. Thus in (*a*) and (*c*) each tangent plane is such that the spheres are situated on the same side of it, hence the *external* common tangents must be drawn.

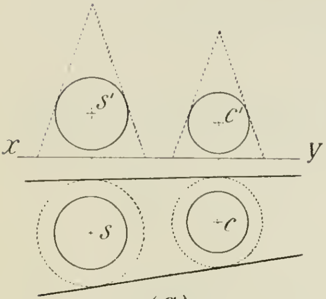
On the other hand, in (*b*) and (*d*) each tangent plane passes between the spheres, consequently the *internal* common tangents must be drawn to give the traces.

Note.—Some of the tangent planes may be coincident or impossible, and the number of solutions may be anything from eight to none, according to the data.

Examples.—**1.** The centres of two spheres which rest on the ground are $2\frac{1}{2}''$ apart; the centre of one is $2''$ from the vertical plane, and that of the other is $1\frac{1}{2}''$. The diameters of the spheres are $1\frac{1}{2}''$ and $1''$ respectively. Determine the traces of a plane touching the spheres and inclined at 55°. Draw the horizontal traces of all such tangent planes.

2. Two points *A*, *B* are respectively $2''$ and $1\frac{1}{2}''$ from each plane of projection and their projectors are $2\frac{1}{2}''$ apart. Determine a plane which shall be inclined at 50°, and $1''$ distant from both *A* and *B*, and which shall lie below *A* and above *B*.

3. Determine all the planes which are $1''$ distant from *A* and *B*, Ex. 2, and which make 50° with the vertical plane.



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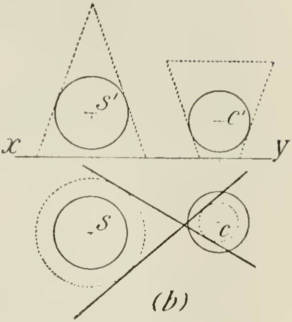

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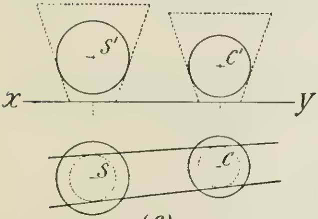

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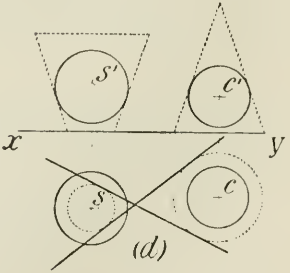


(d)

292. PROBLEM. — **To determine the traces of a plane which shall touch a given cone, axis VA inclined to both planes of projection, and have a given inclination θ.**

Let $r'v's'$ and uvw be the projections of the given cone. Draw the projections of the vertical cone which has its vertex at V and its base angle equal to θ.

Draw the projections of *any* sphere, centre C, inscribed in the given cone, and also the projections of a cone, vertex V_1, similar and similarly placed to the first cone and circumscribing the sphere.

Then nm, an external common tangent to the plans of the bases of the two upright cones, is the horizontal trace of a tangent plane to these cones, and ml is its vertical trace. This plane passes through V, touches the sphere C, and so touches the given cone. And it is inclined at θ.

Note 1.—The inclination θ cannot be less than that of the least inclined generator of the given cone.

Note 2.—There are, in general, *four* planes satisfying the given conditions, two being obtained as shown, and two others from an *inverted* cone (not shown) which circumscribes the sphere C.

Note 3.—Instead of the vertical cone through V we might have taken one circumscribing any second sphere inscribed in the given cone.

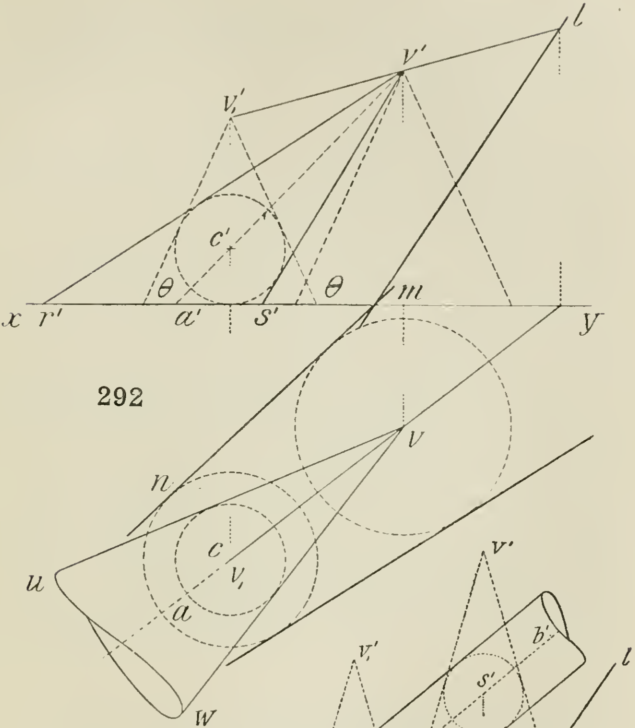
293. PROBLEM.—**To determine the traces of a plane which shall touch a given cylinder, axis AB, and have a given inclination θ.**

Draw the projections of any two spheres, centres S and C, inscribed in the given cylinder.

Now draw the projections of the two upright cones which circumscribe the spheres, each cone having a base angle θ. Determine the plane NML to touch the cones.

Since this plane touches the two cones, it touches both spheres without passing between, hence it must touch the cylinder; and it is inclined at θ.

Note 1.—The value of θ cannot be less than the inclination of the axis of the cylinder.



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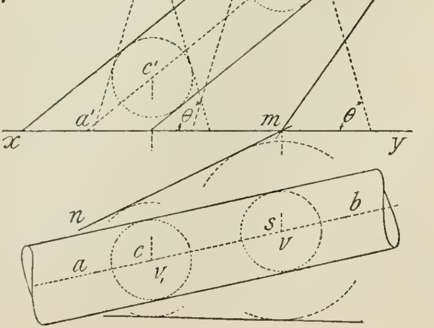

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294. PROBLEM. — **Having given the projections of a cone, axis VA, and p the plan of a point on its surface, to determine the traces of a plane which shall pass through P and touch the cone.**

Let $r'v's'$ and uvw represent the given cone.

First Method.—By Prob. 278 determine p' and also the projections of T (not shown), the point of contact of PV and any sphere inscribed in the cone.

The tangent plane to the sphere at the point T is the required plane; this may be determined by either of the methods in Prob. 288.

Second Method. — Determine p' as in the preceding method, and find Q, the horizontal trace of PV.

If the horizontal trace of the cone be determined, that is to say, an ellipse whose elevation is $r's'$, the tangent drawn at q to this ellipse will be the horizontal trace of the required tangent plane (Art. 284).

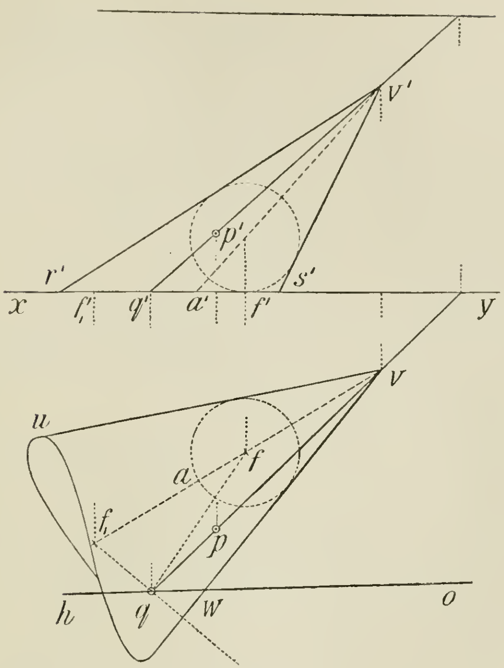
But it is unnecessary to actually draw the ellipse. Determine the foci, f and f_1 in plan, as in Prob. 276.

Join qf and qf_1, and bisect the angle exterior to fqf_1. Then the bisector oh touches the elliptic trace at h and is the horizontal trace of the required tangent plane.

The vertical trace may be determined as in the preceding problems.

Examples on Problems 292 to 294.

1. The plan and elevation of the axis of a cone make angles of 30° and 45° with xy; the vertex is $2\frac{1}{2}''$ above the ground and $1\frac{1}{4}''$ in front of the vertical plane. The sphere which is inscribed in the cone and rests on the ground is $1\frac{1}{4}''$ diameter. Determine the traces of a plane which touches the given cone and is inclined at 60° to the ground.
2. A sphere $1\frac{1}{2}''$ diameter has its centre $1''$ above the ground and $2\frac{1}{2}''$ in front of the vertical plane. A point A, $2''$ to the right of the centre of the sphere, is $1''$ in front of the vertical plane and $3''$ above the ground. Determine the traces of a plane through A which shall touch the sphere and make an angle of 65° with the ground.
3. The plan and elevation of the axis of a cylinder $2''$ diameter



make angles of 25° and 40° with xy. Determine the traces of a plane which shall touch the cylinder and have an inclination of 65.° Draw the horizontal traces of all the planes which satisfy the given conditions. What are the least and greatest possible inclinations of the planes which touch this cylinder?

4. Take the cone Ex. 1. Select a point p $1\frac{1}{2}''$ to the left of the plan of the vertex, and $2\frac{3}{8}''$ from xy. This is the plan of a point on the upper half of the cone. Determine the traces of the plane which touches the cone at P.
5. Determine a plane which shall touch the cylinder of Ex. 3 along a line whose plan is $\frac{1}{2}''$ distant from the plan of the axis.
6. Determine all the planes which touch the cylinder of Ex. 3 and make 60° with the vertical plane.

295. PROBLEM. — **To determine a plane which shall contain a given line AB and touch a given sphere C.**

First Method.—It is obvious that the required plane will touch *any* cone circumscribing the given sphere and having its vertex in *AB*. Let the particular cone with vertex *V* be taken, where *v* is determined by drawing *cv* parallel to *xy*. Such a cone has its axis *VC* parallel to the vertical plane.

Project v' from *v*, and draw the tangents $v'r'$, $v's'$; then $r'v's'$ is the elevation of the cone.

Draw the elevation of the vertical cylinder which circumscribes the given sphere.

The cylinder and cone intersect in two ellipses whose edge elevations are the lines $e'd'$ and $f'g'$ respectively. The plan of each ellipse is the circle centre *c*, since both ellipses are on the cylinder (see Art. 273).

Produce $e'd'$ to meet $a'b'$ in n' and *xy* in *l*; then *TLM* is an inclined plane which cuts the cone in an ellipse, the elevation of which is $e'd'$.

Conceive the tangent plane in position; it intersects the plane *TLM* in a line which passes through *N* and touches the ellipse whose elevation is $e'd'$ (Art. 284). The plan of this line is *nz* (or *nk*), touching the circle which is the projection of the ellipse.

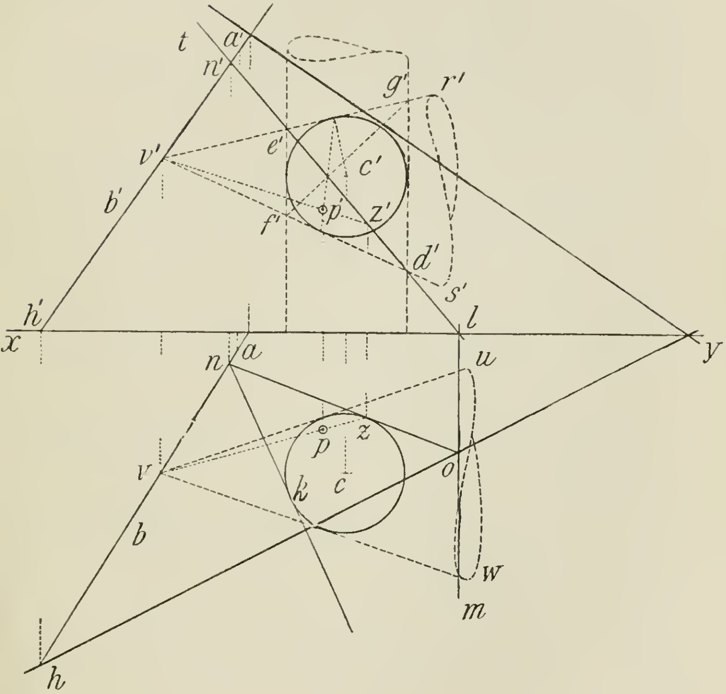
Therefore the required tangent plane contains the lines *NZ* (or *NK*) and *AB*, the horizontal traces of which are *h* and *o*, in plan. Hence *ho* is the horizontal trace of one tangent plane satisfying the conditions of the problem. The vertical trace is found as in previous problems.

Note 1.—The point of contact of tangent plane and sphere is the point *P* where *VZ* meets the circle of contact of cone and sphere.

Note 2.—If the tangent *NK* be taken instead of *NZ* another tangent plane is found. Thus there are *two* solutions.

Note 3.—The inclined plane through $f'g'$ might have been taken instead of the one through $e'd'$, but this would not have met *AB* within convenient limits.

Note 4.—If the position of *AB* be such that it is inconvenient to determine *V*, a new elevation of the line and sphere may be drawn and the same method applied.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Second Method.—A modification of the first method is illustrated in the figure on page 359. The position of AB is altered but the same letters are used.

Draw the elevation of the cone, vertex V, as before. Draw the traces tlm of the inclined plane which contains the circle of contact of cone and sphere. The given line intersects this plane in the point N.

Suppose the tangent plane in position. It will intersect the plane TLM in a line which is a tangent from N to the circle whose elevation is $e'd'$.

An auxiliary plan of the circle and the *two* tangents from N is drawn on $x_1\,y_1$ taken parallel to lt. Here $ij = qc$.

The required plane contains NK or (NZ) and AB; its horizontal trace therefore passes through o and h as before.

Note 1.—Since K is in the tangent plane, the latter may be determined by finding the traces of any two convenient lines through K which meet AB.

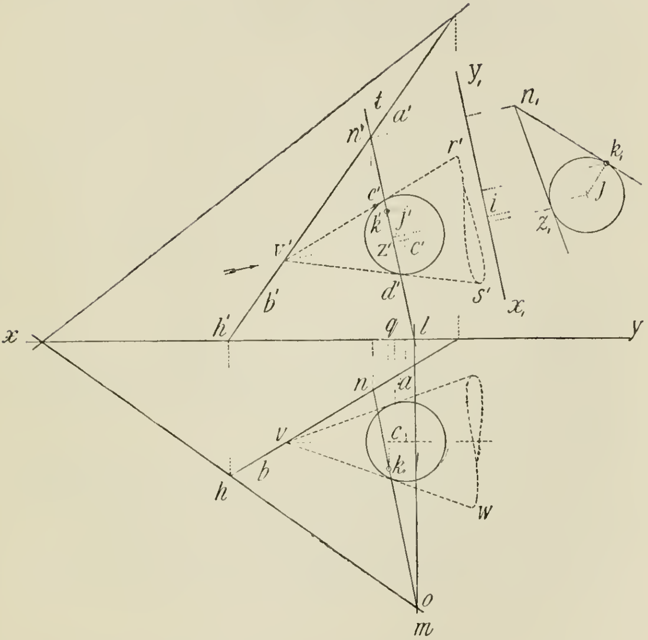
Note 2.—The tangent plane touches the cone along the generator KV, hence it touches the sphere at a point P in KV. When P has been found the problem is reduced to Prob. 288.

Third Method.—Take a cone with its vertex at any convenient point in the given line and circumscribing the given sphere.

Find the horizontal trace of this cone, and draw the two tangents to the trace from the horizontal trace of the given line. These two tangents are the horizontal traces of the required tangent planes.

This method is the more straightforward to apply, but requires the setting out of the curved trace, which is a *conic section.* The other methods require only *circles* and *straight lines* to be drawn.

Example.—Draw a line inclined at 60° to xy, in which take a point a 1″ from xy, and another point b $2\frac{1}{2}$″ from xy; this is the plan of a line AB, whose ends A and B are 3″ and 1″ above the ground. Describe a circle $1\frac{1}{4}$″ diameter, with its centre 2″ from both xy and ab; this is the plan of a sphere whose centre is 2″ above the ground. Determine the traces of a plane containing the line AB and touching the sphere.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296. Problem. — **To determine the traces of a plane which shall touch a given cone, axis VA, and pass through a given external point P.**

First Method.—This problem may be at once reduced to the preceding one, since the required plane must contain the line joining the vertex of the cone to the given point, and touch *any* sphere inscribed in the cone.

There will be *two* planes fulfilling the required conditions.

Second Method.—Let $v'r's'$ and vuw be the projections of the given cone, and p', p those of the point.

The required plane contains PV, hence its horizontal trace passes through the horizontal trace of PV; therefore determine h, the plan of the horizontal trace of PV.

The horizontal trace of the tangent plane must also touch the horizontal trace of the given cone, that is, an ellipse whose elevation is $r's'$.

Determine the foci f_1 and f in plan, and the major axis of this ellipse, as in Prob. 276.

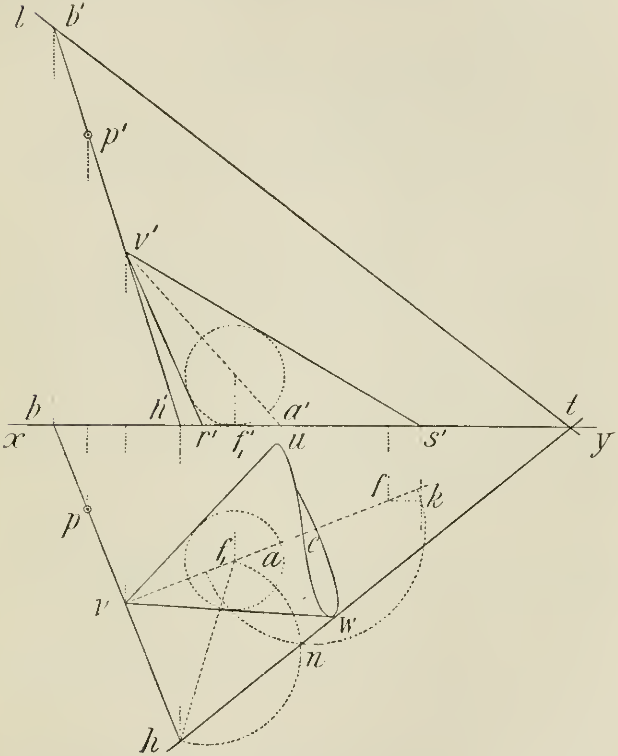
On the major axis as diameter describe a semicircle; and on hf_1 as diameter describe another semicircle, the two intersecting in n.

Then ht drawn through n is a tangent to the ellipse, and is therefore the horizontal trace of a tangent plane. And lt is the vertical trace, lt being drawn through b', the elevation of the vertical trace of PV.

Note.—Two tangents can be drawn from h to the horizontal trace of the cone, hence there are two tangent planes which fulfil the required conditions.

Examples.—1. Determine the traces of a plane which shall touch the cone of Ex. 1, page 354, and pass through a point P, 1″ to the right of the vertex, 2″ in front of the vertical plane, and 1″ above the ground.

2. The axis VA of a cone, vertical angle 25°, coincides with xy. A point P is $1\frac{1}{4}''$ distant from both planes of projection, and the projector pp' contains v, the projection of the vertex. Determine a plane through P to touch the cone. Find also a plane which touches the cone and is inclined at 45°.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297. Problem. — **The projections of a cylinder, axis AB, being given, and p the plan of a point P on the surface of the cylinder, it is required to determine the traces of a plane which shall pass through P and touch the cylinder.**

First Method. — The required plane will touch the cylinder along the generator through *P*.

Determine p' by the method of Prob. 279, and draw the projections of the generator *PD*.

This generator touches any inscribed sphere, centre *C*, at *T*, the projections of which may be determined at the same time that p' is found.

The plane which touches the sphere at *T* is the required tangent plane, and may be found as in Prob. 288.

Second Method.—After having determined the projections of the generator *PD*, the horizontal trace *nm* of the tangent plane may be determined by first obtaining *Q*, the horizontal trace of *PD*, and then drawing a tangent at *q* to the ellipse whose elevation is $r's'$.

This is done in precisely the same manner as was explained in the second method of Prob. 294, *lmn* being the traces of the required plane.

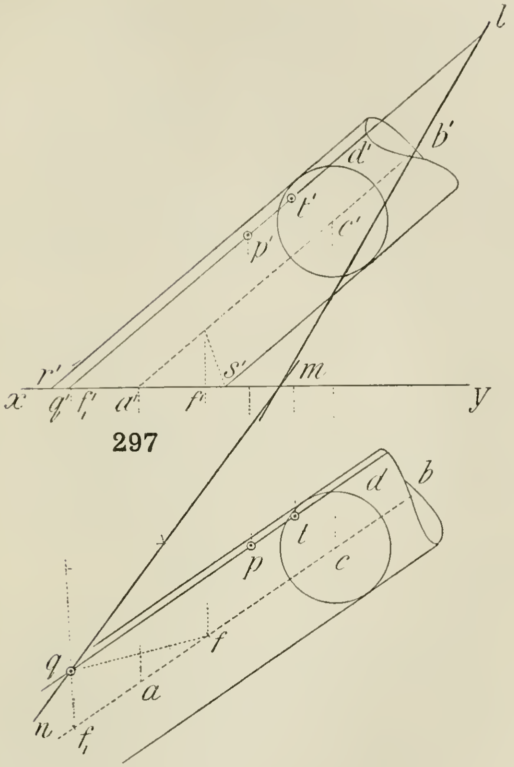
298. Problem. — **To find the traces of a plane which shall touch two given spheres and pass through a given point.** (No figure.)

This may be reduced to Prob. 296 by circumscribing both spheres by the same cone. The plane which passes through the given point and touches this cone must satisfy the given conditions.

There are two such cones, one of which has its vertex between the spheres. Thus there are in general *four* tangent planes, two being found from each cone.

299. Problem.—**To determine the traces of a plane which shall be tangential to three given spheres.**

Take the spheres in pairs, and determine the vertices



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V, W of two cones which envelop any two of the three pairs; then determine a plane which contains VW and touches any of the spheres.

The projections of V and W are found by drawing the common tangents to the projections of the spheres, and the solution of the problem is then the same as that of Prob. 295.

300. PROBLEM. — **To determine the traces of a plane which touches a given surface of revolution, axis vertical, at a given point P on its surface.**

First Method.—The surface considered is the same as that in Prob. 282, o' being the centre of the arc $c'b'd'$. One projection of P being given, the other can be found by means of a horizontal section through the point, viz. $a'p'b'$ in elevation and the dotted circle through p in plan.

Join $o'b'$, meeting $c'd'$ in s'. With s' as centre and $s'b'$ as radius describe a circle. This is the elevation of the inscribed sphere which touches the given surface in the circle passing through P.

And LMN, the tangent plane at P to this sphere, found as in Prob. 288, will also touch the given surface at P (Art. 285).

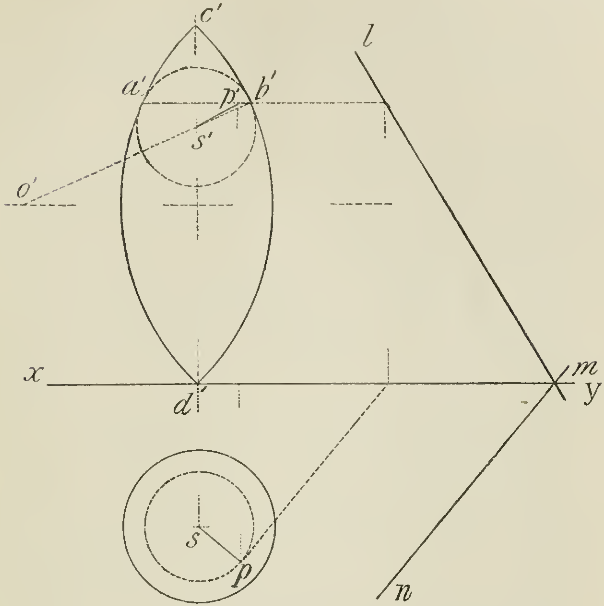
Second Method.— Draw the projections of a vertical cone touching the surface in a circle, of which $a'b'$ drawn through p' is the elevation. Then P lies on the cone, and the plane touching the cone at P also touches the given surface.

This tangent plane may be found as in Prob. 286.

Examples on Problems 297 to 300.

1. The plan and elevation of the axis of a cylinder 2″ diameter make angle of 30° and 40° with xy. Determine the traces of a plane which touches the cylinder at a point whose plan is $\frac{3}{4}''$ from the plan of the axis.
2. Determine the traces of a plane which shall touch the two spheres of Ex. 1, p. 350, and pass through a point $1\frac{1}{2}''$ vertically over the middle point of the line joining the centres of the spheres.
3. Three spheres of 1.5″, 1.0″, and .4″ diameters rest on the ground in mutual contact, and a fourth sphere .8″ diameter rests on the three. Draw the plan of the group and determine the tetrahedron which envelops them, each face touching three spheres.
4. Draw an equilateral triangle abc of 2″ side, and with a, b, c as centres describe circles of 1″, $\frac{5}{8}''$, and $\frac{1}{4}''$ radii respectively. Take these circles as the plans of spheres which rest on the ground, and determine a plane to touch all the spheres.

   Hint.—The horizontal trace of the tangent plane is found at once since the vertices of all the enveloping cones are on



the ground. The inclination of the plane is then readily determined by taking an elevation on a vertical plane perpendicular to the horizontal trace.

5. Take the surface of revolution of Ex. 2, p. 337, with its axis vertical and one end on the ground, and determine a plane which shall touch the surface at a point $2\frac{3}{4}''$ above the ground, the line joining the plan of the point to the plan of the axis making 45° with *xy*.
6. A surface of revolution is shown in elevation at (*f*) on p. 485. Copy the figure double size and draw the plan. Determine a plane which shall touch this surface and make angles of 60° and 40° with the horizontal and vertical planes of projection.
7. In Ex. 6 determine the plan and elevation of the section of the capstan by the tangent plane, and show that the section has a node at the point of contact.

301. Miscellaneous Examples.

***1.** Fig. (*a*). The plan of an inverted right cone with its vertex on the horizontal plane is given. The height of the plane of the base is 25. Determine the scale of slope of a plane containing the given point *P* and tangential to the cone. Unit 0.1″. (1887)

***2.** Fig. (*b*). The plans of two spheres are given and also the plan of a point *P*. Determine a plane touching both spheres, and passing through *P*. (1893)

***3.** Fig. (*c*). The given sphere rests on the horizontal plane. Determine the scales of slope of planes touching it and containing the given line *AB*. Unit 0.1″. (1884)

***4.** Fig. (*d*). *ab* is the plan and *a* the trace of a line inclined at 50° to the horizontal plane. *AB* is the axis of a right cylinder of 1.5″ diameter and 1.5″ in length, its lower base resting on the horizontal plane. Draw the plan of the cylinder, and the horizontal trace of a plane tangent to it and inclined at 70° to the horizontal plane. (1895)

***5.** Fig. (*e*). $a_{20}b_8$ is the axis of a right cylinder of 15 units diameter, and p_{22} is the centre of a sphere of the same diameter. Draw a plane passing over the cylinder and under the sphere, and touching the surfaces of both. Show the point of contact between the plane and the sphere. Unit = 0.1″. (1896)

***6.** Fig. (*f*). Determine a plane inclined at 65°, making 70° with the given plane, and $\frac{3}{4}$″ distant from the given point. Unit = 0.1″. (Honours, 1889)

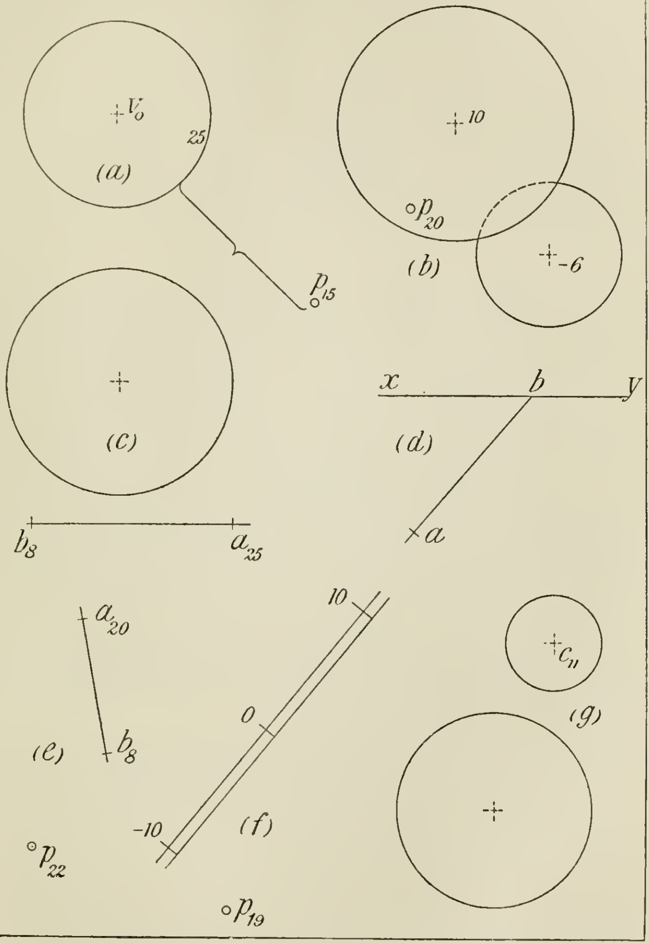
***7.** Fig. (*g*). Two spheres are given, the index of the centre of the smaller being 11, and the lowest point of the larger being level with the highest point of the smaller. Draw a plane inclined at 70° to the horizontal plane, and touching both the spheres. Unit = 0.1″. (Honours, 1891)

8. A sphere 1.75″ in diameter touches both planes of projection. Determine the traces of a plane touching the sphere, and inclined at 60° and 50° to the horizontal and vertical planes respectively. (1878)

9. Three planes are mutually perpendicular, and each touches a sphere 1″ diameter, which rests on the ground; two of the planes are inclined respectively at 35° and 70°. Draw the horizontal traces of the three planes, the plan of their intersections, and find the inclination of the third plane.

10. A right cone, $2\frac{1}{2}$″ high, rests with its base (of $1\frac{1}{4}$″ radius) on the H.P. A sphere of $\frac{3}{4}$″ radius touches it externally. Draw the true shape of the section of the cone, made by a plane passing through the vertex of the cone, inclined at 75° to the H.P. and passing under, and touching the sphere. (1898)

Copy the figures double size.



CHAPTER XV

SURFACES IN CONTACT

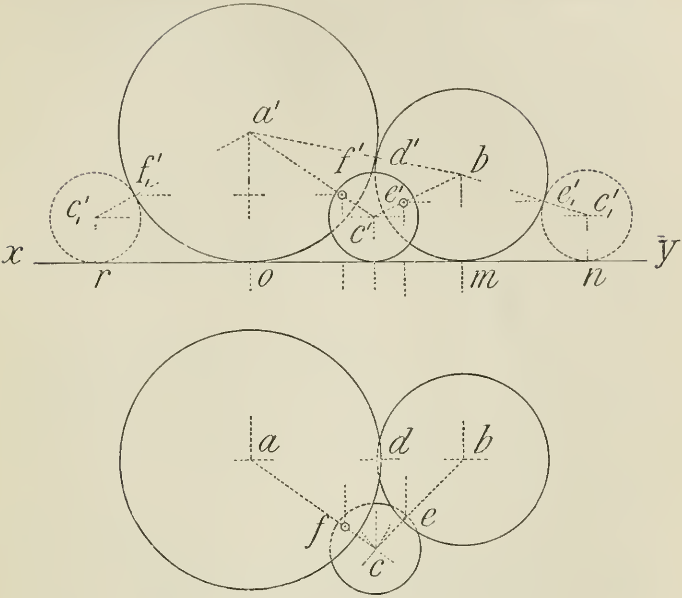
302. General remarks.—Two surfaces which touch will have a common tangent plane and a common normal at the point of contact. The normal at any point on a sphere is the line joining the point to the centre; and at any point P on the surface of a circular cone or cylinder, or any solid of revolution, is the line joining P to the centre C of the inscribed sphere which contains P. See notes to Prob. 278. Any surface which touches the sphere at P also touches the surface of revolution at the same point. Thus problems on the contact of cones and cylinders may often be reduced to those on the contact of spheres, and thereby simplified. See Theorems 23 to 48, Appendix II.

303. PROBLEM.—**To determine the projections of three spheres, centres A, B, C, of given radii, which shall rest on the ground in mutual contact.**

Assume that the spheres with centres A and B are placed so that AB is parallel to the vertical plane.

Describe circles, centres a' and b', to touch xy and each other, their radii being those of the spheres A and B. These circles are the elevations of two of the spheres. Draw their plans with ab parallel to xy.

Describe a circle (centre c_1') to touch xy and the circle with centre b', the radius being that of the third sphere. Describe an equal circle to touch the one with a'



as centre. Then mn and or are the lengths of the plans of BC and AC. Hence c, the plan of the centre of the third sphere, is a point of intersection of two arcs drawn one with centre a, radius or, and the other with centre b, radius mn.

Draw $c_1'c'$ parallel to xy to meet a projector from c in c'. This gives the elevation of C, and the projections of the third sphere may now be drawn.

The point of contact for each pair of spheres is indicated; thus e' is the intersection of $b'c'$ with a line through e_1' parallel to xy; and e projected from e' is in bc.

Example.—Draw the projections of three spheres which touch each other and rest on the ground, the diameters being $2\frac{1}{4}''$, $1\frac{1}{2}''$, $1''$. Show the projections of the points of contact.

304. Problem.—**To determine the projections of a sphere of given radius which shall touch (externally) a given sphere, centre S, at a given point P.**

The centre of the required sphere is on SP produced, therefore produce sp and $s'p'$.

Through p' draw the horizontal line $a'b'$.

Join $s'b'$ and produce to c_1', making $c_1'b'$ equal to the given radius. With centre c_1' describe the circle through b'.

Draw $c_1'c'$ parallel to xy to meet $s'p'$ produced in c', and project c on sp produced. With centres c' and c, radius $c_1'b'$, describe circles; these are the projections of the required sphere.

This construction is suggested by conceiving the required sphere to roll on the given one until SC is parallel to the vertical plane, the point of contact remaining on the horizontal circle through P; their elevations while in this position can readily be drawn. The line $c'c_1'$ is the elevation of the path of C during this motion, and $p'b'$ that of the point of contact.

305. Problem.—**To determine a sphere which shall rest on the ground and touch at a given point P a given sphere, centre S, which also rests on the ground.**

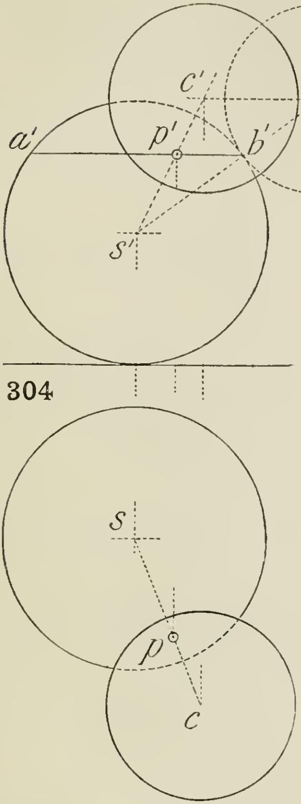
Through p' draw $p'b'$ parallel to xy. Join $s'b'$, and on $s'b'$ produced, determine c', the centre of a circle which touches xy and the circle centre s' (see Prob. 73)

Draw $c_1'c'$ parallel to xy to meet $s'p'$ produced in c'; then c' is the elevation of the centre of the required sphere, the plan c being obtained by projection.

With centres c and c', radius $c_1'b'$, describe circles; these are the projections of the required sphere.

Examples.—**1.** A sphere 2″ diameter rests on the ground, with its centre C $1\frac{1}{2}''$ from the vertical plane. Determine a sphere, $1\frac{3}{8}''$ diameter, to touch the given one at a point P, whose plan p is $2\frac{1}{8}''$ from xy, and $\frac{1}{2}''$ to the right of c.

2. A sphere $2\frac{3}{4}''$ diameter, centre C, rests on the ground. Determine a sphere which shall rest on the ground, and touch the given sphere at a point whose elevation is 1″ above xy and $\frac{3}{4}''$ to the right of c'.



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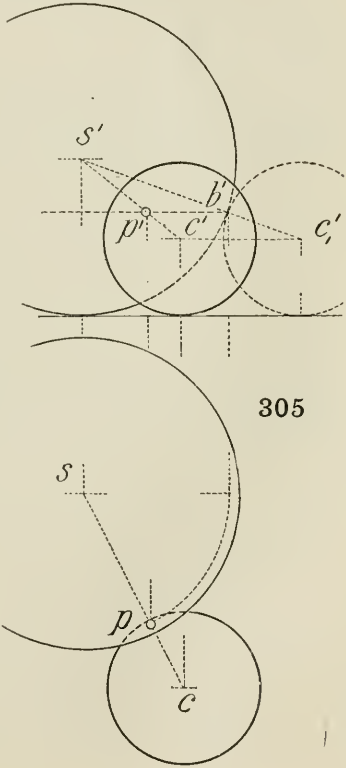

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306. Problem.—**To determine the projections of a sphere of given radius which shall touch (externally) a given vertical cone, at a given point P on its surface.**

Through p' draw $a'b'$ parallel to xy, and draw $b's'$ at right angles to $v'd'$ to intersect the elevation of the axis of the cone in s'. If a circle (not shown) were described with s' as centre and $s'b'$ as radius, it would be the elevation of a sphere inscribed in the cone, and touching it in the circle whose elevation is $a'b'$.

The required sphere must touch this sphere at P, hence its projection may be found as in Prob. 304, and is shown in the figure.

307. Problem.—**To determine the projections of a sphere of given radius which shall touch a given inclined cylinder, axis DE, at a point whose plan p is given.**

Determine the elevation of P (see Prob. 278, note 1), and also the projections of the sphere, centre S, which passes through P and is inscribed in the cylinder.

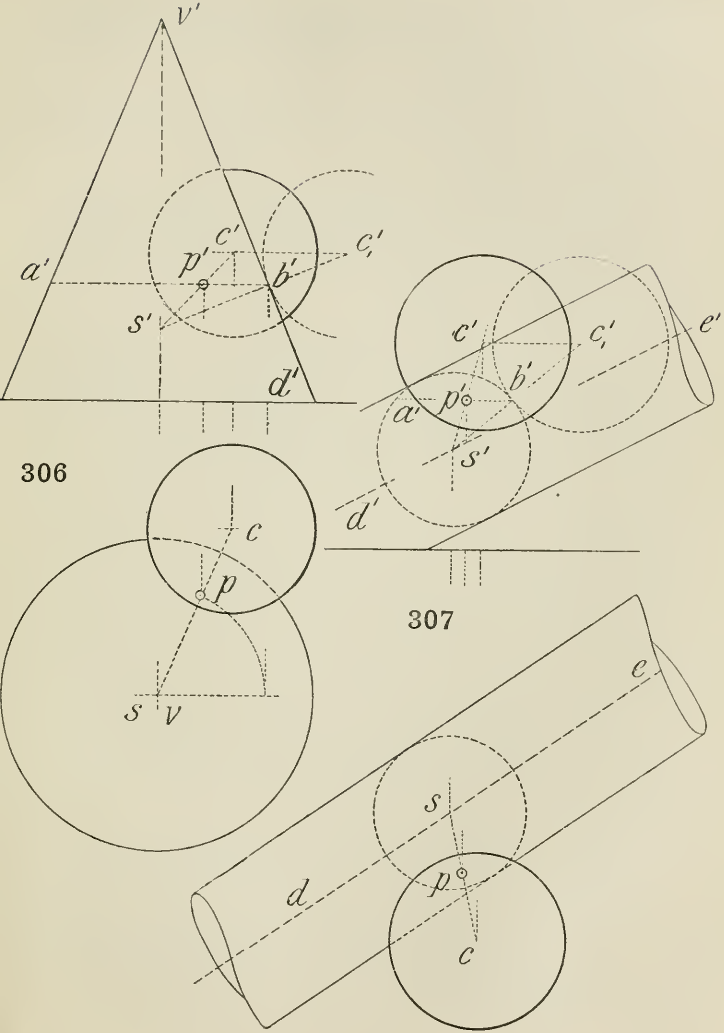
The required sphere must touch this sphere at P; its projections may therefore be found as in Prob. 304, and the construction is shown in the figure.

308. Problem.—**To determine the projections of a sphere which shall rest on the ground and touch a given inclined cone at a given point P** (no figure).

Obtain the projections of the sphere which is inscribed in the given cone and passes through P (see Prob. 278, note 1). The required sphere must touch this sphere at P, and its projections may be found as in Prob. 304.

Examples.—**1.** A cone rests with its base on the ground; diameter of base 2″, height 3″. Draw the plan and elevation of a sphere, $1\frac{3}{8}''$ diameter, which shall touch the given cone at a point 2″ from the vertex; the elevation of this point being $\frac{1}{2}''$ to the right of that of the axis of the cone.

2. The plan and elevation of the axis of a cylinder, 2″ diameter, make angles of 45° and 30° with xy. Determine a sphere, $1\frac{3}{8}''$ diameter, which shall touch the given cylinder at a point whose plan is $\frac{7}{8}''$ from the plan of the axis of the cylinder.



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309. Problem.—**The plans of a sphere, centre C, and of a line AB which touches the sphere are given, the indices of c and a being attached; determine and index the plan of the point of contact.**

Regard *ab* as the plan of a vertical plane; the plane will intersect the sphere in a circle, diameter *de*, to which *AB* must be a tangent.

Take *xy* parallel to *ab*, and project the elevations of *A* and the circle centre *O*, making $mo' = 5$ units. From a' draw the tangent $a'p'$; then this is the elevation of *AB*, and *P* is the required point of contact. Measure $p'n$ and project and index *p* accordingly.

310. Problem.—**The circles, centres v and c, are the plans of a cone and sphere which rest on the ground, the height of V being given. Determine the plan of a cylinder of given radius which rests with a generator on the ground and touches the given cone and sphere.**

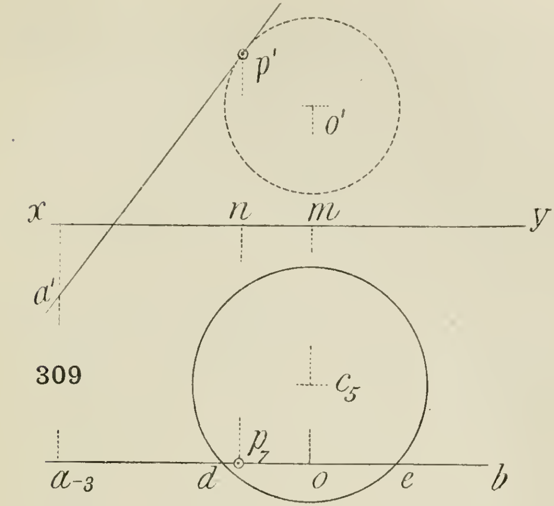
On *xy*, parallel to *vc*, draw the elevations of cone and sphere.

Describe a circle, centre a_1', radius equal to that of the required cylinder, to touch *xy* and the elevation of the sphere. Describe an equal circle to touch *xy* and $e'v'$.

These circles may be regarded as end elevations of the cylinder, and the plan of the axis of the required cylinder is distant from *c* and *v* lengths equal to $a_1'n$ and $b_1'm$. Hence, with centres *c* and *v*, radii $a_1'n$ and mb_1', describe arcs and draw *ab* tangential to these arcs. This is the plan of the axis of the required cylinder, and the outline of the plan can now be at once drawn.

Examples.—**1.** A sphere, 2″ diameter, centre *C*, rests on the ground. A line $a_{-5}b$, $\frac{3}{4}''$ from *c*, is the plan of a tangent to the sphere at *B*; $a_{-5}c = 2\frac{1}{4}''$. Determine and index the plan *b*. Unit 0.1″.

2. A cone rests with its base on the ground, diameter of base $1\frac{3}{4}''$, height $2\frac{1}{2}''$. A sphere $1\frac{1}{4}''$ diameter also rests on the ground, its centre being $1\frac{1}{2}''$ from the axis of the cone. Determine a plan of a cylinder, 1″ diameter, which rests with a generator on the ground and touches the given cone and sphere, and show the points of contact.



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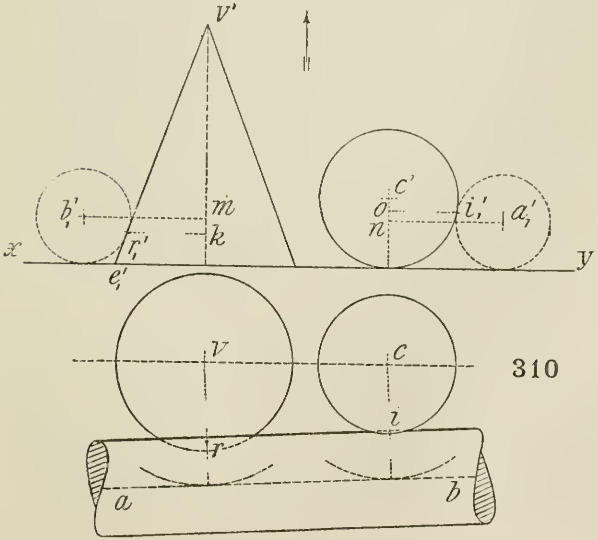

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311. PROBLEM.—**To determine the projections of two cones which rest on the horizontal plane in line contact with one another, the vertical angles being θ and ϕ.**

One axis VZ is taken parallel to the vertical plane.

Since the cones must touch along a common generator, and also lie on the ground, their vertices must coincide.

Draw the isosceles triangles $v't'f'$ and $v't'g_1'$, with the vertical angles at v' equal to θ and ϕ. These are the elevations of the two cones in line contact, with their axes parallel to the vertical plane.

Take any point s' in the axis $v'z'$ and draw $s'a'_1c_1'$ perpendicular to $v't'$. Then S and C_1 are the centres of two inscribed spheres touching each other at A_1.

Project vsz parallel to xy; with centre s, radius $s'a'_1$, describe a circle; the tangents from v to this circle form the plan of the larger cone.

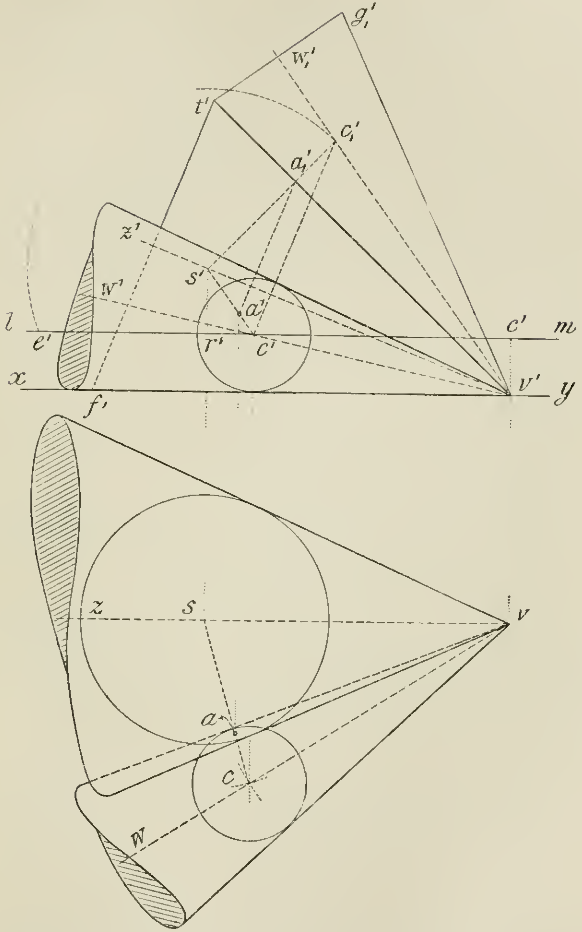
Conceive the cone VW to roll over the other until it lies on the ground; its projections when in this position must now be found.

Draw lm parallel to, and distant $c_1'a_1'$ from xy; also draw $c_1'c'$ perpendicular to $v'z'$, meeting lm in c'. With centre c' describe a circle to touch xy, and from v' draw the upper tangent. The elevation of the smaller cone is thus found.

With centre s', radius $s'c_1'$, describe a circle cutting lm in e'; draw $s'r'$ at right angles to lm. With centre s, radius $r'e'$, describe an arc to cut a projector from c' in c; then c is the plan of C. The plan of the smaller cone is completed by drawing tangents from v to the plan of the sphere, centre C.

To explain the construction. As the cone VW rolls over the other, the triangle SCV turns about SV until C comes into the plane LM; $c_1'c'$ is the edge elevation of the circular path of C. During this motion C remains on the surface of a sphere, centre S, radius $s'c_1'$; hence the plan c is found.

Note.—Draw $a_1'a'$ parallel to $c_1'c'$, and project a on sc. Then A is the point of contact of the spheres, and VA the line of contact of the cones.



Example.—Determine the projections of two cones which rest on the horizontal plane in line contact with one another, the vertical angles being 50° and 25° respectively.

312. PROBLEM.—**The scales of slope of two planes are given, to determine the plan of a cone, vertical angle θ, which lies between the planes so as to touch them.**

The figure to the left shows an elevation of the cone with a sphere of any radius z inscribed in it.

Conceive the cone lying between, and in contact with, the given planes. The vertex V must be at some point on their intersection, and C, the centre of the inscribed sphere, will be on the intersection of two planes respectively parallel to the given planes and distant z therefrom ; C will also be situated on the surface of a sphere, centre V, and radius $v_1'c_1'$.

Determine $r's'$ and $h'k'$, edge views of the given planes, and draw lm' and $f'g'$ respectively parallel to and distant z from them.

Determine a_0b_{10}, the plan of the intersection of the given planes, and also d_0e_{10}, the plan of the intersection of the second pair of planes (see Prob. 244).

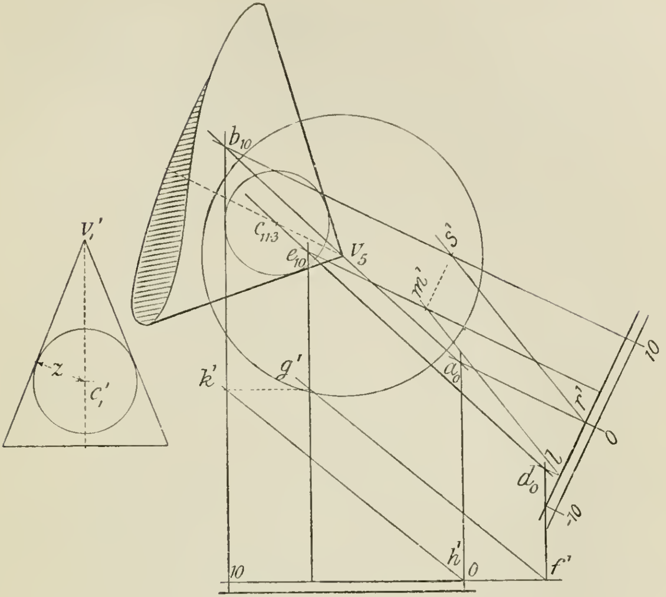
In a_0b_{10} select any point (in AB), say v_5, as the plan of the vertex, and with v_5 as centre and radius $v_1'\ c_1'$ describe a circle ; this is the plan of a sphere on the surface of which C must be situated. But C is also in DE.

Therefore by a construction similar to that in Prob. 309 determine the intersection of DE and this sphere ; the plan of one point of intersection is $c_{11\cdot 3}$. With centre c and radius z describe a circle and draw the tangents to it from v_5 ; the required plan of the cone is thus determined.

Examples.—1. Draw oc and od including 120°. On oc take $or=\frac{3}{4}''$ and $os=2\frac{1}{2}''$. On od take $oc=\frac{1}{2}''$, $oe=2''$. Regard r_0s_{15} and $c_{-5}e_{10}$ as the scales of slope of two planes. Determine the plan of a cone, vertical angle 45°, which lies between the planes so as to touch them. Unit 0.1″.

2. Determine the plan and elevation of a cone of indefinite length, vertical angle 60°, to which the planes of projection are both tangential. Show the lines of contact.

3. The traces vt, th of a plane make 40° and 60° with xy. Find a right-angled cone which touches this plane and the ground.



General Examples.—1. Three spheres of diameters $2''$, $1\frac{1}{2}''$, and $1''$ rest on the ground in mutual contact, and a fourth sphere, $\frac{3}{4}''$ diameter, rests on the three. Draw the plan of the group. Draw also the plan of the triangular pyramid which circumscribes the spheres, showing the three points of contact on each face of the pyramid.

2. A line AB, $3''$ long, has its ends in the planes of projection, and is inclined at 60° to XY and 40° to the ground. Determine a cylinder having XY as its axis, which shall touch the line AB.

3. Determine a sphere of $2''$ radius which shall have its centre in XY and touch the line AB of Ex. 2.

4. A sphere, $3''$ diameter, has its centre C in XY; a second sphere, $1\frac{1}{2}''$ diameter, centre S, touches both planes of projection; $SC = 2\frac{1}{2}''$. Determine a sphere which shall rest on the ground and touch the cone circumscribing the given spheres at a point distant $\frac{3}{4}''$ from both planes of projection.

313. Problem.—**A given cone, axis VZ, lies on the ground; to determine the projections of a cylinder of given radius which shall touch the cone externally at a point whose plan p is given, the direction, mn, of the plan of the axis of the cylinder being also given.**

Suppose the required cylinder in position touching the given cone at *P*, then

(1) There will be two spheres inscribed one in each so as also to touch one another at *P*; call their centres *C* and *S*.

(2) The generators of cylinder and cone which pass through *P* must both lie in the tangent plane common to the cylinder and cone, or the inscribed spheres, at *P*.

Determine, by Prob. 278, the elevation of *P* and the projections of the sphere, centre *S*, which passes through *P* and is inscribed in the cone.

Also by Prob. 278 determine the projections of a sphere, centre *C*, which touches the given cone at *P*, and has a radius equal to that of the required cylinder.

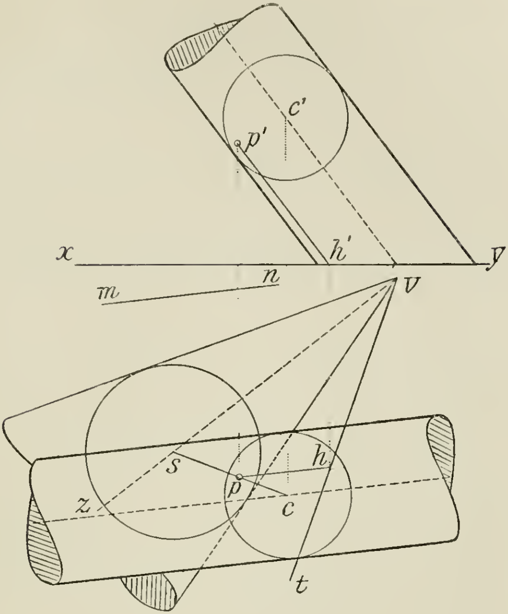
The required cylinder must circumscribe this sphere, hence its plan is obtained by drawing those tangents to the circle, centre *c*, which are parallel to *mn*.

Draw *vt* perpendicular to *sp*. Then *vt* is the horizontal trace of the tangent plane referred to in (2) above.

Draw *ph* parallel to *mn*, and project *h'*; join *p'h'*. Then *p'h'* is the elevation of a generator *PH*. Hence the elevation of the cylinder is obtained by drawing those tangents to the circle, centre *c'*, which are parallel to *p'h'*.

Note.—If it were required that the cylinder should touch the cone along a generator, so as to have *line* instead of *point* contact, the tangents to the circle *c*, giving the plan, would be drawn parallel to *pv*.

Example.—Draw two lines including an angle of 45°; these form the outline of the plan of a cone which touches the ground along a generator. Determine the projections of a cylinder $1\frac{1}{4}''$ diameter, the plan of whose axis makes 30° with the plan of the axis of the cone, which shall touch the cone at a point whose plan *p* is $1''$ and $\frac{1}{4}''$ respectively from the first two lines.



General Examples.—**1.** A sphere $2\frac{1}{2}''$ diameter has its centre C in XY. A point A is distant $1''$ and $2''$ from the vertical and horizontal planes of projection and $3''$ from C. Determine a sphere, centre A, which shall touch the given sphere.

2. A cone, vertex V, vertical angle 60°, has its axis along XY. A point A is distant $1''$ and $1\frac{1}{2}''$ from the horizontal and vertical planes, and $2''$ from V. Determine a normal from A to the cone, and a sphere, centre A, which touches the cone.

3. Draw the projections of any point A, and of any cone with its axis inclined to both planes of projection. Determine the two normals from A to the cone, and the two spheres, centre A, which touch the cone.

4. A cylinder $2''$ diameter and a cone, vertical angle 60°, lie on the ground in contact with their axes at right angles. Draw their plans and determine a sphere of $1\frac{1}{4}''$ radius which shall rest on the ground and touch both.

314.—Miscellaneous Examples.

1. A sphere of $2\frac{1}{8}''$ diameter touches both planes of projection. A second sphere, diameter $1\frac{3}{8}''$, touches the first sphere, and has its centre in the ground line. Draw the projections of the two spheres. (1889)

2. Two spheres, diameters $2.25''$ and $1''$, rest on the horizontal plane touching each other. Draw the plan of the complete locus of the centre of a sphere of $1.4''$ diameter, touching both spheres. (1888)

3. A right circular cone, the vertical angle of which is 35°, rests with a generator in the horizontal plane. A sphere of $1\frac{1}{4}''$ radius also rests on the horizontal plane, and touches the cone in a point $2\frac{3}{4}''$ from the vertex. Draw the plan of the two solids. (1892)

***4.** Draw a sphere, $1\frac{1}{4}''$ radius, resting on the horizontal plane, and touching the two given planes. Determine the points of contact. Unit $0.1''$. (1885)

***5.** Two planes are given by their traces *aob*, *cod*. Draw the projections of a sphere $2''$ in diameter, touching the given planes, and having its centre in the horizontal plane. (1887)

***6.** Draw the plan of any sphere such that the given line *AB* is tangent to it, and that the centre of the sphere is in the line *CD*. Unit $= 0.1''$. (1894)

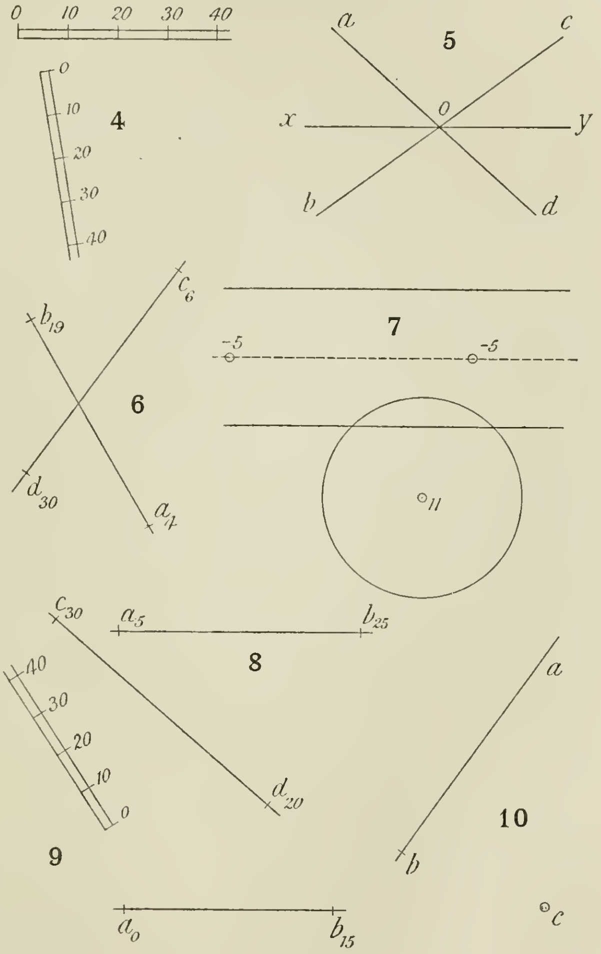
***7.** The plans of a right cylinder and of a sphere are given. A right cone, diameter of base $2\frac{1}{8}''$, height $3\frac{1}{4}''$, stands on the horizontal plane and touches both cylinder and sphere. Draw its plan and show the points of contact. (1886)

***8.** Determine the centre and radius of a sphere to which the two given lines *AB*, *CD* shall be tangent. Unit $= 0.1''$. (1881)

***9.** A plane is given by its scale of slope, and a line *AB* by its figured plan. Determine the centre of a sphere of $1\frac{1}{4}''$ radius, touching the given plane and line, and resting on the horizontal plane. Unit $0.1''$. (1890)

***10.** A right cone is lying on its side upon the horizontal plane, *b* is its vertex, and *ab* is the plan of its axis, which is inclined at 25°. The point *c* is the plan of the centre of a sphere, which also rests upon the horizontal plane, and which is in contact with the cone. Complete the plan. (1882)

Copy the figures double size.



CHAPTER XVI

INTERSECTIONS OF SURFACES, OR INTERPENETRATIONS OF SOLIDS

315. The general problem and its solution.—When two solids penetrate each other, the nature of the line of intersection of their surfaces depends upon that of each of the surfaces. Thus if both surfaces consist of a series of plane faces, they will intersect each other in a series of straight lines; or if one or both of the surfaces be curved, the intersection will consist of one or more curves; these curves may be plane, but generally are *tortuous* curves; that is, such as cannot be contained by a plane.

One solid may completely penetrate the other, entering at a closed curve or zigzag line where their surfaces intersect, and emerging at a second closed curve or gauche polygon. Or the interpenetration may be only partial, in which case the line of intersection generally consists of only one closed figure. For the case intermediate between these two, the surfaces of the solids touch in one or more points; here we must expect to find a node at a point of contact, that is, two branches of the curve of intersection may cross at the point.

Method of sections. — The method most commonly adopted for determining points on the intersection of the surfaces of two solids, or of any two surfaces, is that known as the method of sections. The solids are supposed to be cut by a series of section surfaces, generally plane, but sometimes spherical or otherwise curved. The shapes of the sections are drawn in plan and elevation, and the series of points where these intersect are thus determined. These points are common to the surfaces of the two solids; that is, are points in the required intersection. The section surfaces are chosen, if possible, so that the projections of the sections shall always be straight lines or circles both in plan and elevation, these being the only two forms that can be drawn without trouble.

It would at first appear that the greater the number of sections taken, and hence of points determined, the greater would be the accuracy with which the curves of intersection could be drawn. This, however, would only be true if each point could be located with *absolute* accuracy, which is not possible. So the greater the number of points the more marked will be the result of any slight error in determining any one. To obtain the best results, comparatively few section surfaces should be taken, but their positions should be *very carefully and judiciously chosen.*

In almost all cases there will be **certain important special points** on the line of intersection whose projections ought to be found. Such, for instance, are all points which, in the projections, fall on the outlines of the figures. The projected curve of intersection generally *touches the outline* at such points, and these points often separate a visible from an invisible portion of the intersection. The general method of procedure should be first to select those section surfaces which give us the important special points. These may be sufficient to enable us to plot the whole curve. But should there be any wide intervals, one or two extra sections may be taken to fill up the gaps.

In the problems and figures which follow, the positions of the important sections are generally indicated. A section is projected in detail, and the corresponding points of the curve of intersection determined. The projections of the other sections are omitted to avoid confusing the figures, though these may have had to be drawn to enable us to give the complete intersections.

316. Problem.—**To determine the plan and elevation of the section of the given sphere, centre O, by the given oblique plane LMN.**

Here one of the two intersecting surfaces is a plane.

In determining points on the curves, it is convenient to employ a series of sections by horizontal planes, for these project into straight lines and circles in plan, and of course into straight lines in elevation.

One such section plane is drawn in edge elevation at $p'q'$. It cuts the sphere in a circle of diameter s_1s_1, and the plan of this circle is drawn with centre o. And it cuts the plane LMN in the line whose plan is pq, determined by projecting from p' to p and then drawing pq parallel to mn.

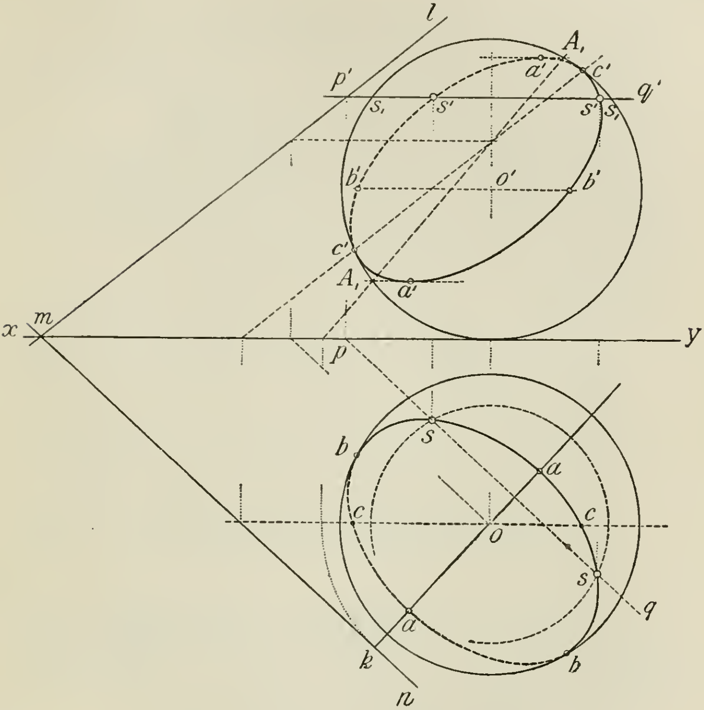
The plans of the sections of sphere and plane intersect in s, s; and the elevations s', s' are found by projecting from s, s on to $p'q'$.

The points S, S are common to the section plane PQ, the plane LMN, and the sphere, and are therefore points on the required intersections of the two latter.

This illustrates the method of determining points by any section plane; we must now select those sections which give rise to the most important points on the curves.

First as to the upper and lower limits of the curve. Draw ok perpendicular to mn; let this be the plan of a line in the plane LMN; the line will evidently intersect the required section in its highest and lowest points. To determine the levels of these we have drawn the elevation $A_1\,A_1$ of the line, supposing it to have been turned into a position parallel to the vertical plane about a vertical axis through O. The points A_1, A_1, where this elevation cuts the elevation of the sphere, give the highest and lowest levels for the section planes. The two sections at these levels give the two points A, A; at a', a' the required curve is horizontal; at a, a the tangents are parallel to mn.

Next as to the points on the outlines. A horizontal plane through the centre of the sphere will cut the latter in a great circle which projects into the outline in plan;



this section gives the two points B, B on the curve, whose plans b, b are on the outline plan of the sphere. Observe that the plan of the curve touches the outline at the points b, b where the two meet; note also that the points b, b separate the full and dotted parts of the curve, which represent the visible and hidden portions.

To obtain the points on the outline in elevation, we must take a section which projects into this outline. This is given by a plane through O parallel to the vertical plane. The elevation of the line where this plane cuts the plane

LMN is shown; it intersects the outline in c', c'. Thus the points *C*, *C* are given by this vertical section plane.

Again observe that the curve and outline touch at c', c', and that the curve changes from a full to a dotted one or *vice versa* when it passes a point on the outline.

The eight points *S*, *A*, *B*, *C* thus determined are almost sufficient to enable the curves to be plotted, especially as the latter are ellipses, to the form of which the eye is well accustomed. Otherwise one or two intermediate section planes will give the required additional points.

Examples.—**1.** Copy Fig. 316 double size, and work the problem.

2. A cone stands upright on the ground, draw the projections of its section by an oblique plane. Diameter of base 2.2″; height 2.6″. Horizontal trace of plane touches base, and both traces make 50° with *xy*.

3. A square pyramid stands upright on the ground, obtain the projections of its section by an oblique plane. Height 2.5″, side of base 1.8″, making 30° with *xy*. Horizontal trace of plane 1.2″ from plan of axis, both traces making 45° with *xy*.

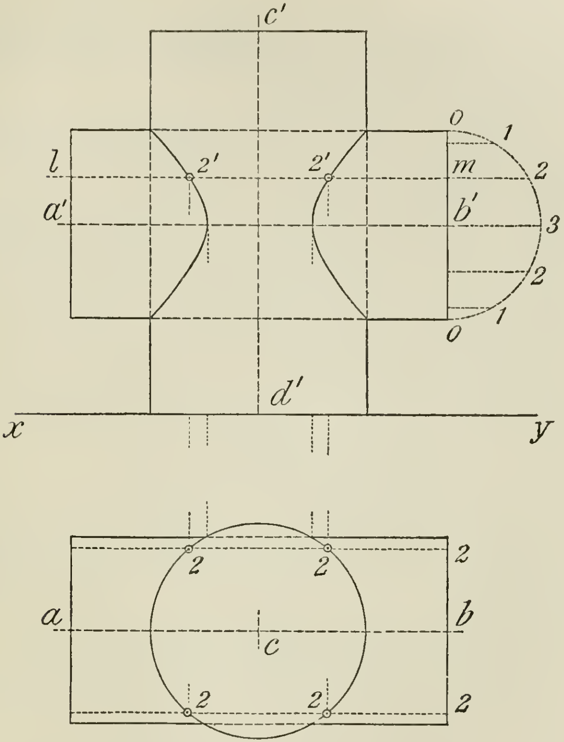
Hint.—Take section planes which contain two long edges of the pyramid, either opposite or adjacent edges, or both, whichever give the best-conditioned constructions.

317. PROBLEM.—**To determine the projections of the line common to the surfaces of two cylinders, the axes AB and CD of which intersect at right angles, CD being vertical.**

Horizontal section planes will be convenient.

Describe the semicircle on *oo* as diameter, and divide its circumference into six equal parts.

Consider the section plane whose elevation *lm* passes through 2. Set off *b*2, *b*2 in plan, each equal to *m*2 in elevation, and through the points 2 thus obtained draw lines parallel to *ab*. These lines form the plan of the section of the horizontal cylinder, while that of the vertical cylinder is the circle, centre *c*. The two sections intersect in plan in the four points marked 2, which must therefore be on the required plan of the line of intersection. Draw projectors



from the points in plan to meet *lm* in $2'$, $2'$, which points are on the elevation of the line of intersection.

Repeat this construction for the planes through the other points of division of the semicircle and draw curves through the points so determined, obtaining two curves for the elevation as shown, while the plan coincides with portions of the circle, centre *c*.

Note.—If the vertical cylinder had been the smaller one, section planes parallel to the vertical plane would have been preferable.

318. PROBLEM.—**To determine the projections of the curve of intersection of a given cone and cylinder, the axes of which intersect each other at right angles, that of the cone being vertical.**

Three Cases are shown.—In (1) the cylinder completely penetrates the cone; in (2) the cylinder and cone circumscribe the same sphere; and in (3) the cone completely penetrates the cylinder.

Case (1).—The projections of the cone and cylinder are given in the figure, the diameter of the cylinder being equal to that of a circle described so as to fall within the elevation of the cone. This circle may be regarded as an end projection of the cylinder.

Horizontal section planes are convenient, since the sections of the cone are circles and those of the cylinder pairs of parallel straight lines in plan. In elevation the sections are overlapping straight lines.

On $g'h'$ describe a semicircle and divide its circumference into six equal parts.

Through any of the points of division, say $1'$, draw lm parallel to xy; this is the edge elevation of a horizontal section plane.

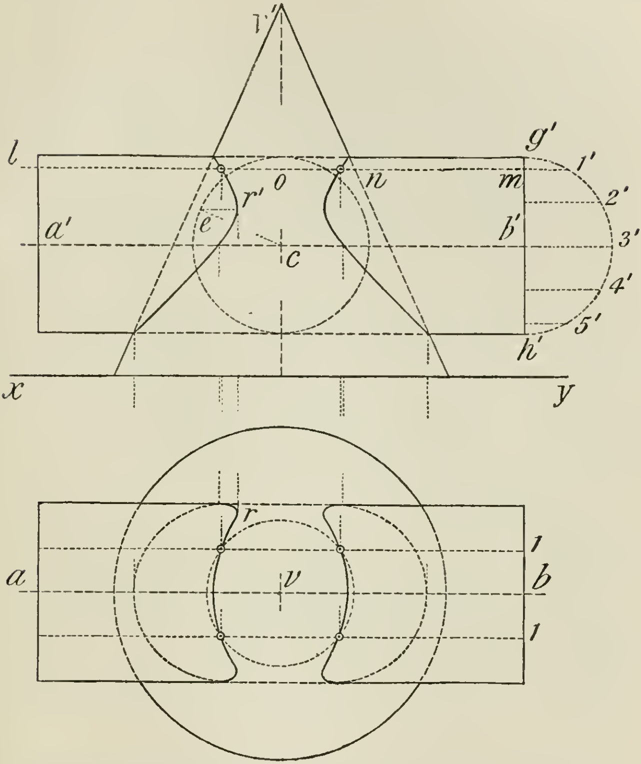
With v as centre and radius on describe a circle; this is the plan of the section of the cone by the plane LM.

Set off $b1 = m1'$ on each side of b as shown, and draw lines through the points 1 parallel to ab; these lines form the plan of the section of the cylinder by the plane LM.

Hence the points in which these sections intersect are points on the required plan, their elevations being found by projectors as shown.

Repeat this construction for the planes through g', $1'$, $2'$ . . . h', and draw a curve through the points thus found.

The planes through $4'$, $5'$, h' give rise to points which in plan are hidden by the cylinder, and the plane through $3'$ will divide the dotted and full portions of the curves in plan. The curve is seen to touch the outline of the cylinder in plan at these points.



Examples on Problem 317.

1. Determine the interpenetration of two cylinders, diameters $2\frac{1}{2}''$ and $2''$; axes intersecting at right angles, and parallel to vertical plane; smaller cylinder horizontal.
2. Work Ex. 1 (*a*) when the axis of the smaller cylinder is horizontal, but makes 30° with the vertical plane; (*b*) when the axis of the smaller cylinder is inclined at 30° to the horizontal plane.
3. A semi-cylinder, $4\frac{1}{2}''$ diameter, rests with its rectangular face on the ground. A cylinder, $2\frac{1}{4}''$ diameter, rests with a generator on the ground, at right angles to the axis of the semi-cylinder. Determine the plan of the curve in which the surfaces intersect.

Case (2).—The diameter of the cylinder is taken equal to that of a circle which touches $v'd'$ and $v'e'$.

Repeat the construction of Case (1). If this be done accurately, the elevation will be found to be two intersecting lines. The intersection of the surfaces therefore consists of two *plane* sections of either surface, that is, of two ellipses. These ellipses project as ellipses in plan. In this case the cone and cylinder circumscribe the same sphere. See Arts. 228 and 273, and Theorems 23 to 48, Appendix II.

An important section plane is that whose elevation passes through f', the point where $v'e'$ touches the circle; this section gives the points where the ellipses intersect in plan and elevation. Observe that at these points the surfaces touch one another, and two branches of their intersection cross one another at each point of contact.

Case (3).—The diameter of the cylinder is taken equal to that of a circle described so as to intersect the lines $v'e'$, $v'd'$, in points such as r', s'.

As important section planes take those whose elevations pass through g' and r'. Another plane between these will be sufficient to determine the upper curve of intersection.

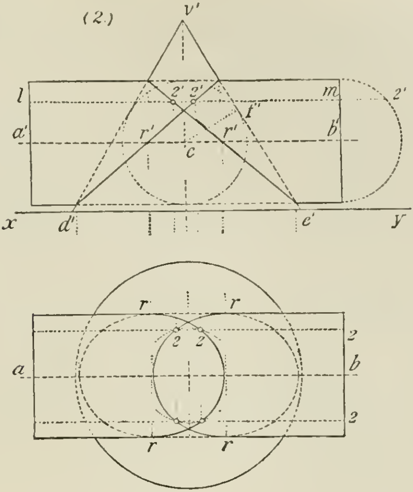
For the lower portion select the two important section planes whose elevations pass through s' and h', and in addition one or two planes between them.

Observe that the lower curve is hidden by the cylinder in plan.

Examples.—1. Draw an isosceles triangle *abc*, with the base $bc = 3\frac{1}{2}''$, and altitude $ad = 3\frac{1}{2}''$; this is the elevation of a cone, vertex *A*, with its base on the ground. On *da* take $de = 1\frac{1}{4}''$. With *e* as centre describe a circle to touch *ab* and *ac*. Also describe two circles concentric with this, but having radii $\frac{1}{16}''$ less and $\frac{1}{16}''$ greater. Each circle is the end elevation of a cylinder. In each case determine the plan and elevation of the curve of intersection of the two surfaces, when the cylinder is turned with its axis parallel to the vertical plane.

2. In Ex. 1 describe a semicircle on *xy* with centre *c* and radius $2\frac{3}{8}''$; this is the end elevation of a semi-cylinder; draw the plan of the curve in which the cone and semi-cylinder intersect.

(2)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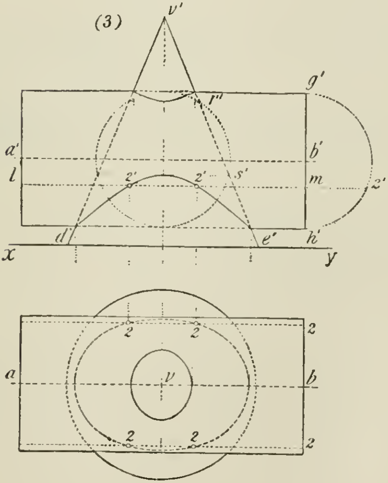
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319. PROBLEM.—**To determine the projections of the line of intersection of a given cylinder and cone, the axes of which are both vertical.**

Horizontal section planes are convenient.

Consider a plane whose elevation is *lm*. With centre *v* and radius *r′s′* describe a circle; this is the plan of the section of the cone. The points *p*, *p* in which this circle intersects the plan of the cylinder are points on the required plan; the elevations *p′*, *p′* are obtained by projection.

Sections which give important points.—Draw *vd* through *v* and *a*, intersecting the circle, centre *a*, in *d* and *c*; then *D* and *C* are the highest and lowest points on the line of intersection. Also draw *ni* through *a* parallel to *xy*; then *i′*, *n′* are on the elevation of the outline of the cylinder. Hence planes should be taken which cut the cone in circles whose radii are equal to *vd*, *vc*, *vn*, and *vi*. Two or three intermediate planes may be required.

Note.—Vertical section planes containing the axis of the cone would be very convenient to take in working this problem; the important points *D*, *C*, *N*, *I* might be readily determined in this manner. Such planes would cut both surfaces in straight lines.

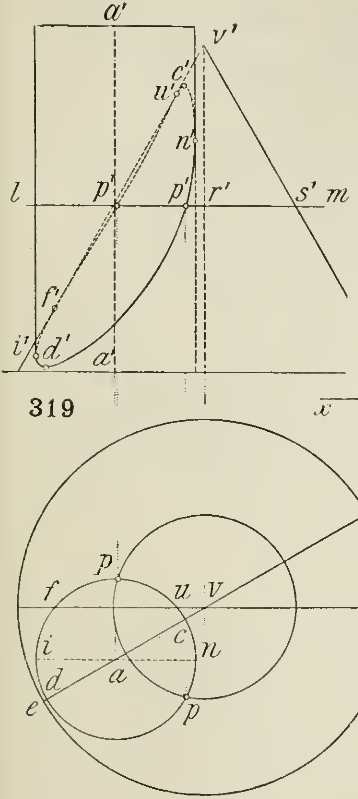
320. PROBLEM.—**To determine the interpenetration of a given vertical cylinder, axis AA, and a given sphere, centre S.**

Take vertical section planes parallel to *XY*. Consider one such plane of which *lm* is the plan.

Draw the circle with centre *s′* and radius *om*; this is the elevation of the section of the sphere. The section of the cylinder is a pair of straight lines, one of which has *c* for its plan and *c′c′* for its elevation. The line and circle intersect as shown in two points *p′*, *p′*, which are on the required elevation, *p* being the plan of the points *P*.

Important section planes are those whose plans pass through *e*, *a*, *s*, and *f*. One or two planes may be taken in addition to these.

It will be observed that the required plan is the arc *erf*.



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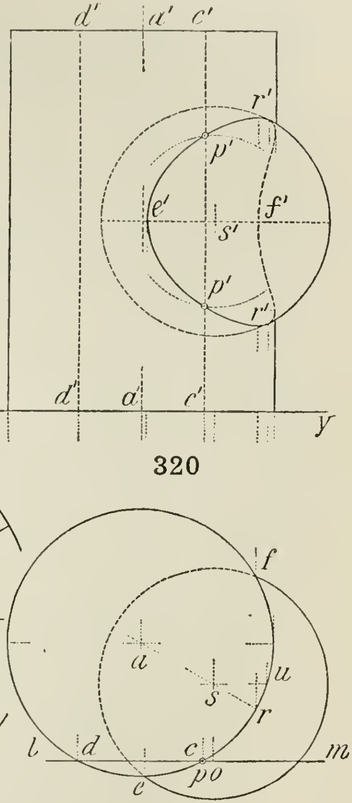

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Example on Problem 319.—Describe two circles with radii $1\frac{1}{2}''$ and $\frac{5}{8}''$, their centres c and v being .85″ apart; these are the plans of a cylinder and cone standing upright on the ground; height of cone 3″. Determine the elevation of their curve of intersection on an xy making 35° with cv.

Obtain the developments of the cone and cylinder, in each case showing the curve of intersection.

321. Problem. — **To determine the projections of the line of intersection of a given sphere, centre S, and a given vertical triangular prism.**

Either vertical or horizontal section planes may be employed, but the former are preferable in determining the important points.

Consider the plane whose plan is *lm*. With s' as centre describe a circle, the diameter of which is *ab*; this is the elevation of the section of the sphere.

Draw the lines $c'c'$, $a'd'$ by projecting from *c* and *d*; these form the elevation of the section of the prism.

The intersections *P* of these two sections are points on the required curve.

The important sections.—Draw *se*, *sn*, *so* respectively perpendicular to the sides of the triangle *uzw*, and take section planes through *e*, *o*, *n*; they give the highest and lowest points on the three portions of the intersection of the sphere and prism. Take section planes through *u*, *z*, *g*, *h*, *s*; the plane through *s* will give the points on the outline of the sphere in elevation.

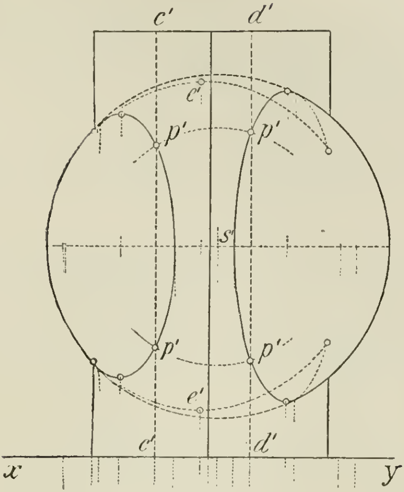
The curves of intersection are ellipses.

Examples on Problems 320 and 321.

1. Describe two circles with radii $1\frac{1}{8}''$ and $1''$, the distance between their centres being $\frac{3}{4}''$; these are the plans of a cylinder and sphere which intersect. Determine the elevation of the curve of intersection on an *xy* which makes 30° with the line joining the centres of the plans.

2. Draw an equilateral triangle *abv* $3\frac{1}{2}''$ side; bisect *av* in *c*. With centre *c* describe a circle to touch *ab* and *av*; this figure is the plan of a cone, vertex *V*, and a sphere, the axis of the cone being horizontal, and the centre of the sphere on the surface of the cone. Determine the plan of the curve of intersection and the elevation on an *xy* parallel to *ab*.

3. Draw a triangle *abc* with $ab = 2.75''$, $bc = 3.2''$, $ac = 2.7''$. Take a point *s* inside *abc* distant $1.1''$ from *ac* and $.9''$ from *bc*; with *s* as centre describe a circle $3.5''$ diameter. The circle and triangle are the plans of a sphere and prism. Determine the elevation of their curve of intersection on an *xy* making 20° with *ab*.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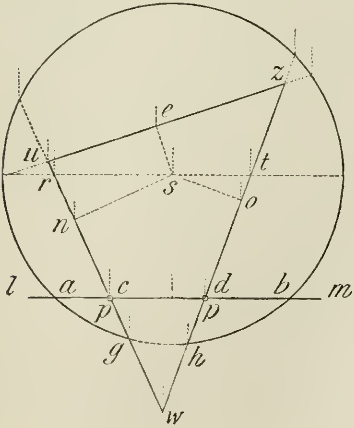
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322. PROBLEM.—**To determine the plan and an elevation of the line of intersection of a pyramid and prism, the indexed plans of which are given; the pyramid rests with its base on the horizontal plane, and the edges of the prism are horizontal.** Unit 0.1″.

Draw an elevation of the solids as seen when looking in a direction parallel to the long edges of the prism; that is, take *xy* perpendicular to the plans of these horizontal edges.

In this example the intersection will consist of a series of straight lines, which may be obtained by determining (1) the points in which the edges of the prism intersect the faces of the pyramid; (2) the points in which the edges of the pyramid intersect the faces of the prism; and then joining these points by straight lines in the proper sequence.

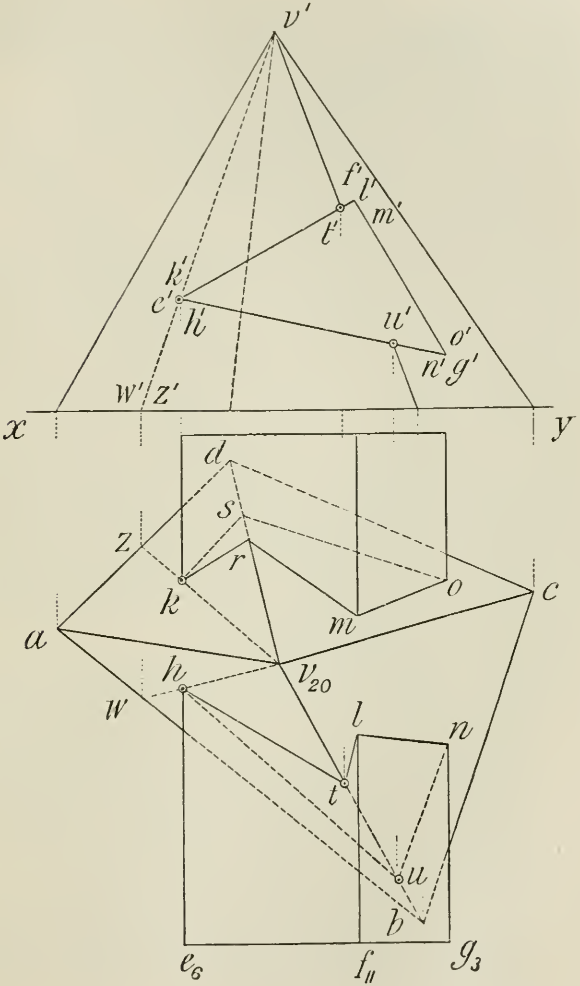
Join *v′e* and produce to *w′*; then *v′w′* is the elevation of two lines, one on each of the faces *AVB*, *AVD* of the pyramid, if *vw*, *vz* are their plans. This may be regarded as a section by a plane containing *V* and the horizontal edge *E*. Hence *H* and *K* are the points in which the edge *E* meets the faces *AVB*, *AVD* of the pyramid.

In a similar manner it is found that the edge *F* meets the faces *CVB*, *CVD* in *L* and *M*, and the edge *G* meets the same two faces in *N* and *O*.

Next consider the edge *VB* of the pyramid. It intersects the faces *EF* and *EG* of the prism in points whose elevations are *t′* and *u′*, the plans *t* and *u* being obtained at once by projection. In like manner the edge *VD* meets the same two faces in *R* and *S*.

When joining the points thus found, each face of the prism may be taken in turn and its lines of intersection with the faces of the pyramid noted.

Thus, taking the face *EG*, it will be seen that we must join *hu*, *un*, *os*, *sk*; all of which are underneath the prism and therefore dotted. On the face *GF* we get *ln*, *mo*; and on the face *FE* we have *ht*, *tl*, *mr*, *rk*.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323. Problem.—**The indexed plan of an irregular tetrahedron is given, the circle representing a cylindrical hole bored vertically through the solid. Draw the elevation of the pyramid on a vertical plane parallel to AB.**

Take *xy* parallel to *ab* and draw the elevation.

The hole passes through the face *ABC* and partly through the faces *ABD*, *CBD*.

A series of vertical section planes passing through *B* will in this case be convenient.

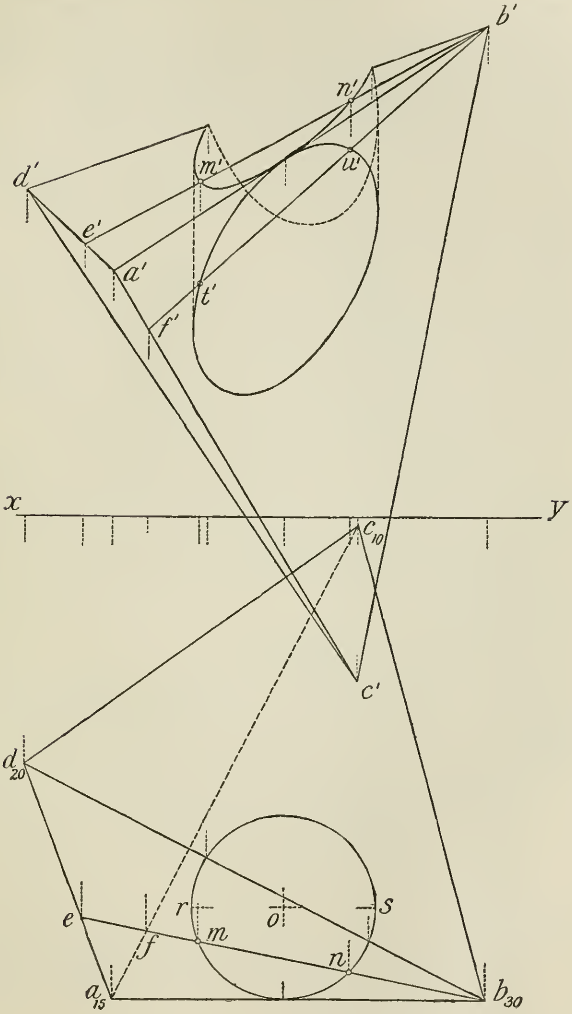
Consider the section plane whose plan is *be*; it cuts the pyramid in a triangle *BFE* and the surface of the hole in two vertical lines whose plans are *m* and *n* respectively. As obtained from the elevations of the lines these two sections intersect in *T*, *U*, *M*, and *N*, which are therefore points on the required curve.

Important Sections.—Through *o* draw *rs* parallel to *ab*. Then the section planes whose plans pass through *b* and each of the points *r*, *s*, *d* are important; also the planes which touch the surface of the hole. In connection with the last two planes it should be observed that the lines in which they intersect the faces of the pyramid are tangential to the curve of intersection. This will appear on drawing the elevation, and the points of contact should be found.

Examples on Problems 322 and 323.

1. Draw a quadrilateral *abcd*, having $ab = 3\frac{3}{4}''$, $bc = 2\frac{5}{8}''$, $ac = 3\frac{1}{2}''$, $ad = 2''$, $cd = 2\frac{1}{2}''$. Take an inside point *v* 1″ from *ab* and $1\frac{1}{2}''$ from *ad*; join *v* to *a*, *b*, *c*, *d*. This is the plan of a pyramid with its base *ABCD* on the ground, and the vertex *V* 3″ high. An equilateral triangular prism with its long edges horizontal and perpendicular to *AC* penetrates the pyramid. A side of the end of the prism is $1\frac{7}{8}''$; one face of the prism is horizontal, its centre being $2\frac{3}{4}''$ vertically below *V*. Determine the plan of the intersection and the elevation on an *xy* parallel to *ac*.

2. A quadrilateral $a_{20}b_{35}c_{0}d_{18}$, with the diagonals *ac*, *bd*, is the plan of an irregular pyramid; $ab = 5''$, $bc = 4''$, $ac = 4\frac{1}{2}''$, $cd = 3''$, and $ad = 2\frac{1}{2}''$. A circle, diameter 2″, with its centre on *ac* and touching *cd*, is the plan of a vertical cylindrical hole cut through the pyramid; determine the elevation of the pyramid and hole on an *xy* parallel to *cd*. Unit 0.1″.



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324. PROBLEM. — **To determine the projections of the intersection of two surfaces of revolution given in elevation, the axes of which intersect each other and are parallel to the vertical plane, one being vertical.**

The given surfaces are generated by the revolution of a circular arc about an axis in the plane of the arc; in one case the axis, CD, intersects its arc; and in the other the axis, AB, does not intersect it. We shall employ the method of *spherical* sections.

Let the axes intersect in O, and with o' as centre describe a circle as shown in the figure; let this be the elevation of a spherical section surface which intersects each given surface of revolution in a pair of circles, each circle of one pair cutting one of the circles of the other pair. Thus the circles whose elevations are $e'f'$ and $g'h'$ cut each other at points which have r' for elevation.

In like manner s' is obtained from the other two circles.

To obtain the plans of these points: with centre o describe a circle, the diameter of which is $g'h'$; draw projectors from r', s' to meet the circle in r, s.

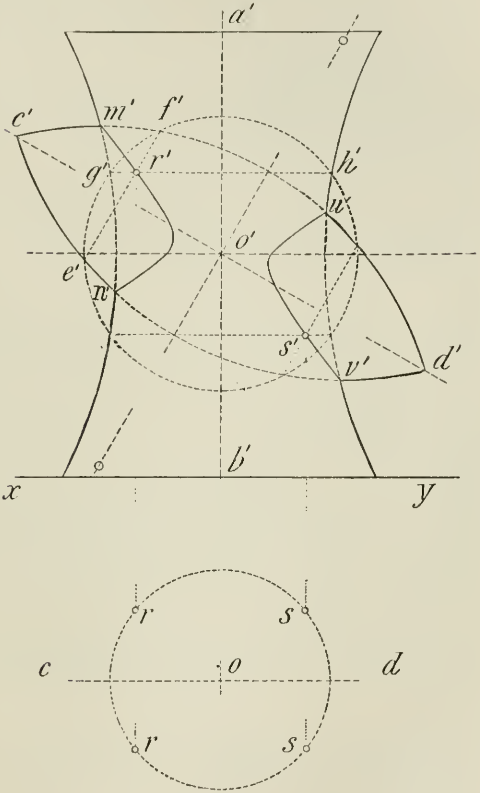
Repeat this construction for two or three other spherical sections, centre o.

The plans of the surfaces and of the curves of intersection are not shown in the figure.

Since the outlines of the surface in elevation may be regarded as sections by a plane containing the two axes, it follows that M, N, U, V are on the curve of intersection.

Examples.—**1.** Determine the plan and elevation of the curve of intersection of two surfaces of revolution like those in the figure, whose axes intersect, the axis of one being vertical and that of the other inclined at 45° to the ground. The least cross-section of the one is $1\frac{1}{2}''$ in diameter, and the greatest cross-section of the other is $1\frac{3}{8}''$. The radius of the curved outline of each is $3\frac{1}{2}''$.

2. The axes of a cylinder and a double cone of indefinite length intersect each other at a point $\frac{3}{8}''$ from the vertex of the latter. Vertical angle of cone 55°, axis vertical. Diameter of cylinder $1\frac{5}{8}''$; inclination of axis 45°. Draw the plan and elevation of their curve of intersection.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325. Miscellaneous Examples.

***1.** Fig. (*a*). The plans of the edges of a four-faced right prism with a horizontal axis are given. The prism is penetrated by a vertical square prism $2\frac{3}{4}''$ high. Draw an elevation on a plane parallel to the axis of the horizontal prism, showing the invisible portions of the intersection by dotted lines. Unit = 0.1″. (1886)

***2.** Fig. (*b*). The elevation of a right cone resting on the horizontal plane is given. The hatched semicircle is the elevation of a hemi-cylindrical portion cut out of the cone. Draw the plan of the remaining portion of the cone. (1893)

***3.** Fig. (*c*). The given pyramid on quadrilateral base *ABCD* in the horizontal plane, and vertex v_{27}, is penetrated by the given triangular prism, the edges of which are horizontal. Determine the intersection of the surfaces of the two solids. Unit = 0.1″. (1890)

***4.** Fig. (*d*). The flat base of a hemisphere of $1\frac{1}{4}''$ radius (centre *O*) rests on the ground, as does also that of a pyramid *abcd*, the height of its vertex (*V*) being $2\frac{1}{2}''$. Draw the plan and elevation on *xy* of the intersection of the pyramid and hemisphere, and develop the faces *BVC*, *DVC* so as to show the development of the intersection. (1896)

***5.** Fig. (*e*). A vertical triangular slot is cut through a sphere. The plan of the sphere and slot is given. Draw an elevation of the sphere. (1879)

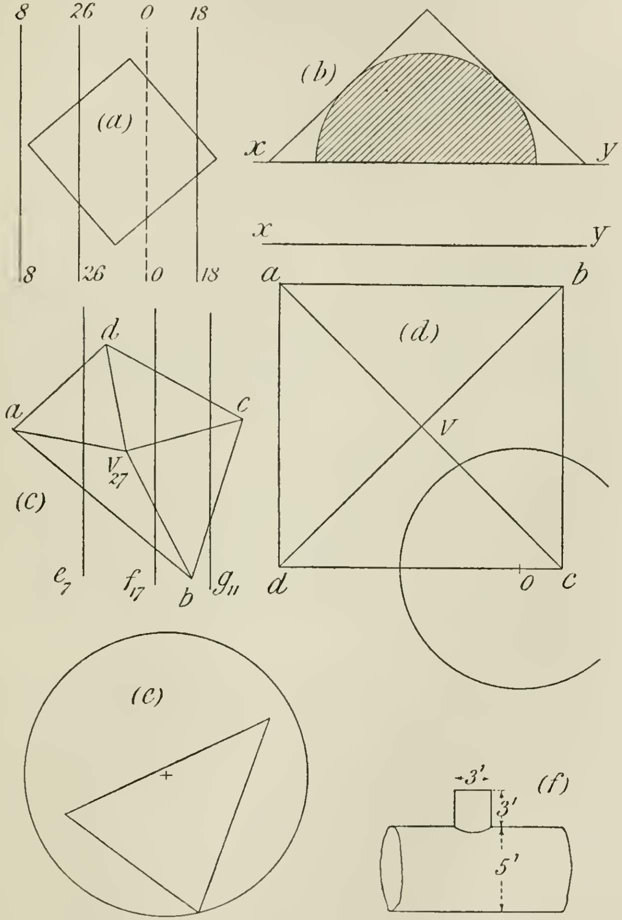
***6.** Fig. (*f*). It is required to fit a cylindrical steam dome on the top of a cylindrical boiler shell. A sketch with the required dimensions is given. Draw the elevation of the curve of intersection of the dome and shell, and obtain the development of the dome. Scale $\frac{1}{24}$. (1880)

7. A horizontal cylindrical hole (diameter $1\frac{1}{4}''$) is bored through a vertical cylinder (diameter $2\frac{3}{4}''$); the axis of the boring cylinder passes $\frac{1}{4}''$ from the axis of the vertical cylinder and is inclined 25° to the vertical plane. Draw the elevation of the vertical cylinder. (1877)

8. A cylinder and sphere of $1\frac{1}{4}''$ and $1\frac{1}{2}''$ diameters respectively rest with their curved surfaces on the ground, the centre of the sphere being in the surface of the cylinder. Draw the plan of the curve of penetration and an elevation of the curve on a vertical plane making 45° with the axis of the cylinder.

9. A cone, height 3″, rests with its base on the ground, diameter of base 3″. The axis of a cylinder 2″ diameter is parallel to the vertical plane and inclined at 45° to the ground, and passes through the vertex of the cone. Determine the plan and elevation of the curve of intersection of the surfaces.

Copy the figures double size.



CHAPTER XVII

CAST SHADOWS

326. Preliminary.—It is a matter of common observation that the rays of light which emanate from any source proceed in straight lines in all directions through space, except so far as they may be intercepted by opaque objects, or otherwise influenced. If a surface receive the rays, a portion of the surface will be deprived of light by the interposition of the opaque body. The space devoid of rays between the body and the surface is the shadow of the body, and that part of the surface deprived of light is the shadow cast by the body on the surface, or the *cast shadow.* Thus a lunar eclipse occurs when the moon enters the region of the earth's shadow, and as we watch the phenomenon we see the earth's *cast shadow* on the face of the moon.

The shadows we see around us are generally of a most complicated nature. Although there may be only one original source, yet all the surrounding objects on which the light falls give back some of the light, and thus become secondary sources of greater or less intensity and of varying size; these reflected lights play a very important part in pictures. We give no account here of this maze of varying light and shade. Our task is comparatively simple; we confine attention to one source of illumination, and this is further supposed to be so small that it may be treated as if it were a point. The corresponding shadows may be termed *geometrical.* The nearest approach to this in nature is perhaps an electric arc light, the cast shadows from which are very sharply defined. When the sun is the source, the shadows are softened at the edges on account of the angular magnitude of the sun's disc.

If the point source is near at hand we have *divergent* rays, but we generally simplify the problems still further, and assume that the source is so far away that all the rays are practically *parallel* to one another.

327. Theorems relating to geometrical shadows.—The opaque object which casts the shadow receives light from the source on one portion of its surface, the other portion being in shade. The line on the surface which divides the two portions is called the *line of separation;* this line is evidently the locus of the point of contact of those rays which *touch* or *graze* the surface; it is further evident that these bounding or extreme rays are those which define the *outline* of the shadow cast on any surface. We thus have

Theorem 1.—*For any object which casts a shadow, the line of separation on its surface is the locus of the point of contact of the bounding rays; and the shadow cast by the line of separation on any surface is the outline of the shadow cast by the object itself on the same surface.*

We may if we like regard the rays of light as projectors, in which case the shadow cast by any body on a plane becomes its outline projection on the plane. The projection is *parallel* (and oblique) or *radial*, according to whether the rays are *parallel* or *divergent*.

Under another aspect the cast shadow may be looked upon as the *section* of a *cylinder* or *cone*, the latter terms being here used in their wider meanings (see Definitions 24 and 27 of the Appendix); the bounding rays become the *generators* of the cylindrical or conical surface, and the line of separation is the directing curve.

Thus theorems relating to cast shadows may often be at once deduced from familiar theorems of projection, or of conic sections, *e.g.*—

Theorem 2.—*The shadow cast by a point on any surface is the trace on that surface of the ray through the point.*

Theorem 3.—*The shadow cast by a straight line on any plane is the straight line joining the shadows cast by its ends.*

Theorem 4.—*The shadow of a straight line on any surface coincides with the trace on that surface of a plane which contains the straight line and the source, for divergent rays, or which contains the straight line and is parallel to a ray, for parallel rays.*

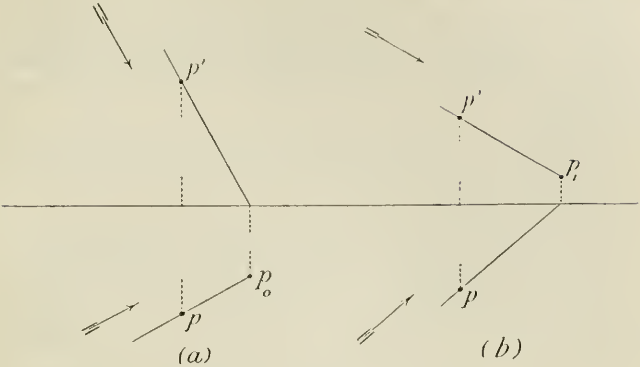
Theorem 5.—*If a series of straight lines be parallel to one another, their shadows on any plane by parallel rays are also parallel to one another and of proportionate lengths. If the rays are divergent the shadows are also divergent.*

Theorem 6.—*If a plane figure be parallel to a plane, its shadow on the plane is equal and similar to the figure if the rays be parallel, and is similar if the rays be divergent.*

This last theorem follows from a property of the cylinder or cone, viz. that parallel sections of a cone (or pyramid) are similar figures, and of a cylinder (or prism) are similar and equal figures. Thus for parallel rays the shadow cast on the ground by a horizontal circle is a circle of the same size, the centre of the latter being the shadow cast by the centre of the former.

In regard to the general method of working problems, in some cases it is necessary to determine the line of separation *before* we can obtain the cast shadow; in others the shadow helps us to determine the line of separation; in all cases the connection between the two should be constantly borne in mind. If the object have any corners, the shadows of these should be found. When a shadow is cast on two surfaces which intersect, such as the two planes of projection, those points on the outline of the shadow which fall on the intersection should be specially determined as *important points.*

The following problems are confined to parallel rays. The direction of the latter may be defined by the projections, or by the indexed plan, of any single ray.



328. Problem.—**To determine the shadow cast by a given point P, (a) on the horizontal plane, (b) on the vertical plane, having given the direction of the parallel rays in plan and elevation.**

(*a*) Draw the projections of a ray through *P*; that is, through *p* draw a line parallel to the plan, and through *p'* a line parallel to the elevation of the given direction of the rays. Determine p_0, the horizontal trace of the ray; then p_0 is the required shadow.

(*b*) Draw the projections of a ray through *P* and determine its vertical trace p_1; then p_1 is the required shadow.

Examples.—1. A point *A* is 2″ in front of the vertical plane and 1″ above the ground. The plans and elevations of the rays make angles of 30° and 45° with *xy*. Determine the shadow of the point on the ground.

2. Take the point *A* in Ex. 1 to be 1″ from the vertical plane and 2″ above the ground; find the shadow on the vertical plane.

3. A point *A* is 2″ above the ground and 1″ from the vertical plane. (1) Determine whether the shadow is cast on the vertical or horizontal plane, the rays being inclined at 40° and their plans making 30° with *xy*. (2) The direction of the plan being unaltered, what must be the inclination of the ray if the shadow of *A* is on *xy*, and what if the shadow is $\frac{1}{4}$″ below *xy*?

329. PROBLEM.—**To determine the shadow cast by a given line AB, (a) on the horizontal plane, (b) on the vertical plane, (c) on both planes of projection, having given the direction of the parallel rays in plan and elevation.**

(*a*) By Prob. 328 determine a_0 and b_0, the shadows cast by A and B on the horizontal plane; then a_0b_0 is the required shadow of the line.

(*b*) Find by Prob. 328 a_1 and b_1, the shadows cast by A and B on the vertical plane; then a_1b_1 is the required shadow.

(*c*) Determine a_0b_0, the shadow cast by AB on the horizontal plane, supposing the vertical plane to be transparent like glass; also determine b_1, B's shadow on the vertical plane. Let a_0b_0 meet xy in i_0; join i_0b_1.

Then the broken line $a_0i_0b_1$ is the required shadow, a_0i_0 being on the horizontal plane and shown in plan, while i_0b_1 is on the vertical plane and is shown in elevation.

Examples.—**1.** A is a point 2″ from each plane of projection; B is 3″ from the vertical plane and 1″ above the ground; AB is 3″ long. Determine (1) the shadow cast on the ground; (2) the shadow cast on a plane parallel to the vertical plane and 1″ in front of it; (3) the shadow cast partly on the vertical plane and partly on the ground, and determine the point on AB whose shadow is in xy. The plan and elevation of a ray make 30° and 60° with xy.

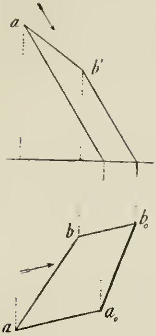
2. $a_{20}b_{10}$ is 2″ long and is the indexed plan of a line AB; c is 2″ from b and 1″ from a. The plan of a ray makes 60° with ab, while the inclination of a ray is 50°. If the shadow of C falls on the shadow of AB, determine the index of c. Unit = 0.1″.

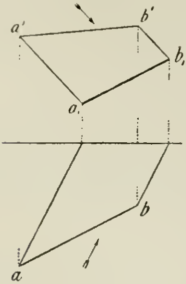
3. The plan and elevation of a ray both make 45° with xy; determine the inclination of the ray. *Ans.* 35.2°.

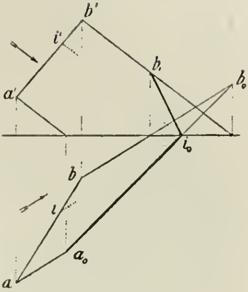
4. A system of parallel rays make 45° with xy in plan, and their inclination is 45°; find their direction in elevation.

5. A triangle ABC has the point A on the ground, and its plan is an equilateral triangle abc of 2.5″ side. The shadow of ABC on the ground is a right-angled isosceles triangle with a as vertex, the parallel rays having an inclination of 45°, and their plan making 45° with ab and 15° with ac. Determine the shadow ab_0c_0; index the points b and c; and find the true shape of ABC.

Hint.—Make use of Prob. 21 to draw the shadow.







Examples on Problems 330 and 331.

1. A cube 2″ edge rests with a face on the ground, one side of the base making 40° with *xy*, the nearer end of this side being 1″ from the vertical plane. Determine the shadow on the planes of projections when the rays in plan and elevation make angles of 45° with and are directed towards *xy*. Indicate the two portions of the line of separation which give rise to the two parts of the shadow, namely, those which are cast on the horizontal and vertical planes.
2. Copy double size the figured plan of the tetrahedron on p. 199, and attach the same indices; unit 0.1″. Find the shadow of the solid on the ground, without drawing an elevation, making use of the following theorem. The rays are inclined at 45°, and in plan are parallel to *ab*.

   Theorem.—For parallel rays, the line joining the plan of a point to its shadow on the ground is parallel to the plan of a ray, and of length proportional to the height (or index) of the point, this length being equal to the height when the inclination of the ray is 45°.
3. An octahedron of 2″ edge rests with a face on the ground. Draw its plan, and by the method of Ex. 2 determine its shadow on the ground, the rays being inclined at 35°, and in plan making 45° with a horizontal edge of the solid.
4. Determine the shadow cast on both planes of projection by the pyramid in Ex. 23, page 201, the vertical plane being parallel to *ad*, the rays inclined at 45° and parallel, in plan, to *av* Indicate the line of separation and the portions of it which cast the shadows on the ground and vertical plane respectively.

330. Problem.—**A given cube rests with one face on the ground; to determine its shadow on the latter from given parallel rays. Also to find the line of separation on the cube.**

The projections of the cube are shown in the figure.

Consider the vertical edge BB. Obtain b_0, the shadow of the upper end B. Join bb_0; then bb_0 is the shadow of BB.

Draw cc_0 and dd_0, each parallel and equal to bb_0: join b_0c_0 and c_0d_0. Then $bb_0c_0d_0dab$ is the outline of the required shadow of the cube.

It will be evident that the outline of the shadow is cast from the vertical edge BB, the upper horizontal edges BC, CD, the vertical edge DD, and the lower horizontal edges DA, AB; hence these edges constitute the line of separation, the vertical faces BC, CD and the base of the cube being in shade.

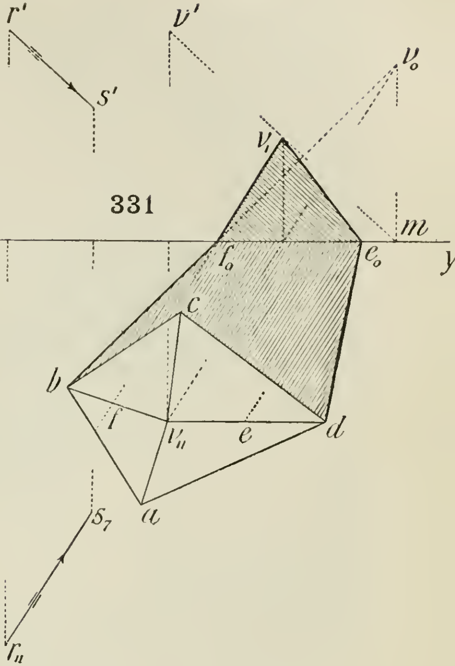
331. Problem.—**The indexed plan of an irregular pyramid, with its base ABCD resting on the ground, is given, and also $r_{11}s_7$, the indexed plan of a parallel ray. To determine the shadow of the pyramid on the ground, and on the vertical plane the plan of which is lm. Also to indicate the line of separation on the solid. Unit 0·1″.**

On lm as a ground line determine the elevations of RS and V. Obtain v_0 and v_1, the horizontal and vertical traces of a ray through V, and join v_0b, v_0d, meeting lm in f_0 and e_0.

The lines v_0b, v_0d, together with ab, ad, would form the outline of the shadow of the pyramid on the ground if the vertical plane LM were transparent or were removed.

Join e_0v_1, f_0v_1; then bf_0e_0dab is the outline (in plan) of that part of the shadow which falls on the ground, and $f_0v_1e_0$ is the outline (in elevation) of the remainder of the shadow, which falls on the given vertical plane.

Since the outline of the shadow is cast from the edges VB, BA, AD, and DV, these edges constitute the line of separation.



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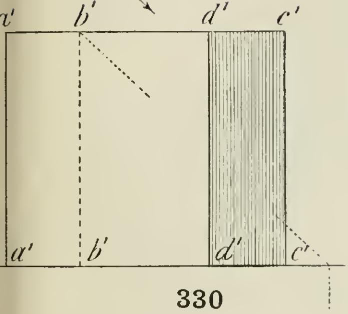

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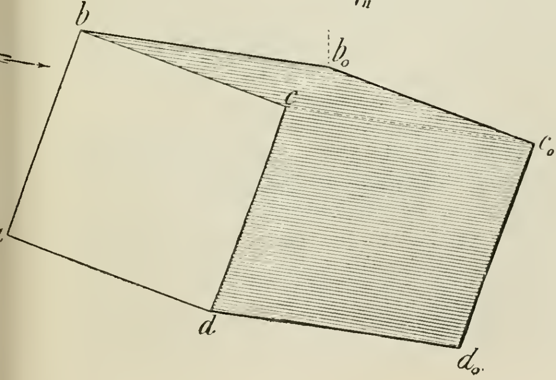


332. Problem.—**A given circle, centre C, is parallel to the vertical plane, to determine the shadow cast by it on the planes of projection, having given the rays.**

We must first find the shadow cast on the vertical plane, on the supposition that the horizontal plane is transparent. That is, determine c_1, the shadow cast by C on the vertical plane, and with c_1 as centre describe a circular arc with a radius equal to that of the given circle; this arc meets xy in d_0 and e_0, and is that portion of the required shadow which falls on the vertical plane. See Theorem 6, Art. 327.

Through d_0 draw d_0d' parallel to the given elevation of a ray, and draw $d'e'$ parallel to xy. It will be obvious that the arc DSE is that which casts the shadow on the vertical plane; the remaining arc DRE throws its shadow on the horizontal plane.

To determine the latter shadow, take any point N on the circle; determine n_0; draw n_0m_0 parallel to xy, making $n_0m_0 = n'm'$; then n_0, m_0 are two points on the required shadow. Repeat this construction for one or two other points on arc RND, and draw the elliptical arc d_0ce_0 through the points so obtained. This will be the required shadow on the horizontal plane.

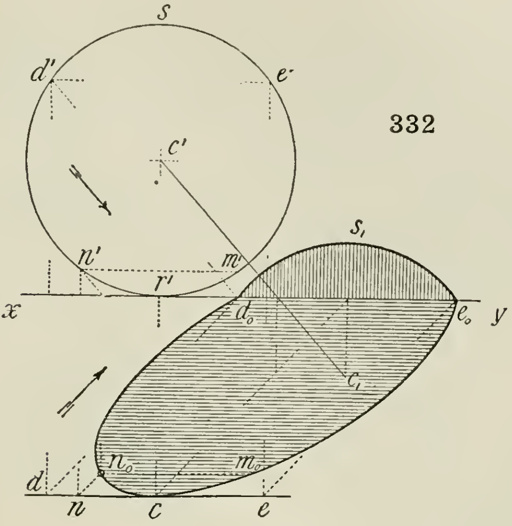
Note that at the point of contact of the circle and ground, the tangents to the shadow and circle coincide.

An important ray is the one which *touches* the circle in elevation at say t' (not shown). The corresponding projector for the shadow *touches* the latter at t_0.

333. Problem.—**To determine the shadow cast by a given sphere on the ground, the given rays being parallel. To find also the line of separation on the sphere.**

Suppose the sphere to be circumscribed by a cylinder, the axis of which is parallel to the given rays, then the horizontal trace of the cylinder will be the required shadow, which may therefore be found as in Prob. 277.

The line of separation is the circle of contact of the cylinder and sphere.



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Examples.—1. A circle $2''$ in diameter has its plane parallel to the vertical plane and $1''$ therefrom, its centre being $1\frac{1}{4}''$ above the ground. Determine the shadow cast on the planes of projection, the rays in plan and elevation making angles of $35°$ and $40°$ respectively with xy.

Determine the projections of that chord of the circle whose shadow coincides with xy.

2. Suppose that the circle in Ex. 1 has its plane horizontal, the centre being $2''$ above the ground and $1\frac{1}{2}''$ from the vertical plane. Determine the shadow cast from it on the planes of projection.
3. A sphere $2''$ radius has its centre $2''$ from each plane of projection. Determine the shadow on the ground if the rays on plan and elevation make angles of $30°$ and $45°$ with xy.
4. Determine the shadow of the sphere in Ex. 3 cast on a plane inclined at $30°$, and $1\frac{1}{2}''$ from the centre of the sphere, the rays being parallel to the vertical plane, and inclined at $40°$ to the ground and $110°$ to the inclined plane.
5. In Ex. 4 what should be the inclination of the plane, so that the plan of the shadow is a circle, the rays being unchanged?

334. PROBLEM.—**To determine the shadow cast on the horizontal plane, (a) by a given cone resting with its base on the ground; (b) by a given cone, axis vertical, with its vertex on the ground; and (c) by a given cylinder resting with its base on the ground. Also to show the line of separation in each case.**

(a) Obtain v_0, the shadow of the vertex on the ground; draw the tangents v_0t_0, v_0r_0; then these tangents together with the arc t_0nr_0 form the outline of the required shadow.

Draw the radii vt_0, vr_0 at right angles to the tangents; then the generators VT, VR and the arc TRN form the line of separation on the cone, the portion $VTNRV$ being illuminated, while the remaining surface, including the base, is in shade.

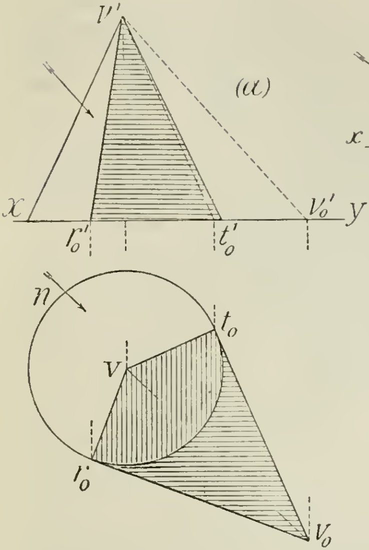
(b) Determine c_0, the shadow of the centre of the base of the cone, and with c_0 as centre describe a circle with a radius equal to that of the base; draw the tangents vt_0, vr_0. These tangents together with the arc $t_0n_0r_0$ form the outline of the shadow on the ground.

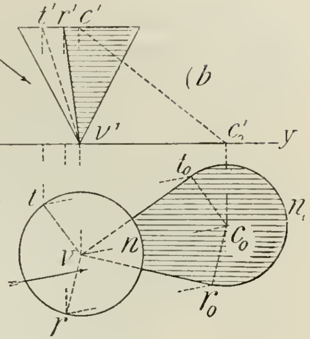
Draw the radii c_0t_0, c_0r_0 perpendicular to the tangents; and draw vt, vr respectively parallel to these radii; then VT, VR and the arc TNR constitute the line of separation, and the portion $VTNRV$ of the conical surface is in shade.

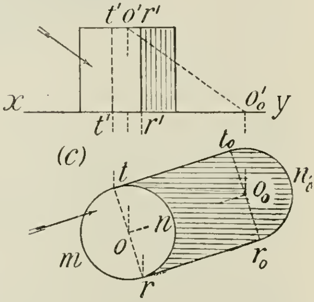
(c) Determine o_0, the shadow of the centre of the upper end, and with o_0 as centre describe a circle with a radius equal to that of the cylinder. Draw the tangents tt_0, rr_0; then these tangents together with the semicircles rmt, $r_0n_0t_0$ form the outline of the required shadow.

Draw the diameter rt at right angles to oo_0, then the generators RR, TT, the upper semicircle RNT, and the lower semicircle RMT form the line of separation; one-half of the cylinder, including the base, is in shade.

Note.—Observe that the shadows v_0r_0, v_0t_0 in (*a*) and (*b*), and rr_0, tt_0 in (*c*) are the horizontal traces of the planes parallel to the rays which *touch* the surfaces along VR, VT, or RR, TT. See Theorem 4, Art. 327.







Examples.—**1.** A cone rests with its base on the ground; diameter of base 2″, height 3″, and the centre of the base $2\frac{1}{2}''$ from *xy*. Determine the shadow on the ground, the rays in plan and elevation making angles of 30° and 45° with *xy*. Indicate the line of separation, and the portion of the surface of the cone which is in shade.

2. Suppose the cone in Ex. 1 to be cut by a horizontal plane 2″ high; find the shadow of the frustrum on the ground.

3. Determine the shadow on the ground of the cone in Ex. 1 when resting with the axis vertical and the vertex on the ground. Show the line of separation.

4. A cylinder 2″ diameter, length of axis 3″, rests with one end on the ground, the centre of the base being $2\frac{1}{2}''$ from *xy*; if the rays are as in Ex. 1, determine the shadow on the ground. Indicate the line of separation.

335. PROBLEM.—**A given cylinder, axis AC, rests on the ground with its axis at right angles to the vertical plane. Determine the projections of its shadow on the planes of projection from given parallel rays. Show also the line of separation on the cylinder.**

We may regard the required shadow as consisting of three portions, one from the *curved* surface of the cylinder, and two others from the *flat* circular ends. If these three were found separately they would be seen to intersect each other; only the outline is required for the shadow.

To determine the shadow cast by the curved surface, draw the two tangential rays in elevation, touching the circle at t' and r', from which tt and rr may be obtained by projection. Then TT and RR form the line of separation for the curved part of the cylinder.

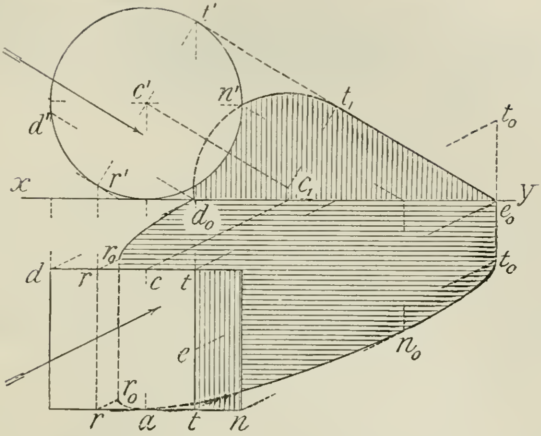
Obtain r_0r_0 and $t_0e_0t_1$, the shadows of RR and TT on the horizontal and vertical planes (Prob. 329); in determining these use may be made of the fact that r_0r_0 and t_0t_0 are each equal and parallel to rr or tt.

To obtain the shadow cast by the circular end nearer to the vertical plane, determine c_1, the vertical trace of a ray through C, and with c_1 as centre and radius equal to that of the cylinder describe a circular arc; it will touch e_0t_1 at t_1 and meet xy in d_0. This arc is that portion of the shadow of the circle which falls on the vertical plane.

Through d_0 draw d_0d' parallel to the elevation of a ray; then the arc DR is the only part of the circular end which casts a shadow on the ground. To obtain this shadow it will be sufficient to consider one point on DR between D and R (not shown), then by finding the shadow of this point we shall be enabled to draw the elliptical arc r_0d_0.

The shadow from the other circular end consists of an elliptical arc extending from r_0 to t_0 as shown, and passing through a. Points on this arc may be found by taking a few points on the semicircle TNR and determining their shadows on the ground.

Observe that r_0r_0, t_0t_0 are tangents to the elliptic arcs at



r_0, r_0, t_0 ; and that e_0t_1 touches the circle at t_1 ; note also that *at* and nn_0 are tangents to the ellipse at *a* and n_0.

The line of separation consists of the generators *RR*, *TT*, together with the semicircles *RNT*, *RDT*. This is the line whose shadow gives the outline.

Examples.—1. A cylinder, 2″ diameter, length 2″, rests on the ground along a generator which is at right angles to the vertical plane. The nearer end of the cylinder is $\frac{5}{8}''$ from the vertical plane. The rays, in plan and elevation, make angles of 30° and 45° with and are directed towards *xy*. Determine the shadow on the planes of projection ; indicate the line of separation, showing the two parts of it which cast the shadow on the horizontal and vertical planes respectively.

2. A semi-cylinder, 2.5″ diameter, 2″ long, rests with a semicircular end on the ground, its curved surface touching the vertical plane, and its rectangular face parallel to the latter. Draw its plan and elevation and determine the shadow cast on the planes of projection, the parallel rays making 45° with *xy* in both plan and elevation, and being directed towards *xy*.

3. A hemisphere 3″ diameter rests with its curved surface on the ground, its flat face being horizontal, and its centre 2″ in front of the vertical plane. Obtain the shadow cast on the planes of projection, the rays being as in Ex. 2.

336. PROBLEM.—**A square slab as shown in the figure has a square hole cut through its centre ; to determine its shadow on the ground, the parallel rays being given. Also to indicate the line of separation.**

The shadow cast by the outer edges of the slab is easily obtained and requires no description.

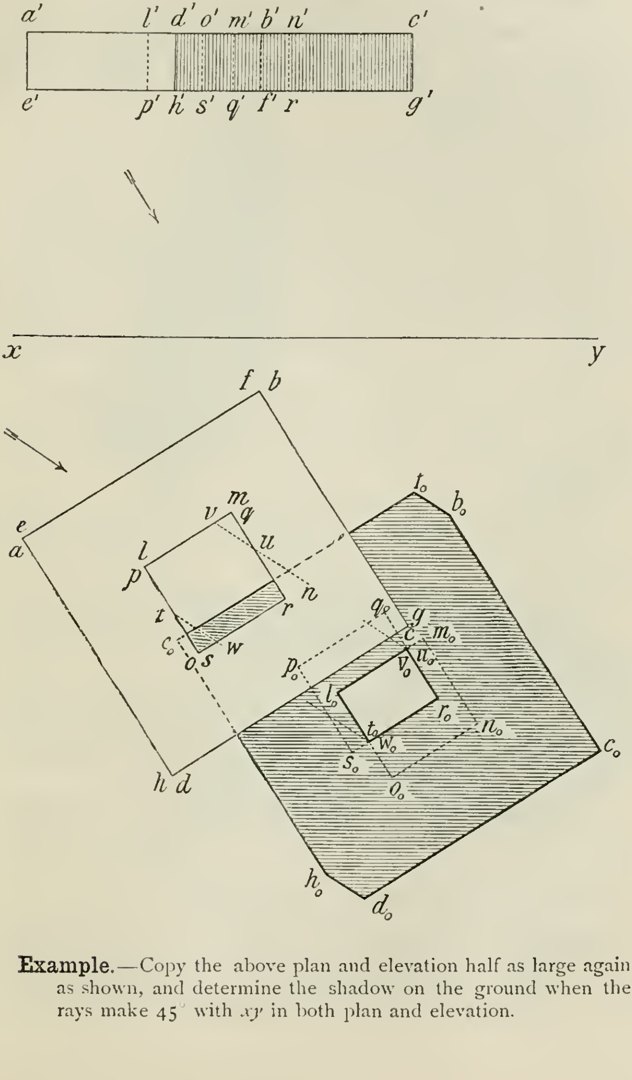
For the hole, the shadows from the upper and lower squares, supposing that they existed alone, would be $l_0m_0n_0o_0$ and $p_0q_0r_0s_0$ respectively. These squares intersect at two points, denoted by u_0, v_0 and t_0, w_0 ; from which we infer that the two rays which pass through them respectively must intersect the sides of *both* the upper and lower squares of the hole. Draw the plans of the rays through v_0 and t_0 intersecting the plan of the hole in v, u and t, w.

It will now be obvious that the outline of the shadow from the edges of the hole is cast by TL and LV on the upper square, and UR, RW on the lower, and is composed of the *innermost* lines of the squares $p_0q_0r_0s_0$, $l_0m_0n_0o_0$. On the contrary the outline of the shadow of the outer edges of the slab might be obtained by first determining the shadow from the upper and lower squares and selecting the *outermost* lines for the outline of the shadow.

It should be noticed that the shadow from VM would be cast on the vertical face $MNRQ$ of the hole and would be a straight line from U to M; this is not shown. Similarly, the shadow from the line TO would be a straight line WO on the face $ONRS$.

The line of separation therefore consists of the edges EF, FB, BC, CD, DH, HE, and also the lines LM, MU, UR, RW, WO, OL. The portions VM, TO cast shadows on the internal surface of the hole, and therefore do not contribute to the outline of the shadow on the ground.

The student should learn from this instructive example that the points in which two shadows intersect may assist in tracing the line of separation, which always passes round the object in a circuit without break.



Example.—Copy the above plan and elevation half as large again as shown, and determine the shadow on the ground when the rays make 45° with *xy* in both plan and elevation.

337. Problem.—**To determine the projections of the line of separation on the surface of a given solid of revolution, axis vertical, the rays being parallel to the vertical plane, and hence to obtain the shadow of the solid on the ground.**

The solid considered is the same as that in Prob. 281. Let o' be the centre for the arc $a'c'b'$.

Draw any radius $o'd'$ intersecting $a'b'$ in s'; with centre s' draw the circle through d'. This is the elevation of a sphere inscribed in the given figure, and touching it in a circle whose elevation is $e'd'$. Through s' draw a diameter perpendicular to the given elevation of a ray, and meeting $e'd'$ in p'. Then P is a point on the required line of separation.

For the ray through P evidently *touches* the sphere at P; hence it must also touch the surface of revolution at P. (See Art. 285.) P is thus a point on the required line of separation.

The plan of P may be found by describing a circle with s as centre, and diameter equal to $e'd'$; a projector from p intersects this circle in p, p, which are the plans of *two* points on the line of separation.

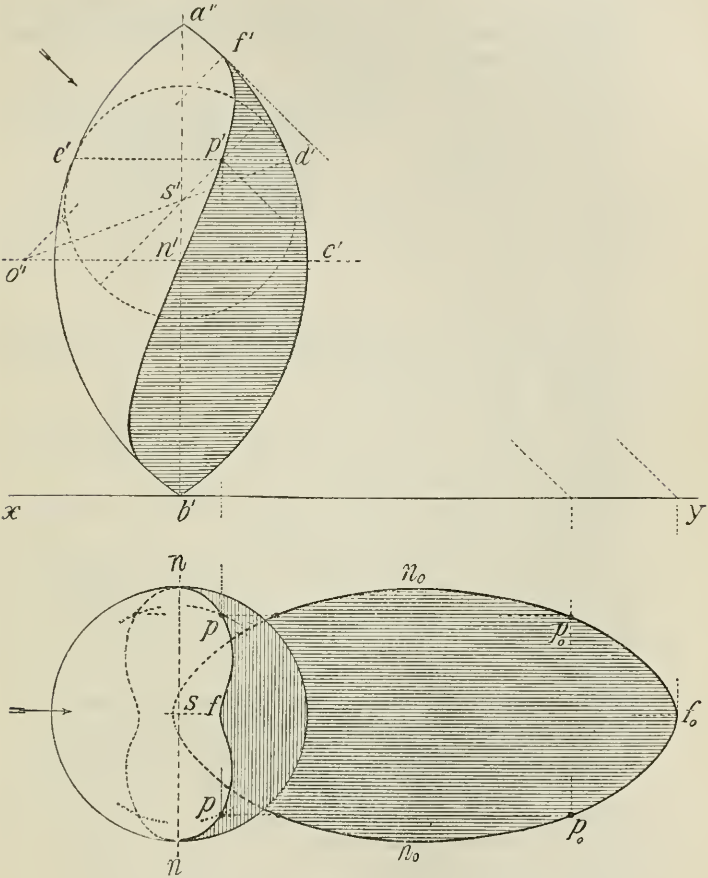
Three or four other spheres should be taken, and the above construction repeated, a curve being drawn through the points thus found.

The points N will be on the line of separation. Also, if a radius $o'f'$ be drawn at right angles to the elevation of a ray, the highest point F of the curve will be found, and this will be on the outline in elevation.

The shadow on the ground is obtained by considering rays which pass through the points on the line of separation. For example, the points P give the shadows p_0, p_0.

One half of the surface is illuminated.

Examples.—1. Determine the shadow cast on the horizontal plane by the surface of revolution in Ex. 1, Prob. 282, when its axis is vertical, one end being on the ground. The rays are parallel to the vertical plane, and inclined at 45°.



2. Draw a line *ab* parallel to *xy*, $1\frac{3}{4}''$ above it and $2\frac{1}{4}''$ long. With *ab* as radius, and *a* and *b* in turn as centres, describe arcs commencing at *b* and *a* and terminating in *xy*. This is the elevation of a surface of revolution; determine the shadow cast on the ground, the rays being parallel to the vertical plane, and inclined at 40°.

338. Problem.—**To determine the shadow cast by the hexagonal head of the given bolt on the cylindrical shank, from given parallel rays.**

In this example the plan of the shadow surface is also an edge view of the surface, so that the plan of the shadow is in this edge view.

Draw the tangents ra, tb parallel to the given plan of the ray; these represent the extreme rays, and the plan of the shadow on the shank is the semicircle rpt. It is evident that the shadow is cast from the portion ACB of the lower hexagon.

We may choose any point Q (i.e. q, q') in ACB; then to determine the corresponding shadow P draw qp the plan of the ray QP, and project from p to p' on to the elevation of the ray drawn through q'.

We ought now to select the rays which give the important points. There is one through the angular point C which gives the angular point S of the shadow.

Another is drawn through n in plan, and this determines n', a point on the outline in elevation.

Next, to find the highest points in the two elliptic arcs of the shadow. For the arc RNS draw ou perpendicular to ac and uv the plan of a ray; obtain $v'u'$; then U is the highest point of the curve RNS, and the tangent at u' is horizontal. A similar construction would determine E, the highest point in ST, the tangent at e' being horizontal.

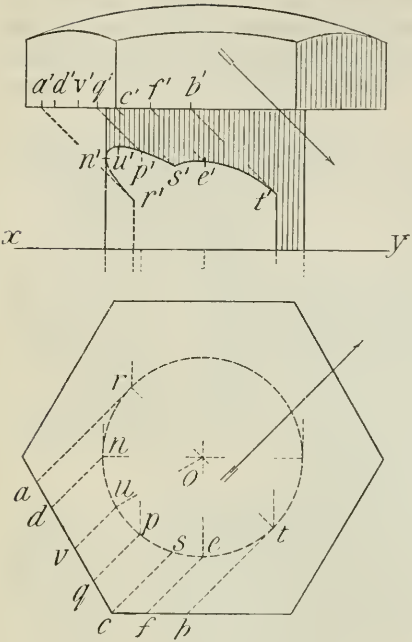
The two extreme rays AR and BT illustrate the following theorem relating to shadows which fall on curved surfaces:

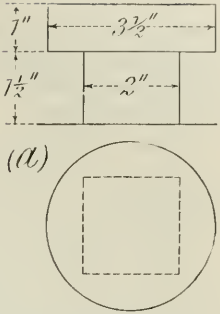
Theorem.—A ray which is tangential to the shadow surface is also tangential to the shadow; and the plan and elevation of the ray are respectively tangential to the plan and elevation of the shadow.

Thus $b't'$, $a'r'$ touch the curves at t' and r'.

Examples on Problems 338 and 339.

1. Copy the figure of Prob. 338 double size, the edge AC being placed perpendicular to the vertical plane. Then work the problem, the rays making 45° with xy in plan and elevation.





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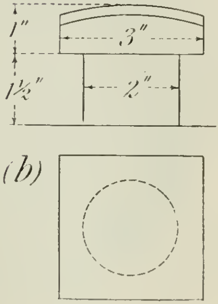


(b)

2. Draw Figs. (*a*) and (*b*) to the dimensions given above, and determine the shadows of the heads on the shanks, as in Prob. 338; the rays making 45° with *xy* in plan and elevation.

3. Draw the projections (*a*) and (*b*) when the plans are turned through 45°. Then work Ex. 2.

4. The shank of a rivet is a cylinder 1" diameter, and the head is a cone, diameter of base $1\frac{7}{8}''$, height 1". Determine the shadow of the rivet on the planes of projections, the axis of the rivet being vertical and $2\frac{5}{8}''$ from the vertical plane; the rays in plan and elevation making angles of 30° and 40° with *xy*. Show also the shadow of the head on the shank, and the line of separation on the solid. Scale, double size.

5. Work examples 1 to 4, supposing in each case the solid shank to be replaced by a thin hollow shank with the front half cut away and removed.

339. PROBLEM.—**To determine the shadow cast by the head of the given rivet on the shank, and by the rivet on the planes of projection, the direction of the parallel rays of light being given. Also to indicate the line of separation on the rivet.**

First, determine the shadow cast by the cylindrical shank on the ground, supposing the vertical plane to be transparent; it consists of the two tangents kk_0, ll_0 and the semicircles $k_0m_0l_0$, kgl.

Next, obtain the shadow cast by the head on the ground; it consists of the circle, centre c_0, and its tangents v_0r_0, v_0t_0. Draw t_0t and r_0r parallel to the plan of a ray; then VT and VR form the line of separation on the curved surface of the cone. The shadow of the cone is therefore that cast by the generators VR, VT, and the arc $RFHT$.

Determine v_1r_1, the shadow cast by VR on the vertical plane; join v_1e_0. Then $v_1e_0t_0$ is the shadow cast by VT on the two planes of projection.

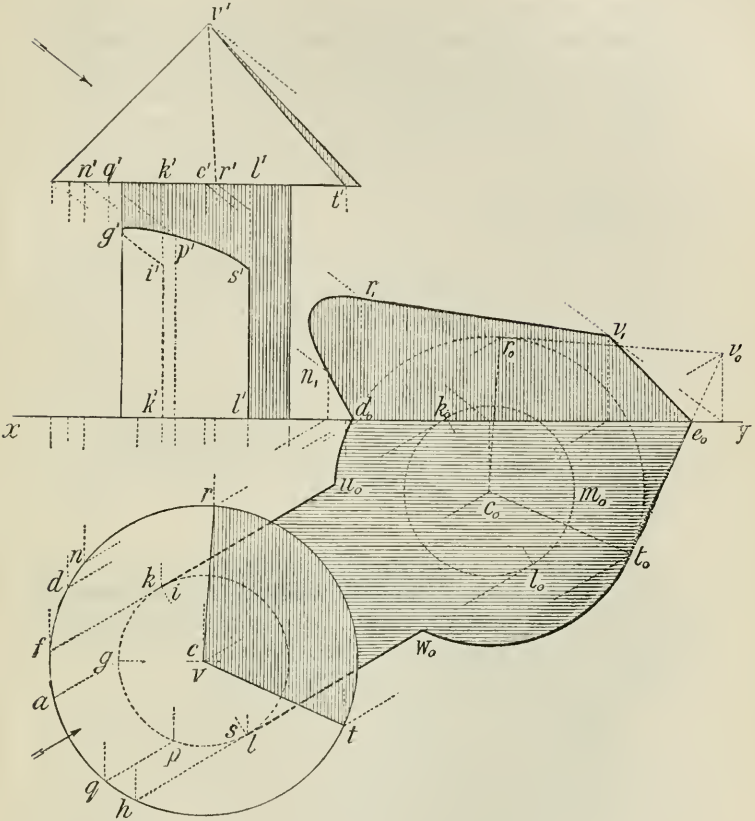
Draw d_0d parallel to the plan of a ray; then the arc DR casts its shadow on the vertical plane, and this shadow may be found by taking one or two points in DR and obtaining the shadows cast by them; the construction is shown for one such point N, which gives the shadow n_1.

The outline of the shadow on the horizontal plane is $ku_0d_0e_0t_0w_0lgk$, and on the vertical plane is $d_0n_1r_1v_1e_0d_0$.

To obtain the shadow of the head on the shank, draw the tangents lh and kf parallel to the plan of a ray; these are the limiting rays, and the arc FH is that which casts the required shadow.

Take any point q on fh and draw qp, the plan of a ray through Q. This ray is intercepted by the surface of the cylinder at a point on the generator whose plan is p; draw the elevation of this generator, and through q' draw $q'p'$ parallel to the elevation of a ray and meeting the elevation of the generator at p', then p' is on the elevation of the shadow.

Repeat this construction for a few other points on fh,



not forgetting to determine the important points *i'*, *s'*, *g'* and the highest point on the curve. (See the remarks towards the end of the last problem.) The elevation of the shadow on the shank is the curve *s'p'g'i'*.

The line of separation consists of two detached portions, each closed, viz. *LSGIKL* and *VTHRV*.

340. PROBLEM.—**Having given any two lines AB and CD, and the direction of the rays of light; to find the two points F and E in which a ray intersects both lines; that is, to find the shadow of the one line on the other.**

Method.—Find the shadows of both lines on any convenient surface. Project the ray through the point where the shadows intersect. This ray will cut the given lines in the required points.

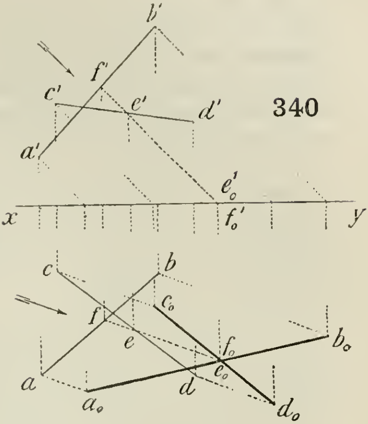
In the figure the shadows a_0b_0, c_0d_0 on the horizontal plane are found; these intersect in the point marked e_0f_0: a ray through this point is seen to cut the given lines in the points F and E. Then the point E is the shadow of AB on CD, and F is the point in AB which casts the shadow.

Note.—The student should make careful note of the principle underlying this problem, and of the corresponding construction. The lines AB and CD may be curved. AB typifies the line of separation, and CD a selected line on the shadow surface; the construction shows how to find the point where the shadow cuts CD. See applications in following problems.

341. PROBLEM.—**Having given a line AB, and a thin circular plate, centre C, parallel to the ground, to determine their shadows on the ground. Also to obtain the shadow of the line on the plate, and the portions of the line which cast the shadows on the plate and ground.**

Obtain a_0b_0, the shadow of the line on the ground, on the supposition that the plate is removed; find also the circular shadow, centre c_0, cast by the plate on the ground. These shadows intersect each other in e_0, f_0, and g_0, h_0. Thus the outline of the required shadow on the ground consists of the circle, centre c_0, and the lines a_0f_0 and b_0h_0.

Through e_0 and g_0 draw the lines e_0ef and g_0gh parallel to the plan of a ray; then, as in Prob. 340, the rays which meet the ground in e_0 and g_0 intersect the circular plate in E and G, and the line AB in F and H. Hence EG is the required shadow on the plate, cast by FH; and AF and BH throw their shadows on the ground.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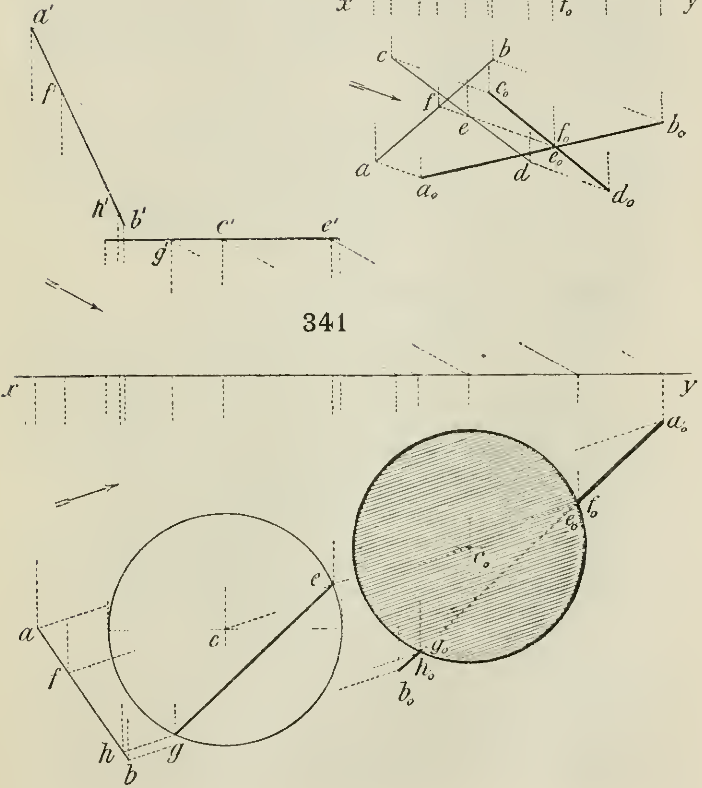
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342. Problem.—**The plans of a line EF and of a square pyramid VABCD are given, the points v, e, f being indexed, and the base ABCD resting on the ground. Determine the plan of the shadow cast by the line EF on the ground and on the pyramid, rs being the plan of a ray, the inclination of the rays being α.**

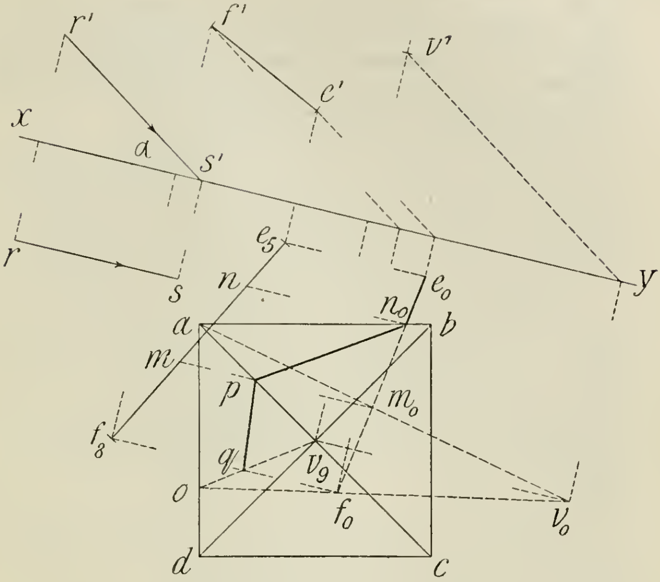
Draw xy parallel to rs, and project the elevations $r's'$, $e'f'$, and v'. Obtain e_0f_0, the shadow of EF on the ground, supposing the pyramid to be removed; also determine v_0a, the shadow of VA on the ground. These two shadows intersect in m_0, and the corresponding point M on EF casts the shadow P on VA.

The shadow of F will obviously fall on the face VAD, and to determine its projection join v_0f_0 and produce to meet ad in o; join ov, meeting f_0f in q. Then q is evidently the plan of the shadow of F on the pyramid.

The complete shadow of EF is e_0n_0pq in plan.

Examples on Problems 340 to 343.

1. The x, y, z co-ordinates of two points A and B are ($1''$, $1''$, $1''$) and ($3''$, $3''$, $2''$), and those of C and D are ($2''$, $1''$, $1''$) and ($1''$, $3''$, $2''$). If the plan and front elevation of a ray make angles of 30° and 45° with the axis of Y, determine the projections and co-ordinates of the points E and F in which a ray intersects both of the lines AB and CD.
2. Determine the projections of the line parallel to the vertical plane of ZX, which intersects the lines AB and CD in Ex. 1, and is inclined at 60° to the ground plane.
3. The two ends A and B of a line $3''$ long are respectively $2''$ and $1''$ from each of the two planes of projection. A horizontal circular plate, $1\frac{3}{4}''$ diameter, has its centre $1''$ vertically beneath the mid-point of AB. Determine the portion of the line AB which casts a shadow on the plate, the plan and elevation of a ray making 45° and 60° respectively with xy.
4. An equilateral triangular plate, $1\frac{1}{2}''$ side, is in a horizontal position $2''$ above the ground, the nearest corner A being $1\frac{1}{2}''$ from the vertical plane, and one side making 70° with the plane. A square plate, $2''$ side, is also in a horizontal position with one side at right angles to the vertical plane, one end of this side being $1''$ therefrom, $1''$ to the right of A, and $1''$ below it. Determine the shadow of the two plates on the



ground, and the shadow of the one on the other, the rays making 30° and 45° with *xy* in plan and elevation.

5. A square pyramid, side of base 2″, height $2\frac{1}{2}$″, has its base on the ground, one side making 30° with *xy*, the nearest corner of the base being 1″ from *xy*. The end *E* of a line *EF* is $3\frac{1}{2}$″ to the left of the vertex, $\frac{1}{2}$″ from the vertical plane, and 3″ above the ground; the end *F* is 2″ to the left of the vertex, 3″ from the vertical plane, and 2″ above the ground. Determine the plan and elevation of the shadow of *EF* on the ground and pyramid, the rays being parallel to the vertical plane and inclined at 40° to the ground.

6. Describe two circles with *a* and *b* as centres and radii $\frac{7}{8}$″ and 1″, *ab* being 2″. The circle with centre *a* is the plan of a circular plate 3″ above the ground, the other circle being the plan of a sphere whose centre *B* is $1\frac{1}{2}$″ above the ground. Determine the portion of the edge of the plate whose shadow falls on the sphere, the rays being inclined at 50°, their plans making 10° with *ab*.

348. Problem.—**The projections of a straight line RT, and a sphere, centre S, are given; to determine the projections of the portions of the line which cast their shadows on the sphere and ground respectively, the given rays being parallel to the vertical plane.**

The rays which touch the sphere will generate a cylinder the elevation of which is obtained by drawing the tangents $g'l$ and $h'a'$ parallel to the elevation of a ray; this cylinder touches the sphere in a circle whose elevation is the diameter $g'h'$, and this circle is the line of separation on the sphere.

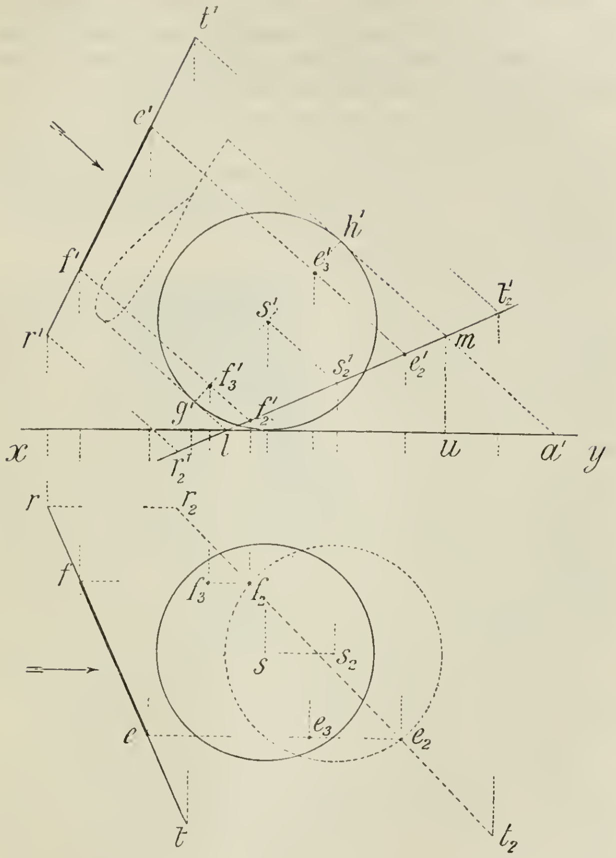
Make $lu = g'h'$; draw um perpendicular to xy and join lm. Regard lm as the edge elevation of a plane; this plane will cut the cylinder in an ellipse whose major axis LM is parallel to the vertical plane, the minor axis being equal to the diameter of the cylinder. Now since it has been arranged that the plan of the major axis is equal to the diameter of the cylinder, the plan of the ellipse will be a circle the centre s_2 of which is projected from s_2', the middle point of lm.

The shadow of the sphere, that is of its line of separation, on the plane LM is the ellipse referred to; hence the *plan* of the shadow is the circle, centre s_2.

Next, obtain the plan of the shadow of the line RT on the plane LM. That is, draw $r'r_2'$ and $t't_2'$ parallel to the elevation of a ray; draw the projectors $r_2'r_2$ and $t_2't_2$ to meet rr_2 and tt_2 in r_2 and t_2 respectively, then r_2t_2 is the plan of the shadow of RT on the plane LM.

The two shadows intersect in e_2 and f_2. Now since e_2 is on the circle, centre s_2, the ray through E_2 *touches* the sphere; also since e_2 is on r_2t_2 the ray through E_2 intersects RT. Similar remarks apply to f_2. Hence the rays which pass through E_2 and F_2 respectively *touch* the sphere and intersect RT. The projections of these rays are e_2e_3e, $e_2'e_3'e'$ and f_2f_3f, $f_2'f_3'f'$, E_3 and F_3 being the points of contact with the sphere.

Therefore the portion EF casts its shadow on the sphere, and ET, FR cast their shadows on the ground.



Example.—Describe a circle, centre c, radius 1″, and draw a line adb to touch the circle at d; make $ad = 1''$, and $db = \frac{1}{2}''$. Attach indices of 30, 20, and 15 to a, b, and c. Find the part of the line AB whose shadow falls on the sphere C, the rays being inclined at 45° and making 60° with ba in plan. Determine also the shadow cast on the sphere. Unit 0.1″.

344. Problem.—**The projections of a truncated cone with a circular slab placed centrally upon it are given; it is required to determine the projections of the shadow cast on the cone. The parallel rays are given.**

The shadow is cast from the lower edge of the slab, but the exact *portion* of this edge which casts the shadow on the cone is not so readily found as was the case with FQH in Prob. 339. The method now adopted is of general application, and might have been used in Probs. 338 and 339, had it been necessary to do so.

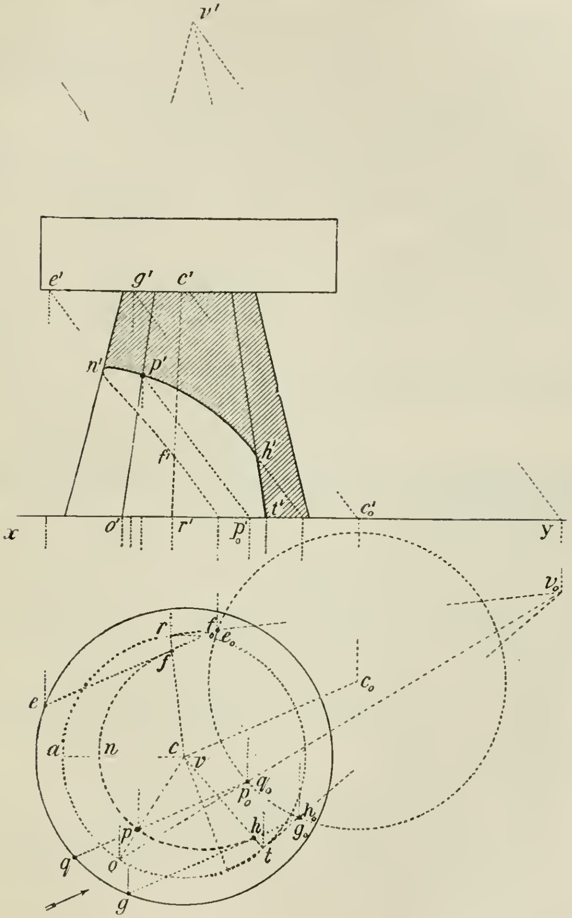
Obtain v' the elevation of the vertex of the cone supposing the truncated portion to be extended; and by Prob. 334 determine the projections of the two generators VT, VR, which form the line of separation on the cone; this is done by first obtaining the shadow of the cone on the ground. Now draw the shadow cast from the lower circular edge of the slab; it is the circle with centre c_0. This circle intersects the shadow of the cone in the points marked $e_0 f_0$ and $g_0 h_0$.

Draw $e_0 fe$ and $g_0 hg$, the plans of the rays which meet the ground in e_0 and g_0 respectively; then, by the principles explained in Prob. 340, these rays intersect the generators VR, VT in F and H, and the lower edge of the slab in E and G respectively. Consequently the arc EG is that which casts the required shadow on the surface of the cone.

Draw ov, $o'v'$ the projections of any generator OV, and draw ov_0 the plan of the shadow of OV; let $p_0 q_0$ be the point where ov_0 intersects the shadow of the arc EG.

Draw $p_0 pq$ parallel to the plan of a ray, then the ray through p_0 intersects OV in P and the lower edge of the slab in Q; and therefore P is a point on the outline of the required shadow. The elevation p' may be found by projecting from p on to $o'v'$ at p', or, as this is not very well conditioned, by projecting p_0 to p_0', then drawing $p_0'p'$ the elevation of the ray through p_0'.

This construction should be repeated for other generators similar to OV. The generator AV will give n on the outline in elevation. The two extreme rays EF, GH touch



the surface of the cone, and their projections are tangential to the projections of the shadow at *f* and *g*. The projections of *N*, *F*, and *H* should be obtained as important points.

The preceding construction may be slightly varied in the following manner. Choose q, draw qp_0, v_0p_0o, ov, the latter meeting qp in p. Draw p_0p_0' and $p_0'p'$ to meet a projector from p in p'; in this manner, however, N would not be determined unless it were found accidentally.

The elevation of the outline of the shadow on the cone is the curve $f'n'p'h'$, and the plan is the curve $fnph$, while HT and FR are portions of the line of separation on the conical surface.

Instead of taking generators such as OV, circles may be selected on the surface of the cone, in which case their shadows must be drawn and the points obtained in which these intersect the shadow of the lower edge of the slab. By this method, however, the limiting rays EF and GH could not be accurately determined.

345. Problem.—**To determine the shadow cast by a given point P on a given oblique plane VTH, from given parallel rays RS.**

We must find the point where the ray through P meets the given plane. This may be done as in Prob. 199, or by either of the following methods.

(**a**) Through p' draw $p'l'm$ parallel to $r's'$, and draw mn perpendicular to xy. Then $l'mn$ are the traces of an inclined plane which contains the ray through P.

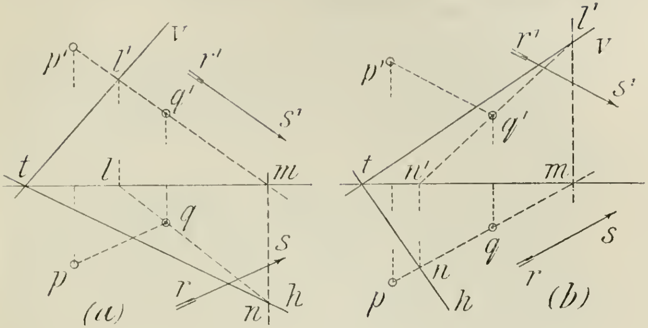
Obtain nl, the plan of the intersection of the planes VTH, LMN; and draw pq parallel to rs to meet ln in q. Project from q to q'.

Then Q is the intersection of the ray PQ and the plane VTH, and is the required shadow on the plane.

(**b**) Draw pnm parallel to rs, and ml' perpendicular to xy. Project from n to n'; join $l'n'$. Draw $p'q'$ parallel to $r's'$; and project from q' to q.

Then Q is the required shadow of P on the plane VTH.

In this case a vertical plane containing the ray PQ has been substituted for the inclined plane of (a).



Note.—The problem of finding the intersection of a given line and plane is generally best solved by applying the construction of either (*a*) or (*b*).

Examples on Problems 344 and 345.

1. A cone rests with its base on the ground; diameter of base $2\frac{1}{2}''$, height $5''$. It is cut by a horizontal plane which bisects the axis, and the upper portion is removed. A circular slab, $3''$ diameter and $\frac{3}{4}''$ thick, rests centrally on the frustum of the cone. Determine the plan and elevation of the shadow cast from the slab on to the conical surface, the rays in plan and elevation making 30° and 50° with *xy*.
2. The traces *vt*, *th* of an oblique plane make angles of 60° and 40° with *xy*. A point *P*, in a plane through *t* at right angles to *xy*, is $2''$ from each plane of projection. Determine the shadow of *P* on the oblique plane, if the rays in plan and elevation make angles of 30° and 45° with *xy*.
3. Taking the plane *VTH* and the point *P* as in Ex. 2, let *PQR* be an equilateral triangular plate, $1\frac{1}{2}''$ edge, in a horizontal position with *PQ* parallel to *xy* and directed towards the plane. Determine the shadow of the plate cast on the ground and on the plane, the direction of the rays being unaltered.
4. An equilateral triangular plate *ABC*, $2''$ edge, has one edge *AB* on the ground at right angles to the vertical plane, the nearer end of the edge being $1''$ from *xy*; the plate makes 40° with the ground. A line *EF* has its end *E* $1\frac{1}{2}''$ to the left of *AB*, $2''$ above it, and $2''$ from the vertical plane. The end *F* is $\frac{1}{2}''$ to the left of *AB*, $1''$ above it, and $3''$ from the vertical plane. If the rays in plan and elevation make angles of 30° and 45° with *xy*, determine the shadow of the line *EF* on the ground and on the triangular plate.

346. Miscellaneous Examples.

1. A tetrahedron of $1\frac{1}{2}''$ edge has its three lowest corners 0.1″, 1″, and 1.3″ respectively above the horizontal plane. Show the shadow thrown by it upon that plane, assuming the rays of light to be inclined at 45°, and their plans to make an angle of 45° with the horizontal trace of the plane containing the three given points. (1881)

***2.** Fig. (*a*). The plan of an octagonal prism, $\frac{1}{2}''$ thick, with a square hole cut through it, is given. Assuming the height of the lower surface of the prism above the horizontal plane to be $1\frac{1}{2}''$, obtain the complete outline of the shadow thrown on the horizontal plane. *N.B.*—The rays of light are parallel, and their direction is given in plan and elevation. (1893)

***3.** Fig. (*b*). The elevation and plan of a chimney are given; ab is the plan of one of the parallel rays of light. Determine in plan the shadow cast upon the roof. Unit = 0.1″. (1891)

***4.** Fig. (*c*). Draw the projections of the given cylinder and truncated cone *four* times the size of the diagram, and determine the outline of the combined shadow of the two surfaces, thrown by parallel rays, the direction of which is given, on the horizontal plane. The shadow of the cylinder on the cone is not required. (1890)

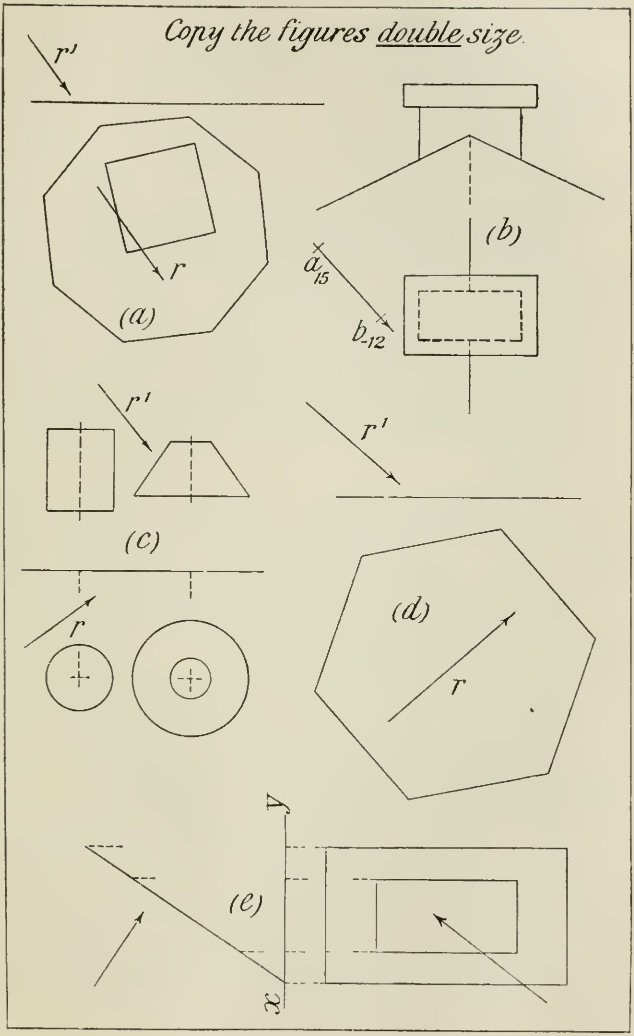
***5.** Fig. (*d*). The plan is given of a solid prism $1\frac{1}{2}''$ deep resting on the horizontal plane. r is the plan and r' the elevation of a ray of light, to which the other rays are parallel. Show the shadow thrown on the horizontal and vertical planes. (1882)

***6.** Fig. (*e*) shows the plan and elevation of an inclined square plate out of which is cut a rectangular piece. Determine the portions of the shadow cast on the planes of projection. The plan and elevation of one of the parallel rays are shown.

7. Determine the shadow cast on the horizontal plane from the capstan shown in Fig. (*f*), p. 485. Take the rays parallel to the vertical plane and inclined at 35° to the ground; also take the axis of the capstan 2″ from the vertical plane. Show the line of separation on the surface.

***8.** Fig. (*c*). Copy the figure *four* times the size shown. Determine the shadow of the cylinder on the cone if the rays are parallel to the vertical plane and slope downwards to the right at 30°. Also obtain the shadow of the cone on the cylinder, if the rays are parallel to xy and directed to the left.

9. A hollow cylinder 2″ long, 3″ and $2\frac{1}{2}''$ external and internal diameters, is cut into two halves by an axial plane. One of the portions is placed with its end on the ground and the middle generator of its convex surface against the vertical plane.



Copy the figures double size.

Determine the shadows on the concave surface, and on the planes of projection, the parallel rays making 45° with *xy* in plan and elevation.

*10. Fig. (*a*). The figure represents a fluted column with a circular cap. Draw on the elevation the shadow thrown on the column by the cap. The direction of the parallel rays of light is given by arrows. (1897)

*11. Fig. (*b*). The diagram, represents a horizontal square block, *ABCD*, intersecting a square-based pyramid *VEFGH*. (The overlapping portions of the solids are left dotted on the diagram.) Determine the intersections of the solids in plan and elevation ; and draw the outlines of the shadows thrown by them, one on the other, and by both on the horizontal plane. The arrows indicate the direction of the parallel rays. (1895)

*12. Fig. (*c*). *abc* is the plan of the base of an octahedron lying in a horizontal plane $1\frac{1}{2}''$ above the ground ; *pqms* that of a pyramid resting on the horizontal plane, its vertex (O) being $2.35''$ above the ground. The direction of parallel rays of light is shown in plan by r, their inclination to the horizontal plane being 44°.

Show in plan the shadow of the octahedron cast on the ground and on the pyramid. Shade lightly all portions of the latter not illuminated. (1896)

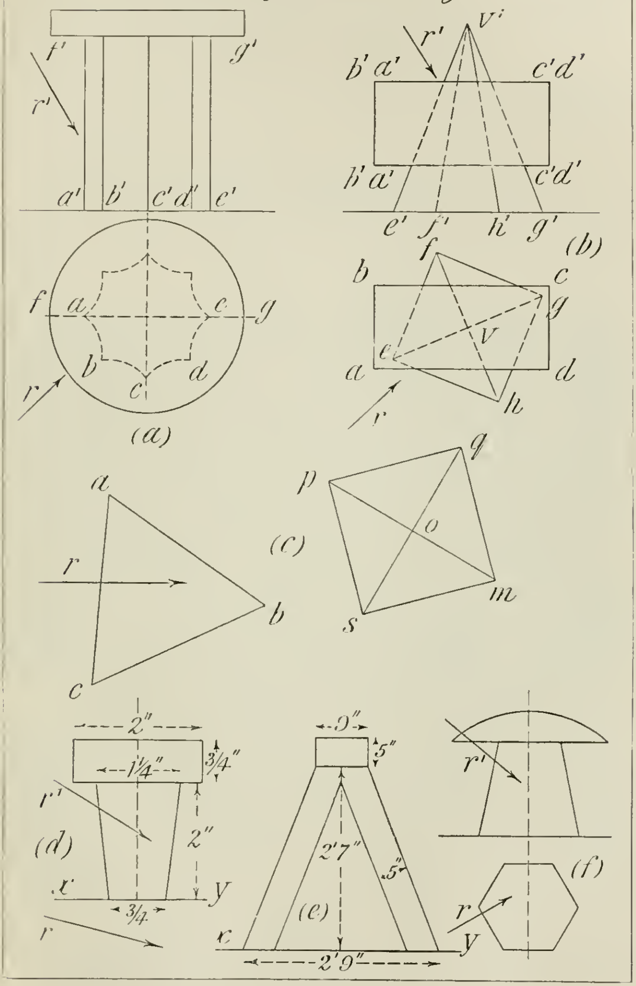
*13. Fig. (*d*). The given figure is the elevation of a square-headed bolt with a conical shank standing on the horizontal plane. The projections r, r' of one of the parallel rays of light are given. Draw the bolt full size, and obtain the shadow cast on the horizontal plane and on the shank. (1887)

*14. Fig. (*e*). Draw the plan, and determine the shadow cast on the ground by the trestle of which an end elevation is shown. The length of the top block is 4′ 6″, and it projects 6″ beyond the legs at each end ; the legs are square in section. Let the plan and elevation of one of the parallel rays of light be parallel to each other and make 45° with *xy*. Scale $\frac{1}{12}$.

*15. Fig. (*f*). A spherical segment rests on the top of a truncated hexagonal pyramid. The elevation and plan of the base are given. Draw the figure *four* times the given size, and determine the portion of the solid in shadow, as seen in elevation. The direction of the parallel rays of light is indicated. (1888)

16. A right circular cone, 3″ high, rests with its base (3″ diameter) on the ground. A sphere $1\frac{1}{2}''$ diameter has the vertex of the cone as its centre. Determine the shadow cast by the sphere on the cone, the rays being parallel to the vertical plane and inclined at 45° to the ground.

Copy the figures double size.



CHAPTER XVIII

METRIC PROJECTION

347. General explanation.—We have seen how by an indexed plan the form or position of an object may be defined by one projection only. In this chapter we develop another method of representation by means of a single view. And in this case, as in the former one, the projection of the object is one that can be readily scaled for the purpose of ascertaining the dimensions of the parts. Each system is associated with a class of examples coming within its special province, and with which it is well adapted to deal.

Thus the shape of an irregularly curved surface is well exhibited by the method of figured plans. We have a well-known example in maps with indexed contours, which indicate the configuration of the hills and valleys. A similar series of sections or contours is employed by naval architects in representing the shape of the surface of a vessel.

Metric projection is well fitted to show the forms and dimensions of what may be termed *rectangular solids*, such as are many examples of wood work; they are bounded mainly by three systems of planes mutually perpendicular, intersecting in three systems of parallel lines in directions also mutually at right angles. The metric projection of such an object resembles a perspective view, and like the latter conveys a realistic impression of the form even to the uninitiated. The metric view may look distorted; but there is compensation in that it is a *scale drawing*.

348. Metric scales and axes.—Let *oa* be the projection of a line *OA* on any plane. On *OA* set off any scale, say a scale of inches, and project the scale on *oa*.

The latter scale may be used *either* (1) to ascertain the length of any line parallel to *OA* by measuring the length of its projection on the plane, *or* (2) to set off the length of the projection, knowing the length of the line.

Such a scale is called a *metric scale;* the plane of projection is called the *metric plane;* the direction *oa* in the plane is a *metric direction;* any line in the plane and parallel to *oa* is a *metric line;* and any line of reference parallel to *oa* and in the plane is a *metric axis.*

Suppose an object like a building brick is projected on the metric plane. The edges of the solid form three systems of parallel lines; as each system will have its own scale, we shall require *three scales* in measuring all the lines of the projection. These are known as the *trimetric scales.*

Let any three axes of reference OX, OY, OZ be taken in space parallel to the three systems of lines, then their projections ox, oy, oz on the metric plane are called *metric axes* of the projection. As was explained in Chapter VII., the axes OX, OY, OZ serve to define the position of a point in space. Thus if the co-ordinates of a point P in space are X, Y, Z, the point may be reached from O by going first a distance X along OX, then a distance Y parallel to OY, and finally a distance Z parallel to OZ. And it is easily seen that the projection p of the point P may be plotted on the metric plane by stepping off first from o a distance x along ox, then at the end of this a distance y parallel to oy, and finally a distance z parallel to oz, where x, y, z are the co-ordinates X, Y, Z, measured on the metric scales.

Examples.—1. If a line 2.31″ long be inclined at 38.3° to the metric plane, find the length of its metric projection.
Ans. 1.81″.

2. Construct a half-size metric scale for a system of parallel lines inclined at 55° to the metric plane. The scale is to read inches and eighths of an inch.

349. Isometric projection—the axes and scale. — Suppose that in a trimetric system the three directions of the lines in space are all *equally inclined* to the metric plane, then the three scales become identical, and only one scale is required in measuring *all* the lines of the system in the metric projection.

In this case we have an *isometric* or equal-scale system; the projection on the metric plane is called an *isometric projection*, and the scale used is the *isometric scale.*

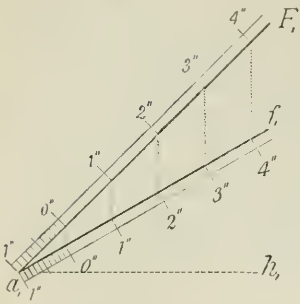
The simplest example of such a solid is a cube; and the edges of a cube will evidently be all equally inclined when a diagonal of the solid, that is, a line joining two opposite corners, is perpendicular to the metric plane. This projection of a cube may be readily determined by ordinary Descriptive Geometry, but is still more easily obtained by the method of isometric projection as follows.

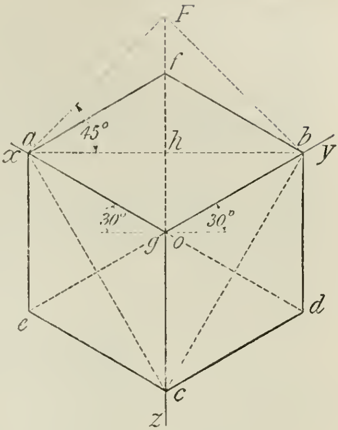
To project a cube isometrically.—From considerations of symmetry we see that the three edges of the cube which radiate from the corner O farthest from the metric plane will project into three lines radiating from a point o at equal angles of 120°; these are very conveniently drawn with the 30° and 90° angles of the set-square, as shown at ox, oy, oz in the figure. These lines define the three *isometric directions;* and they, or any other concurrent parallel set, may be taken as *isometric axes.*

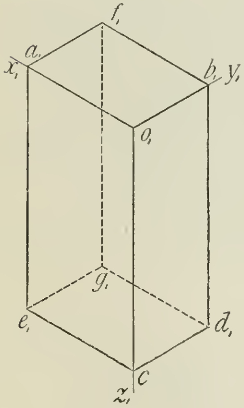
Next, since the edges OA, OB, OC of the cube are equal and equally inclined, their projections are equal in length. Therefore mark off along ox, oy, oz three equal lengths oa, ob, oc. This length is at present undefined.

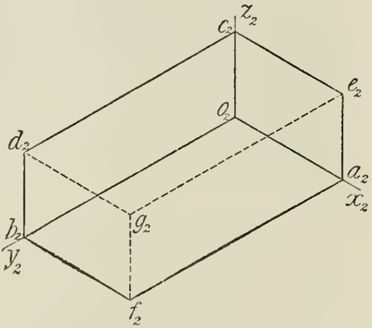
Finally, to complete the isometric projection of the cube, we make use of the principle that parallel lines have parallel projections. Thus to complete the face of which oa, ob are sides, draw af and bf parallel respectively to ob and oa, and intersecting in f. The corners d, e, g, with the sides radiating from them, are similarly determined.

The outline of the projection is seen to be a regular hexagon.









We must now find the length of the edge of the cube, of which the figure just obtained is the isometric projection.

The corners *A*, *B*, *C* of the cube are evidently equidistant from the plane of projection, for they are the ends of lines which all start from a common point *O*, and are equal and equally inclined. Thus the triangle *ABC* is parallel to the metric plane, and *abc* gives its true shape. Thus any side of the latter, say *ab*, is the *true length* of a diagonal of one face of the cube.

Draw aF, bF at 45° to ab; then aF is a side of the square of which ab is a diagonal; so aF is the required length of the edge of the cube. The following method of constructing an isometric scale is thus suggested:—

To construct an isometric scale.—From any point a_1 draw a_1F_1 parallel to aF, that is, with the 45° set-square; also draw a_1f_1 parallel to af, that is, with the 30° set-square. On a_1F_1 set off a true length scale, and project the scale on a_1f_1 by lines parallel to Ff, that is, lines drawn with the 90° set-square. Then the projected scale a_1f_1 is the required *isometric scale*. Measuring oa on this scale, the edge of the cube is seen to be 2.95″.

To project a rectangular prism.—Employing this scale, we have at the lower part of the figure, drawn half size, two isometric projections of an ordinary building brick $9'' \times 4\frac{1}{2}'' \times 3''$. The process is quite simple. From any point draw three isometric lines; set off along them from the point the length, breadth, and thickness of the solid, using the isometric scale; complete the figure by drawing the system of parallel lines, as explained for the cube.

To express the isometric scale numerically.—Since the angles Fah, fah are 45° and 30°, we have—

$$\frac{Fa}{ah}=\frac{\sqrt{2}}{1}, \text{ and } \frac{fa}{ah}=\frac{2}{\sqrt{3}},$$

hence
$$\frac{af}{aF}=\frac{2}{\sqrt{3}}\div\frac{\sqrt{2}}{1}=\frac{\sqrt{2}}{\sqrt{3}}=.817=\frac{9}{11} \text{ nearly.}$$

This gives the ratio of the isometric length to the true length.

We are not compelled to employ an isometric scale in setting out an isometric projection; an ordinary scale may be used. But in this case the projection will be that of an object larger in the inverse ratio of the isometric scale. However, if this be recognised, and the ordinary scale used in measuring the projection, no error or ambiguity will result, and labour will be saved by using an ordinary scale.

350. Examples.—Represent in isometric projection the following ten objects :—

1. A cube of 3″ edge.

2. A building brick 9″ × $4\frac{1}{2}$″ × 3″. Scale $\frac{1}{3}$.

3. A 24″ square slab, 3″ thick. Scale $\frac{1}{8}$.

4. A 1″ square prism, 3″ long.

5. A box without lid, 6″ long, 4″ wide, 2″ deep inside, and $\frac{1}{2}$″ thick throughout. Scale $\frac{1}{2}$.

6. An instrument box with the lid open at right angles. Dimensions of box outside : length 6″, breadth $4\frac{1}{2}$″, depth of box $1\frac{1}{4}$″, depth of lid $\frac{3}{4}$″. Thickness of wood at sides and ends $\frac{3}{8}$″; at top and bottom $\frac{1}{4}$″. Scale $\frac{1}{2}$.

7. The slab of Ex. 3, when pierced with 6″ square hole through its centre.

8. The chimney shown in Fig. (*b*), page 439.

9. The trestle, Fig. (*e*), page 441. Scale $\frac{1}{8}$.

10. The column with cap, Fig. (*a*), page 441.

11. Represent in isometric projection the three co-ordinate planes of Fig. 160, page 179, and in this view plot—

(*a*) The point A whose co-ordinates are (2″, $1\frac{1}{2}$″, 1″).

(*b*) The point B whose co-ordinates are (2, 1.5″, – 1″).

(*c*) The line CD joining the points (1″, 2″, 3″), (3″, 2.5″, 2″).

(*d*) The line EF joining the points (1″, – 2″, 3″), (– 3″, 2.5″, – 2″).

(*e*) The irregular tetrahedron whose angular points G, H, K, L are (2″, $1\frac{1}{2}$″, 1″), (1″, 2″, 3″), (1″, 2.5″, 1.5″), (1.5″, .5″, 3.5″).

(*f*) The traces of the plane whose intercepts OA, OB, OC are 2″, 3″, and 4″ respectively.

12. A room is 24 feet long in the direction north and south, 18 feet wide east and west, and 12 feet high. The following are points within the room :—

(*a*) A, situated 4 feet above the floor, 7 feet from the north wall, and 5 feet from the west wall.

(*b*) B, 3 feet high, 6 feet and 8 feet from the south and east walls.

(*c*) C, 4 feet below the ceiling, and 15 and 9 feet from the south and west walls.

(*d*) D, 5 feet below the ceiling, and 3 and 12 feet from the north and east walls.

Draw the room in isometric projection, and in this view plot the projections a, b, c, d of the points A, B, C, D.

Note.—An isometric scale should be constructed and used in some of the above examples. Afterwards an ordinary scale may be employed to set off isometric dimensions.

351. Problem.—**Required the isometric projection of a cube of given edge which has a sphere of given radius in contact with the centre of one face, and a circle inscribed in the same face; a cone of given axis with its base inscribed in a second face; and a cylinder of given length with its base inscribed in a third face.**

The cube.—Let *OA* be the given edge. Determine *Oa* the isometric length of the edge. Set out the projection *a*, *a*, . . . of the cube, as explained in the last article.

For convenience refer to the visible faces as the upper, and the right and left front vertical faces. We shall require the centres *c*; the isometric bisectors *dd*; and the diagonals *aa* of these faces.

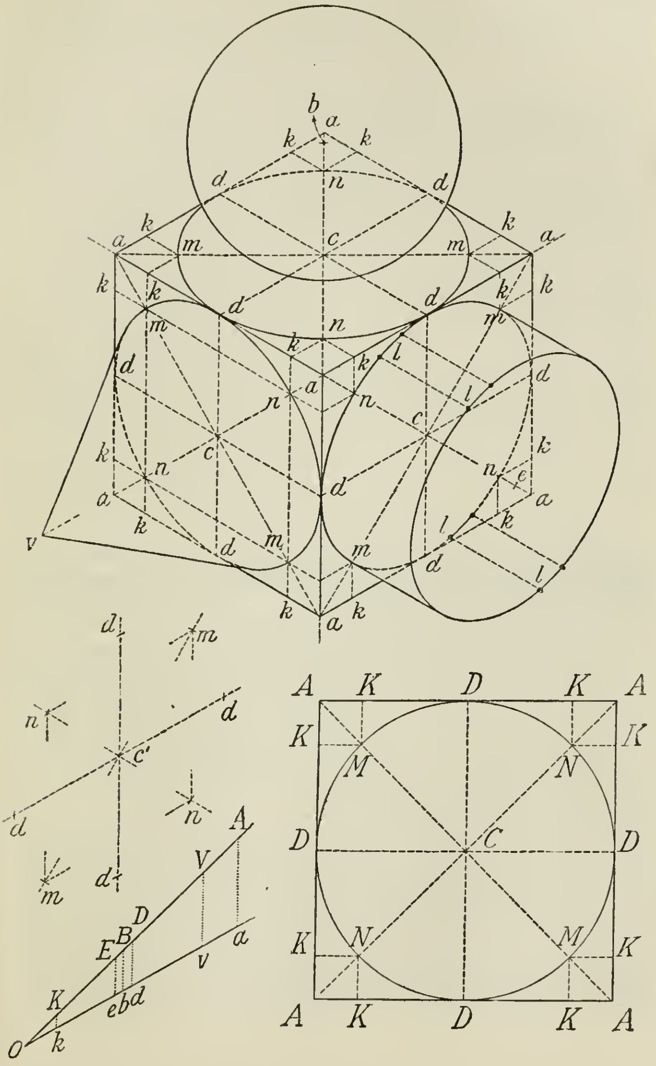
The sphere.—Let the sphere rest on the centre of the top face, and let *OB* be its given radius. Obtain the isometric radius *Ob*, and set this up vertically from *c* to *b*. With *b* as centre, radius *OB*, describe a circle. This circle is the required isometric projection of the sphere.

Note 1.—The diameters of the sphere and its projection are equal.

The circle.—Draw a circle *DD* of the given diameter *OA*, and circumscribe this by a square *AA*; draw the diagonals *AA*, *AA* intersecting the circle in *M*, *M*, *N*, *N*. Draw the equal perpendiculars *MK*, *NK* and obtain their isometric length *Ok*. Mark off this length at *ak*, *ak*, . . . from the corners of the upper face along the sides as shown, and through the points *k*, *k*, . . . draw the isometric lines intersecting in *m*, *m*, *n*, *n*. The ellipse drawn through the eight points *m*, . . *n*, . . *d* . . is the required projection of the circle inscribed in the face. The lines *dd* may be called the *isometric diameters* of the circle.

Note 2.—Observe that this is an application of co-ordinates. The eight points *M* . . *N* . . *D* . . in the circle are referred to the sides of a circumscribing square *AA*. The co-ordinates are reduced by the isometric scale, and then plotted on the projection *a*, *a* of this square. Any plane curve can be similarly referred and plotted. And any irregular solid figure can be referred to a circumscribing rectangular prism, and its points then plotted on the isometric projection of the prism.

The cone.—Let *OV* be the given length of axis of the cone, and let the base be inscribed in the left front vertical face. Draw the ellipse inscribed in the projection of the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face in the manner just explained. Draw cv equal to Ov, and in the proper isometric direction, to represent the axis CV of the cone. Draw the tangents to the ellipse from v, the projection of the vertex.

The cylinder.—Let the cylinder have its base inscribed in the right front vertical face of the cube, and let OE be the given length of the cylinder. Inscribe the ellipse in the projection of this face; this is the projection of one end of the cylinder. To obtain that for the other end, conceive the ellipse to be moved parallel to itself in the isometric direction proper to the axis CE of the cylinder, through a distance Oe equal to isometric length of the axis. The construction suggested is to draw isometric lines through a number of points on the first ellipse, marking off on them lengths each equal to Oe; we thus obtain points on the second ellipse. Some of these lines are shown at ll in the figure. The projection of the cylinder is completed by drawing the common tangents to the two ellipses.

Note 3.—*Second method of projecting a circle.*

Required the isometric projection of a circle, centre c', radius OE, the plane of which is parallel to the right front vertical face of the standard cube.

All the lines dd, mm, nn on any face of the cube may be drawn with one or other of the 30°, 60°, and 90° edges of the set-square.

First, through c' draw the isometric lines dd, dd (whose directions are readily seen by reference to a standard cube); and along them set off the four isometric radii $c'd$, each equal to Oe. Next draw the lines through c' in the directions of the major and minor axes. On the former (that bisecting the acute angles between dd, dd) set off $c'm$, $c'm$ each equal to the true radius OE. Finally through m, m draw the isometric lines intersecting in n, n.

Then mm, nn, are the major and minor axes, and $mnmn$ is an inscribed square. We thus obtain the projection by drawing the *isometric diameter*, the *major axis*, and the *inscribed square*. Before sketching the ellipse through the eight points thus found, it is well to draw short lengths of the tangents at these points as shown.

Note 4.—If an ordinary scale be used to set off isometric dimensions, the radius of the circle which represents a sphere, or the semi-major axis of the ellipse which represents a circle, are larger than the actual radius in the inverse ratio of the isometric scale.

352. Examples.—Represent by an isometric projection:—

1. A sphere of $1\frac{1}{2}''$ diameter.

2. A circle $2''$ diameter.

3. A cone, diameter of base $2.2''$, length of axis $1.7''$.

4. A cylinder $2.5''$ diameter, $.6''$ long.

5. A grindstone, $24''$ diameter, $5''$ thick, with a $6''$ square hole through its centre. Scale $\frac{1}{8}$.

6. A rectangular slab of stone, $18''$ long, $12''$ broad, $3''$ thick, with a circular hole $5''$ diameter through its centre. Scale $\frac{1}{6}$.

7. The frustum of a cone, the diameters of the ends $2.5''$ and $1.5''$, length $1''$.

8. A square pyramid, base $2\frac{1}{2}''$ side, axis $3\frac{1}{2}''$ long.

9. A hexagonal pyramid, base $1.5''$ side, axis $2.5''$ long.

10. The frustum of the pyramid, Ex. 9, obtained by a plane bisecting its axis at right angles.

11. A hemisphere, $3.1''$ diameter.

12. A hemispherical bowl, $20''$ diameter inside, $\frac{3}{4}''$ thick. Scale $\frac{1}{8}$.

13. A semicircle, and a quarter circle, $3''$ diameter.

14. A half cone, $2.5''$ base, $2''$ axis, obtained by a cutting plane through the axis of the cone.

15. A half cylinder, $2''$ base, $1.5''$ long, obtained by a plane containing the axis.

16. A quarter cone and a quarter cylinder, of dimensions as in Exs. 14 and 15, obtained by planes containing the axis.

17. A quarter sphere, and an eighth of a sphere, obtained by two or three mutually perpendicular planes through the centre of a sphere $4''$ diameter.

18. A cone, base $2\frac{1}{2}''$ diameter, axis $1\frac{1}{4}''$, resting with its base concentrically on one end of a cylinder, $1\frac{1}{2}''$ diameter, $2''$ long.

19. A ring, $2''$ internal diameter, of section $\frac{1}{2}''$ square.

20. A ring, $2''$ internal diameter, of circular section, $\frac{1}{2}''$ diameter.

21. A regular tetrahedron of $3''$ edge.

22. A regular octahedron of $2''$ edge.

Hint.—In examples like 21 and 22 there is a choice as to which three perpendicular lines connected with the solid shall be taken for the three principal directions. In the tetrahedron we might select the base and altitude of one triangular face, and the direction perpendicular to the face. In the octahedron the three mutually perpendicular diagonals of the solid might be chosen.

23. A cone of indefinite length having a vertical angle of 50°.

24. The surface of revolution, page 337, double the size shown.

Hint.—Employ inscribed spheres in Exs. 23 and 24.

Note.—In the above examples the student may use either an isometric or an ordinary scale in setting off dimensions in the isometric views.

353. Problem.—**To determine the isometric projection of the solid given in plan and elevation.**

Draw the isometric axes a_1x, a_1y, a_1z, and set off a_1b_1, a_1d_1, and a_1e_1 respectively equal to the isometric lengths (as obtained by a scale) of the lines ab, ad, and $a'e'$. In like manner obtain the projections of the other edges parallel to AE. The four arcs joining the upper extremities of these projections must now be determined.

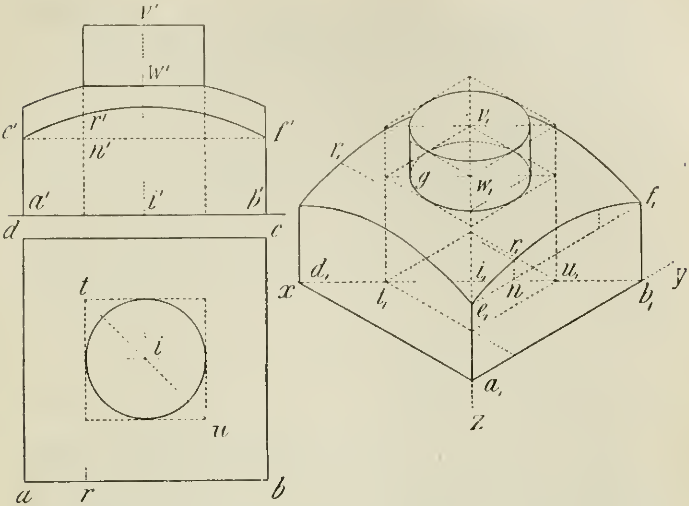
Divide the circular arc $e'f'$ into a number of equal parts, say four. Join e_1f_1. Consider one of the points, say r'; draw $r'n'$ perpendicular to $e'f'$. Set off e_1n and nr_1 respectively equal to the isometric lengths of $e'n'$ and $n'r'$; this determines r_1, and the corresponding point on the arc parallel to $e_1r_1f_1$ is obtained by drawing r_1r_1 equal and parallel to a_1d_1. By a repetition of this construction the four arcs may be obtained.

To determine the isometric projection of the cylindrical portion of the solid we may proceed as directed in the last problem or as follows: Suppose that the cylinder is circumscribed by a square prism which extends to the base of the solid; its plan and elevation are shown in the figure, and its isometric projection may be thus obtained—

Draw the diagonal b_1d_1, and from its middle point i_1 set off i_1t_1 and i_1u_1 respectively equal to it and iu. (TU is not shortened by projection.)

Make i_1w_1 and i_1v_1 respectively equal to the isometric lengths of $i'w'$ and $i'v'$. It will then be seen that the three parallelograms with i_1, w_1, and v_1 for their centres can be completed.

The ellipses which are the projections of the circular ends of the cylindrical portion of the solid must now be inscribed in the parallelograms of which w_1 and v_1 are the centres. Make w_1g equal to the true length of the radius of the cylinder, and complete both ellipses in the manner explained in Prob. 352; two common tangents to the ellipses will complete the required projection of the solid.



Examples.—Represent in isometric projection—

1. A sphere of 3″ diameter penetrated by a cone of base 3″ diameter, axis 4″ long, the centre of the sphere being at the middle point of the axis of the cone.
2. The wedge-shaped figure formed from a cylinder 3″ diameter, 3″ long, by two sloping planes containing a diameter of one end, and touching the circle at the other end at opposite points, thus cutting away the sides.
3. The instrument box of Ex. 6, p. 447, with the lid opened at 130°. Scale $\frac{1}{3}$ or $\frac{1}{2}$.
4. One of the four quarters of a hollow sphere, formed by two perpendicular planes containing a diameter. External and internal diameters 3″ and 2″.
5. The separated parts (*c*) and (*d*) of the dovetailed joint, p. 457, represented (fitted together) by ordinary projection at (*a*) and (*b*).
6. The bolt with hexagonal head, p. 425, to double the size shown. The upper surface of the head is spherical.
7. The bolts with round and square heads, p. 425, to the dimensions given. The upper surface of the square head is spherical.

Note. — In working these examples, the use of an isometric scale is optional.

354. PROBLEM.—**Having given a set of trimetric axes, to determine the corresponding trimetric scales.**

Let ox, oy, oz to the left be the given axes. Take any point d in oy, and draw df, fe, ed respectively perpendicular to ox, oy, oz. Now ox, oy, oz are the projections of three lines OX, OY, OZ in space mutually perpendicular, and df, fe, ed are the traces of the planes YZ, ZX, XY on the plane of projection, or on a parallel plane.

On ed as diameter, describe a semicircle cutting oz in o_0; join o_0e, o_0d. Thus eo_0d is a right angle, and the triangle eo_0d is the rabatment of EOD into the horizontal position about ed. Thus eo_0 and do_0 are the true lengths of the lines of which eo, do are the metric projections.

Again, the right-angled triangle of which the line fgo is the plan, is shown rabatted at fgo_1; so that fo_1 is the true length of the line of which fo is the metric length.

The construction for the trimetric scales is thus at once obtained. Draw Ox, Oy, Oz parallel to the given axes, and draw OX, OY, OZ respectively parallel to o_0e, o_0d, o_1f. On OX, OY, OZ set off true length scales, and project these scales on Ox, Oy, Oz by lines parallel to oo_0, oo_0, oo_1.

The scales thus projected on Ox, Oy, Oz are the trimetric scales required.

Note.—Having given the trimetric scales l, m, n, or their ratios $l : m : n$, to determine the trimetric axes.

The values l, m, n are the ratios of the projected to the true lengths.

Construct a triangle xyz, the sides yz, zx, xy of which are respectively equal or proportional to l^2, m^2, and n^2. Determine o the centre of the circle inscribed in this triangle, and join ox, oy, and oz; these three lines are the required trimetric axes.

It can be shown that $l^2 + m^2 + n^2 = 2$.

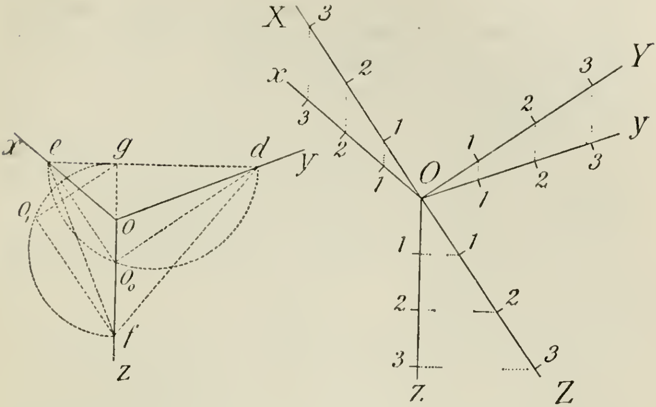
Examples.—**1.** The angles xoz and yoz between given trimetric axes are respectively 135° and 120°. Construct the trimetric scales. Measure the representative fractions of these scales.

Ans. l, m, n for the scales ox, oy, oz are .838, .932, .649.

Note.—The trimetric axes in this example may be drawn with the 90°, 45°, and 30° edges of the set-squares.

2. If the trimetric scales l, m, n are in the ratios $1 : \frac{7}{8} : \frac{3}{4}$, determine the trimetric axes and measure the angles between them.

Ans. 107°, 138.5°, 114.5°.



Examples on Problems 354 to 357.

Represent in trimetric projection the following six objects, the metric axes being such that oz is vertical on the drawing-paper, the angle xoz 135°, and the angle yoz 120°. All the lines may be drawn with the ordinary set-squares.

1. A cube of 3″ edge, with a circle inscribed in each visible face.

2. A building brick $9'' \times 4\frac{1}{2}'' \times 3''$. Scale $\frac{1}{3}$.

3. A sphere of 2″ diameter.

4. Three lines, each 3″ long, which mutually bisect one another at right angles.

5. The rivet on p. 427 to double the size shown.

6. The instrument box of Ex. 6, p. 447, with the lid open at 120°.

7. Refer to Art. 357 and the figure on p. 459. Draw the axes oa, ob, oc respectively horizontal, vertical, and with the 30° edge of the set-square. Take the scales for lines parallel to oa and ob full size, and the scale for lines parallel to oc half size. Then draw the projection of a cube of 2.5″ edge, with a circle inscribed in each visible face.

8. Take the axes and the scales as in Ex. 7, or alter them in any manner that may seem desirable, and represent in metric projection the model or the three planes of reference as shown in Fig. 161 (*b*), making OX, OY, and OZ each 4″.

In this view plot the point A, the line CD, and the plane ABC of Ex. 11, p. 447.

9. Take the axes and scales as in Ex. 8, and draw the metric projection of the steps, Fig. (*a*), p. 461.

355. Problem.—**Having given the trimetric axes ox, oy, oz, to draw the trimetric projection of a cube of given edge with a circle inscribed in one face.**

First determine the trimetric scales as in Prob. 354. The axes and scales in Prob. 354 are used again.

To project the cube, set off the lengths *oa*, *ob*, *oc* for the edges, each to its proper scale. Complete the figure by drawing the parallel lines as in isometric projection.

Let the circle be inscribed in the right front vertical face; its projection is the principal ellipse inscribed in the projection of this face. The ellipse may be set out as in Prob. 88, or as follows:—

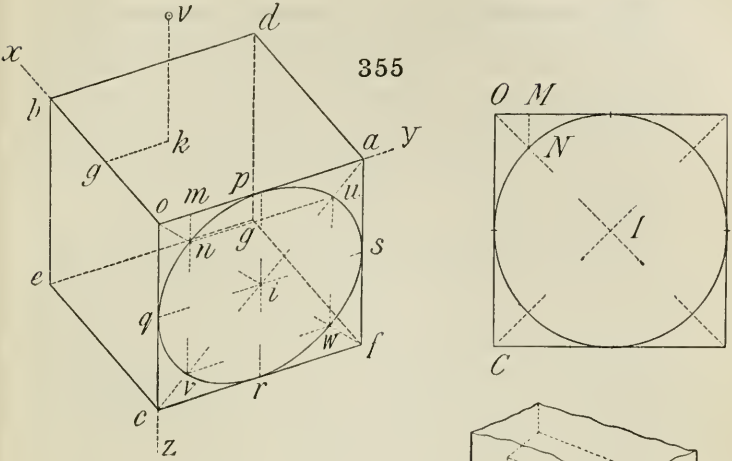
Draw the two diagonals *ac*, *of*, and through their intersection *i* draw the two trimetric diameters *qs*, *pr* parallel to *oy*, *oz*; then *p*, *q*, *r*, *s* are four points in the ellipse. Draw the separate figure, centre *I*, and then locate *n* by plotting the co-ordinates *om*, *mn* equal to *OM*, *MN* reduced by the scales. The other points *u*, *v*, *w* on the diagonals are then determined by drawing the projection of the inscribed square. The ellipse can now be traced through the eight points found. The tangents at all these points are known.

Note. — Observe that the diameters *uv*, *nw* which lie on the diagonals *ac* and *of* are *not* the major and minor axes of the ellipse, as was the case with isometric projection. The major axis is the line through *i* perpendicular to *ox*, of length equal to the true diameter of the circle, for this line is parallel to the plane of projection. And generally, a line in or parallel to one face, say *xoy*, is parallel to the metric plane, when its projection is at right angles to the perpendicular axis *oz*.

356. Problem.—**Two views (a) and (b) of a dovetailed joint are given. To draw a trimetric projection of each portion of the joint, the trimetric axes and scales being taken the same as in Prob. 354.**

The required projections are shown at (*c*) and (*d*).

After the detailed descriptions previously given, the method of obtaining these should not require further explanation. The lines *EH* and *FL*, not being parallel to a trimetric axis, their projections cannot be scaled. Their ends are located by the method of co-ordinates. See the line *RH*.



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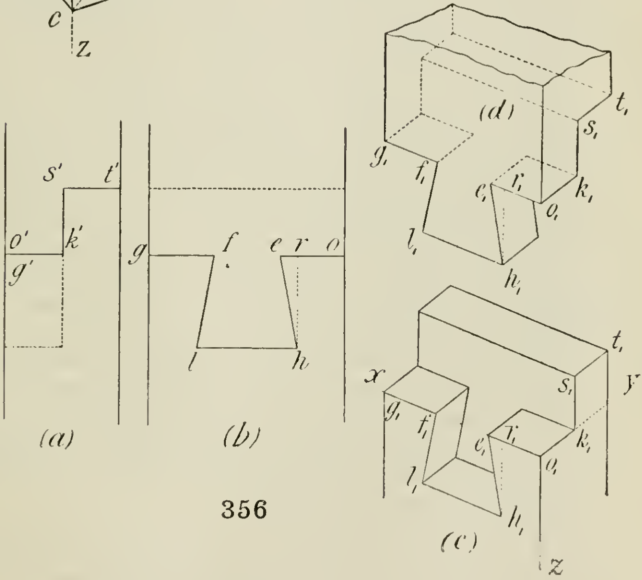

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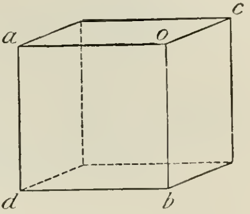
357. Generalisation of the foregoing methods.—The object of metric projection is to define a form having three dimensions by one pictorial view drawn to scale. In the examples hitherto considered this has been effected by *orthogonal* projection. We may, however, remove the restriction that the projection shall be orthographic, and still secure the object in view, and at the same time gain considerably in freedom.

Radial projection is not permissible, because a perspective view cannot be scaled; but we may employ *oblique parallel* projection, since in this case *parallel lines project into parallel lines, all to the same scale.* Now if the projectors, instead of being perpendicular, may be inclined, and in any direction, we gain *two degrees of freedom* in arranging for the projection. We may now select what two angles we like for the latitude and longitude, or the altitude and azimuth, so to speak, of our projectors.

Let us now see how this helps us. In trimetric orthogonal projection we may choose the axes, and then we require to determine the corresponding scales. In trimetric oblique projection, with two more degrees of freedom, we may choose the axes and any two of the scales; or what is equivalent, the axes and the ratios $l : m : n$ of the scales, leaving the *absolute scale* to be determined by a geometrical construction, if this should ever be required. But in practical applications we are not concerned with the absolute ratio which the size of the object bears to the size of its projection; thus in isometric projection we generally use the full size instead of the isometric scale, and do not trouble ourselves as to how much bigger the object would have to be in order to actually project into the view drawn. Discarding consideration for the absolute scale, the general proposition may be thus stated:

Proposition.—In trimetric oblique projection we may take the three axes in any directions we like, and we may take the three scales whatever we like, and we shall have a true projection of the object, or of one similar in form.

As a simple illustration, draw *oa*, *ob* equal and perpendicular to one another, and draw *oc* in any direction and of any length. Complete the figure as shown by drawing the series of parallel lines. Then we are at liberty with perfect propriety to take this figure as the projection of a cube. Thus let the cube rest on the plane of projection, with one face coinciding with *oadb*. The line *oc* is the projection of the perpendicular edge *oC*. Therefore *Cc* must be one projector, and the oblique projectors are parallel to *Cc*. But generally it would not be so easy as in this case to locate the position of the object in space in regard to its projection. We may, however, employ the projection without solving this latter problem.



We may remove another restriction. The three-directional system of lines in space need not be perpendicular to one another, but may have any directions. The proposition in its most general form may then be stated :—

Theorem.—Let OA, OB, OC be any three lines of definite length in space, and oa, ob, oc any three lines of definite length, in one plane : then the former lines can be projected into a figure similar to the latter by parallel projection.

In applying this general method to practical cases, some regard must be had to the effects of distortion. Any projection would appear right if viewed in a direction parallel to the oblique projectors. But a picture is generally looked at from somewhere near the front ; if the projectors were very oblique this front view would appear very distorted; so the axes and scales must be kept within reasonable limits.

As in ordinary trimetric projection we can plot the projections of irregularly situated points or lines from their co-ordinates, and circles may be plotted by projecting the circumscribing square or parallelogram, and drawing the principal inscribed ellipse.

358. Miscellaneous Examples.

*1. Fig. (*a*). Draw the isometric projection of the object represented orthographically. A section on AB is shown. Scale $\frac{1}{48}$. (1896)

*2. Fig. (*b*). Draw the letter P from the dimensions given in the sketch. Assuming the thickness of the material from which it is cut to be $\frac{1}{4}''$, make an isometric view of the letter.
N.B.—An isometric scale is not to be used. (1893)

*3. Fig. (*c*). Make an isometric view of the cross. (1879)

*4. Fig. (*d*). Make an isometric view of the bolt head, the elevation and half plan of which are given.
N.B.—An isometric scale is not to be used. Lengths to be transferred direct from the given figure to the isometric lines. (1886)

*5. Fig. (*e*). The two lines ab, ac form one of the right angles of a face of a cube of $2\frac{3}{4}''$ edge. Complete the plan of this cube. Unit 0.1″. (1891)

*6. Fig. *(f)*. Two elevations of a gable cross are given. Make an isometric view of the cross. An isometric scale is not to be used. (1892)

7. Make an isometric view of the bolt, Fig. (*d*), p. 441, drawing it full size. An isometric scale is not to be used. (1887)

8. Particulars of a trestle are given in Ex. 13, p. 440. Represent the trestle in isometric projection. Scale $\frac{1}{8}$. Use of isometric scale is optional.

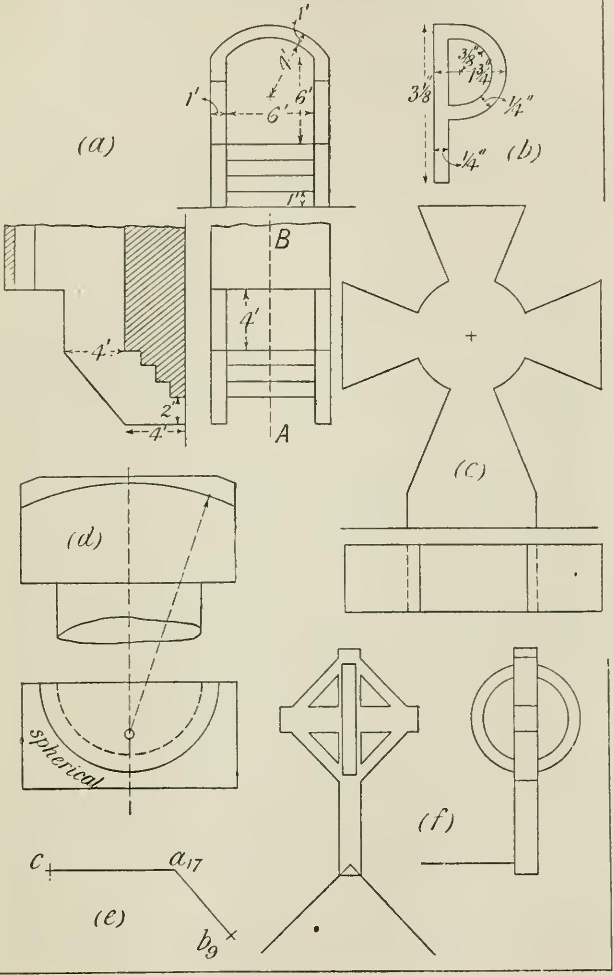
9. Draw an isometric projection of the desk shown in figure, p. 203. Use of scale optional.

10. On each face of a cube of 2″ edge stands an equal cube. Make an isometric view of the solid formed by these seven cubes. An isometric scale is *not* to be used. (1878)

11. Draw the isometric projection of the object in Fig. (*a*), p. 441, standing on the horizontal plane, one isometric plane to be taken parallel to a diagonal such as *fg*, passing through two opposite edges of the fluted column. An isometric scale is not to be used. (1897)

12. Draw three lines meeting at a point and making angles of 120°, 130°, 110°, with each other. These lines are the plans of the edges of a cube of 3″ edge. Complete the plan of the cube and draw its elevation on any plane parallel to no side or diagonal. (1879)

Copy the figures _double_ *size.*



CHAPTER XIX

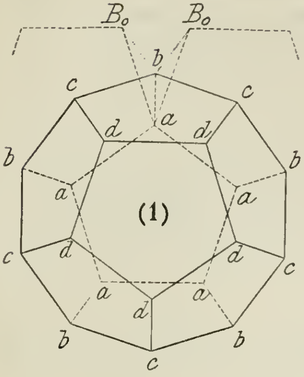
MISCELLANEOUS PROBLEMS

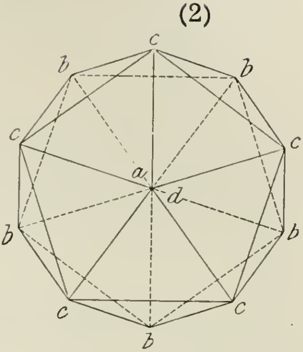
359. The five regular polyhedra. — It is shown in treatises on pure solid geometry that there are *five*, and only five *regular polyhedra*. These are the *tetrahedron*, *cube*, *octahedron*, *dodecahedron*, and *icosahedron;* see the Appendix, Definitions 18 to 22. We have already had occasion to represent the first three in projection; the remaining two will now be considered.

The regular dodecahedron, Fig. (1). — This solid has twelve equal and regular pentagonal faces. The projection easiest to determine is that on the plane of one face.

To obtain this, draw the regular pentagon *aaaaa* to represent the bottom face. On two adjacent sides construct regular pentagons, and regard these as the rabatments of two adjacent faces. Thus aB_0, aB_0 are the rabatments of the *same* edge about the two axes *aa*, *aa*. From B_0, B_0 draw perpendiculars to the axes intersecting in b; then ab is the plan of a sloping edge. The plan may now be completed from considerations of symmetry

From the corners *a*, *a*, *a*, *a* draw the plans *ab* of the other four similarly-situated edges AB. For the top face, draw the regular pentagon *ddddd*, with sides parallel to *aaaaa* and circumscribing the same circle. Draw the five lines *dc* each equal to *ab*, to represent the five edges sloping from the top face. Complete the plan of the solid by drawing the outline, which is a regular decagon.





The distances of the points B, C, and D from the lower face A are readily found since we have the plans, and we know the true lengths of the lines AB, BC, and CD; we can thus draw an elevation on any vertical plane.

If we draw an elevation on a plane parallel to a *diagonal* AD of the solid (a diagonal being a line which joins opposite corners), then we can measure the length of the diagonal; and from the elevation we may project a plan with the diagonal vertical. This plan might also be drawn first-hand from considerations of symmetry.

The regular icosahedron, Fig. (2).—This solid is bounded by twenty equilateral triangular faces. Each angular point of the solid is the common vertex of five triangles, the bases of the triangles forming regular pentagons. We shall obtain the plan of the solid when a diagonal is vertical.

Draw two regular pentagons b, b . . . c, c . . . circumscribing the same circle and with sides parallel. Join all the points b and c to the centre, as shown in the figure, and also join the adjacent points b and c so as to obtain the regular decagonal outline. This figure is the projection of the solid on a plane perpendicular to the diagonal AD.

We may obtain the heights of the various points, the length of the diagonal, the distance between parallel faces, and the projection on the plane of a face, by methods suggested for the dodecahedron.

360. PROBLEM.— **To determine the spheres inscribed in and circumscribing a given regular polyhedron.**

Let the given solid be a regular tetrahedron, of which *abcd* in the plan.

The inscribed sphere. General method.—First draw a view of the solid on a plane perpendicular to an edge, so as to project two adjacent faces in profile. Next obtain the projection o' of the centre of the solid. Finally with centre o' draw the circle which touches the profile projections of the two faces. This will be the projection of the inscribed sphere.

Thus take *xy* perpendicular to *ab*, and draw the elevation $a'b'c'd'$. Draw the perpendicular $d'm'$ and the bisector $a'n'$ to intersect in o'. With centres o' and *o*, radius $o'm'$, draw two circles.

These circles are the projections of the inscribed sphere.

The circumscribing sphere. General method. — First obtain the projection of the solid on a plane equidistant from the centre *O* and two corners. Then with centre o' draw the circle through the projections of the corners. This will be projection of the circumscribing sphere.

Take *xy* parallel to *ocd*, and draw the elevation $a'b'c'd'$. Obtain o' as before. With centres o' and *o*, radius $o'd'$, draw two circles.

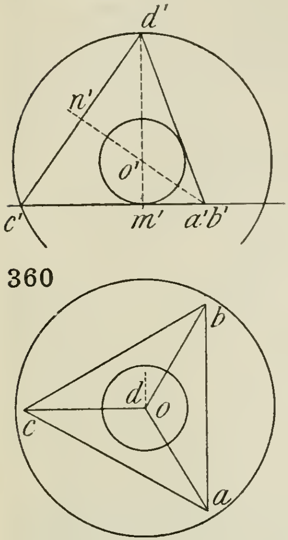
These circles are the projections of the circumscribing sphere.

Note.—**To determine the spheres inscribed in, and circumscribing a given irregular tetrahedron.**

For the *inscribed sphere*, first determine the three planes which bisect any three dihedral angles of the solid. Then determine the sphere which has its centre at the point of intersection of the planes, and which touches any face of the solid.

For the *circumscribing sphere* first obtain the three planes which bisect at right angles any three edges of the solid. Then determine the sphere which has its centre at the point of intersection of the planes, and passes through any corner of the solid.

It is seen that this problem is the same as that of finding a sphere which shall touch four given planes, or contain four given points.



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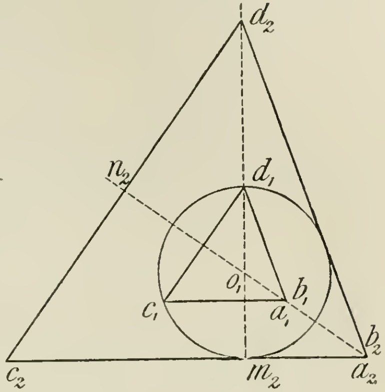


361

361. Problem.—**To determine any regular polyhedron inscribed in or circumscribing a given sphere.**

Let the circle, centre o_1, be a projection of the given sphere, and let the required figure be a tetrahedron.

First work the last problem for a tetrahedron of any assumed edge, that is, draw Fig. 360.

Through o_1 draw lines parallel to $d'm'$ and $a'n'$.

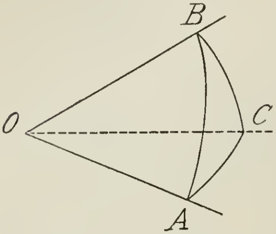
The inscribed tetrahedron.—Through d_1 draw d_1a_1, d_1c_1 parallel to $d'a'$, $d'c'$, and join c_1a_1, which will be parallel to $c'a'$.

The circumscribing tetrahedron.—Draw the tangents a_2d_2, a_2c_2 parallel to $a'd'$, $a'c'$, and through d_2 draw d_2c_2 parallel to $d'c'$.

The lengths of the edges of the inscribed and circumscribing tetrahedra are d_1c_1 and d_2c_2 respectively.

A similar method is used for any regular solid.

362. Trihedral angles and spherical triangles. — Let O be the apex of any trihedral angle, and suppose a section of the angle to be made by any spherical surface whose centre is at O. The sections of the three faces will be arcs of great circles, forming a *spherical triangle* ABC on the surface of the sphere.



We may let A°, B°, C° stand for the *angles*, and a°, b°, c° for the *sides*. By the former we mean the angles between the tangents to the curved sides at the corners, and by the latter the angles subtended by the curved sides at the centre of the sphere. Thus the *angles* A°, B°, C°, and the *sides* a°, b°, c° of the spherical triangle are respectively equal to the three dihedral angles and the three plane angles or faces of the solid trihedral angle.

The polar triangle.—If through the centre of the sphere, lines OA_1, OB_1, OC_1 be let fall respectively perpendicular to the faces OBC, OCA, OAB, *and all directed outwards* (or all inwards), meeting the spherical surface in A_1, B_1, C_1, we have a second trihedral angle formed, with its corresponding spherical triangle $A_1B_1C_1$. The latter is called the *polar triangle* of the triangle ABC.

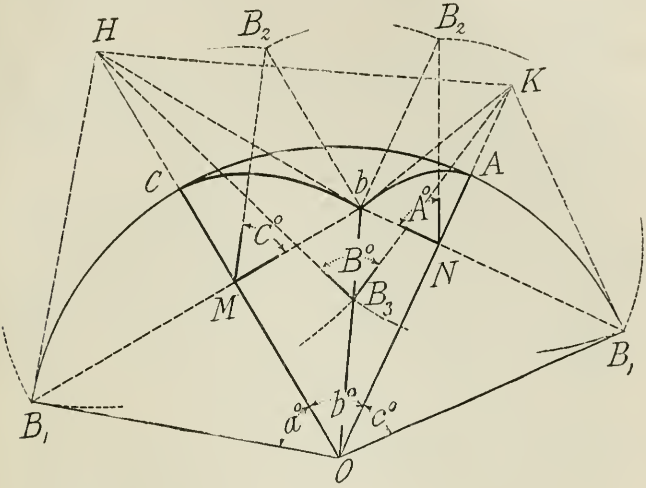
We have not space to establish the well-known relations between the sides and angles of the triangles ABC, $A_1B_1C_1$, therefore we shall merely state them. Denoting the sides and angles of the polar triangle $A_1B_1C_1$ by a_1°, b_1°, c_1°; A_1°, B_1°, C_1°, it may be shown that the *angles* of the one triangle are supplementary to the *sides* of the other; that is, we have

$$\begin{aligned} A_1^\circ &= 180^\circ - a^\circ & \qquad a_1^\circ &= 180^\circ - A^\circ \\ B_1^\circ &= 180^\circ - b^\circ & \qquad b_1^\circ &= 180^\circ - B^\circ \\ C_1^\circ &= 180^\circ - c^\circ & \qquad c_1^\circ &= 180^\circ - C^\circ \end{aligned}$$

The relations between the two triangles are reciprocal; each triangle is the polar of the other.

The principal use of the polar triangle is in the solution of spherical triangles. It reduces, to one-half, the number of cases necessary to be dealt with.

Problems on spherical triangles are equivalent to problems on trihedral angles. In solving problems on the former we shall require to draw projections of the latter.



363. PROBLEM.—**To solve a spherical triangle or trihedral angle, having given:—I. Three sides,** *or* **three angles. II. Two sides and the included angle,** *or* **two angles and the adjacent side. III. Two sides and one angle opposite to one of them,** *or* **two angles and a side opposite to one.**

Case I.—Let the three given sides be a°, b°, c°.

Begin by drawing a *development* of the trihedral angle. That is, draw any circle, centre O, and set off the angles B_1OC, COA, AOB_1 equal respectively to the given sides a°, b°, c°. Through B_1, B_1 draw perpendiculars to OC, OA intersecting in b; then b is the plan of the point B when the two outer sides or faces a° and c° of the solid angle are turned into their true positions, about OC and OA.

Join Ob. We have now obtained a projection of the solid angle on the face OAC. The projection AbC of the spherical triangle is also shown. Several simple methods of setting out the elliptic arcs Ab, bC will suggest themselves to the student.

Through B_1, B_1 draw tangents to the circle, that is perpendiculars to OB_1, OB_1, meeting OC and OA produced in H and K.

The three required angles can now be found. The angle B° is obtained by the rabatment HB_3K of the triangle HBK. The angles A° and C° are equal to bNB_2, bMB_2, determined by the rabatments of the triangles bNB, bMB. The problem is thus solved.

If the three angles A°, B°, C° were given, the problem could be reduced to the one just worked, by means of the polar triangle. We should first subtract each angle from 180°, and thus obtain the *sides* of the polar triangle. Then by the construction above we could find the angles of the polar triangle, the supplements of which would give us the required sides of the original triangle.

Note 1.—Observe the properties of the figure.

There are five rabatments of the point B about five different axes, from which we have the relations

$$MB_1 = MB_2\,;\ NB_1 = NB_2\,;\ bB_2 = bB_2\,;\ HB_1 = HB_3\,;\ KB_1 = KB_3.$$

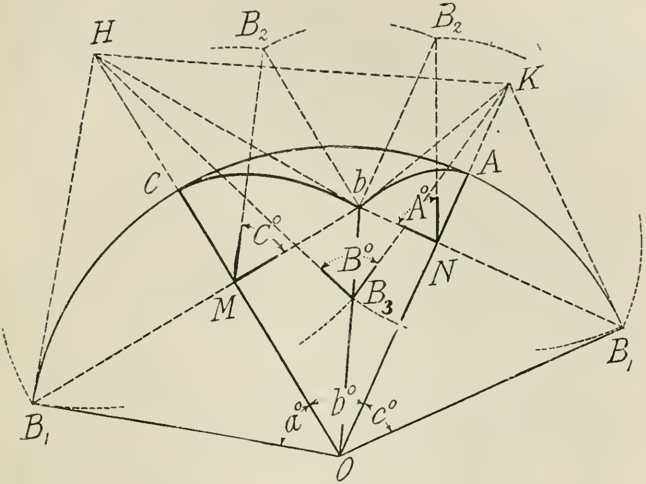
Since the plane HBK is perpendicular to the line OB, the trace HK is perpendicular to the projection Ob. Hence also B_3 falls on Ob.

Note 2.—A simple and effective **model** can be made by cutting out the shape OB_1HKB_1O in paper, drawing on it the arc B_1CAB_1, and indenting and folding along OH, OK. The three triangles HKB_3, bMB_2, bNB_2 are also cut in paper, with margins along their bases for attachment to the model by glue or paper-fasteners.

Case II.—Let the given sides be a°, b°, and the included angle C°.

First draw the rabatted sides B_1OC, COA equal to a°, b°. Through B_1 draw a perpendicular to OC, intersecting the latter in M. Set off the angle bMB_2 equal to C°, and make $MB_2 = MB_1$. Through B_2 draw Bb perpendicular to BM produced. Then b is the plan of B. Through b draw bNB_1 perpendicular to OA. We thus find the side c°, and the angles A° and B° are readily determined by rabatments.

If we are given two angles and an adjacent side this can be at once reduced to the above by means of the polar triangle. Or the problem can be easily worked directly.



Case III.—Let a°, b°, A° be the given sides and angle.

First draw B_1OC, COA the development of the two given sides or faces.

Next set out the elliptic arc Ab, of indefinite extent, which shall be the projection of the arc AB_1 when the latter is turned about OA until the face OAB is inclined at the given angle A° to the plane OAC.

Then through B_1 on the side a° draw a perpendicular B_1M to OC to intersect the elliptic arc in two points b, b.

The plan of B is thus found, and the solution can now be completed as in the preceding cases. There are two solutions, so that Case III. is ambiguous.

Note 3.—The drawing of the elliptic arc may be avoided and the points of intersection found by the method of Prob. 95.

The case where two angles and a side opposite to one of them are given reduces at once to the present case by means of the polar triangle.

364. PROBLEM.—**It is required to fit a cylindrical shell eccentrically on a hemispherical dome, as shown in the figure. Draw the elevation of the intersection of the surfaces, and develop the plate for the cylinder.**

As the development is required, take equidistant generators on the cylinder, and find the points where these meet the sphere.

Through c, a draw the diameter of the plan of the cylinder; divide the semicircles each into the same number (six) of equal parts, figuring the points as shown.

Turn $c3$, $c3$ into the position $c3_1$ parallel to xy; project from 3_1 to $3_1'$; draw the horizontal line through $3_1'$ to meet the projectors from 3, 3 in $3'$, $3'$.

Repeat the construction for the other points, and draw a fair curve through the points $1'$, $2'$, . . . thus found. This is the required elevation of the intersection.

The development.—On any line set off HM, MK each equal to $\frac{1}{4}$ the circumference of the cylinder, Prob. 113. Divide HK into six equal parts. Erect perpendiculars of lengths $M3_1 = m'3'$, $N2_1 = n'2'$, . . . Draw a fair curve through the points 0_1, 1_1, 2_1, . . . as shown.

The development of a symmetrical half of the cylindrical shell plate is thus obtained.

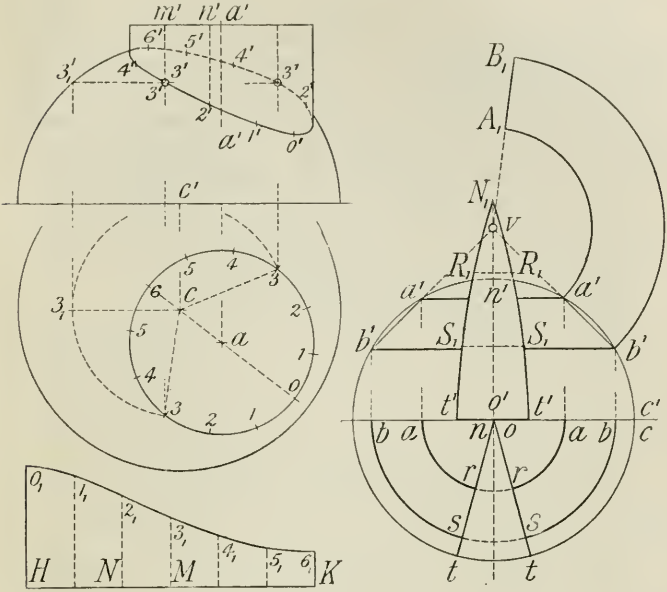
365. PROBLEM.—**To develop the surface of a sphere approximately (a) in zones; (b) in lunes.**

The centre of the sphere is taken in XY, and the vertical diameter is the polar axis.

A *zone* is the surface included between two planes, perpendicular to the axis; as between two circles of latitude. A *lune* is the surface included between two planes meeting along the axis; as between two semicircles of longitude.

(**a**) Divide the arc $n'c'$ into three (or other number) of equal parts. Draw $a'a'$, $b'b'$. Join $b'a'$, $b'a'$ and produce to meet in v'. Then the frustum AB of the cone VB coincides approximately with the zone AB.

Develop this frustum. One half only is shown. Arc $b'B_1$ = semicircle bsb. Prob. 113.



The other zones must be similarly developed.

(**b**) Make $tt = a'b'$. Join ot, ot. Then tot is the plan of a lune of $\frac{1}{12}$ the surface.

Set off $oN_1 = \text{arc } n'c'$ (Prob. 113). Divide $o'N_1$ into three equal parts by R_1R_1, S_1S_1, making $R_1R_1 = rr$, $S_1S_1 = ss$. Draw the curves $t'S_1$, R_1N_1 as shown.

This is the development of the upper half of the lune.

Examples.—**1.** Develop approximately the surface of a hemisphere (*a*) in four zones; (*b*) in sixteen lunes.

2. Two copper pipes 10″ diameter, with their axes at right angles in one plane, are connected by an elbow pipe in the form of a quarter annulus, the mean radius of which is 10″. Develop approximately the plates for the elbow (*a*) in twelve zones; (*b*) in four equal lunes. Scale $\frac{1}{4}$.

366. Problem. — **The shape of the surface of a piece of ground is given by contour lines indexed in feet, and a horizontal scale of feet. The indexed plan of a flat surface ABCD is also given. The plot is formed partly by cutting and partly by embankment, the slope of the former being 45° and of the latter 40°. It is required to draw the contours and boundary of the completed earthwork.**

The contours or lines of level on any inclined plane surface are straight lines perpendicular to the line of slope. Their distance apart in plan for any difference of level may be found by the following simple construction:—

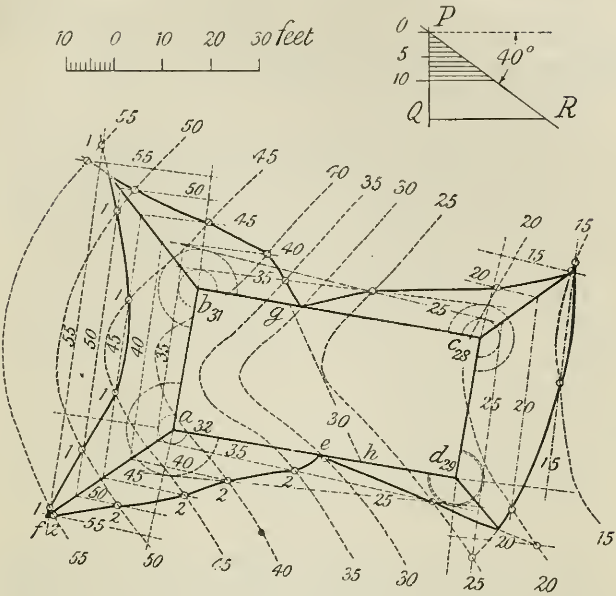
Draw two intersecting lines, one vertical, the other inclined at the angle of slope, say 40°. From their intersection P set off PQ vertically to represent any difference in level, and draw QR horizontally to meet the inclined line in R. Then QR is the distance between the contour lines in plan.

When, as in the above cutting, the slope is 45°, we have $QR = PQ$, or the horizontal distance is equal to the vertical distance; the construction is not then required.

We shall begin the solution by determining the contours at 5-feet intervals on the cutting which springs from AB. With a_{32}, b_{31} as centres, draw circles respectively of 3-feet and 4-feet radius to scale. The common tangent to these circles is the 35-feet contour on the cutting. The other contours are then drawn parallel to this line at distances of 5 feet. A curve drawn through the points marked 1, where these contours intersect the given contours at the same levels, gives the upper boundary of the cutting on this side.

Next consider the cutting which rises from AD.

With d_{29} as centre, radius 6 feet to scale, describe a circle; the tangent common to this circle and to the one with a_{32} as centre is the 35-feet contour on this cutting. The remaining contours are drawn parallel to the one just found at intervals of 5 feet. Their points of intersection with the given contours are marked 2 in the figure. The freehand curve through the points 2 is drawn, and



intersects the first curve in *f* and the edge of the plot in *e*. These points represent the highest and lowest points of the cutting on this side.

AF is the line of intersection of the two cuttings. At the point *E* the embankment begins.

The student should now be able to complete the problem without further detailed description.

For the embankments, which slope at 40°, the radii of the circles and the distance apart of the contours are determined by the horizontal lines of the scale *PQR*, as explained at the beginning.

The line *gh* is the contour 30 across the plot; *BG* and *DH* are each one-third the length of the plot.

This is a good example of the use of figured plans.

367. Problem.—**A given parallelogram pqrs is the projection of a certain square, determine the length of the side and the inclination of the plane of the square.**

Draw the bisectors jj_1 and ii_1.

By Prob. 87 determine aa_1, bb_1, the major and minor axes of the ellipse of which jj_1, ii_1 are conjugate diameters.

This ellipse will be the projection of the circle inscribed in the required square, and the major axis is the projection of that diameter of the circle which is horizontal. Thus aa_1 is the length of the side of the square.

Draw xy perpendicular to aa_1, and from b draw a projector to meet, in b', an arc with c' as centre and ca as radius. Then the angle $b'c'x$ is the required inclination of the plane of the square.

The edge elevation of the square will be parallel to $c'b'$, and we can draw it when we know the height of some point connected with the square.

The rabatment of the square about AA_1 into a horizontal position is easily found if required.

368. Problem.—**A given triangle abc is the projection of a triangle ABC, the latter being similar to a given triangle A′B′C′; it is required to find the actual size of ABC, and the inclination of its plane.**

On any side, say $B'C'$, of the given triangle $A'B'C'$ describe a square; join A' to one corner P' of this square, and meeting $B'C'$ in R'.

In the corresponding side bc obtain r such that

$$br : rc = B'R' : R'C'.$$

Join ar and produce to p such that

$$ar : rp = A'R' : R'P'.$$

Join bp and draw cq, pq parallel to bp, bc.

Then $bcqp$ is the plan of the square attached to BC, the triangle ABC and the square $BCQP$ lying in the same plane.

Determine as in the preceding problem the inclination of this plane, then by a rabatment obtain the actual size of the triangle ABC.

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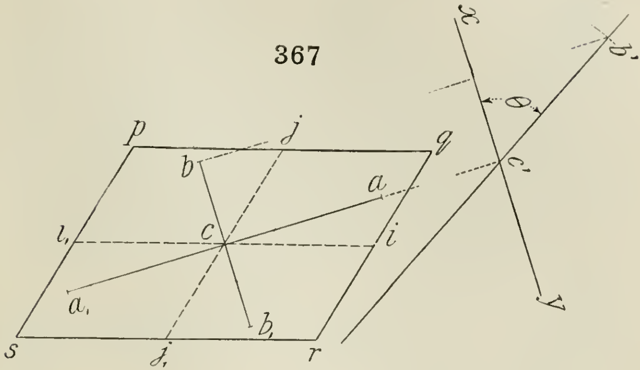

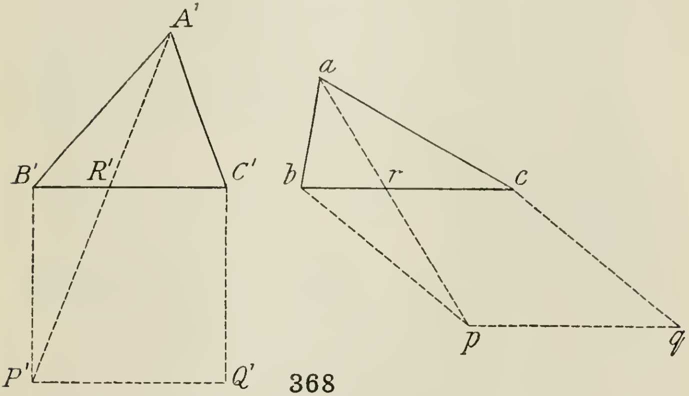

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Note 1.—The points *r* and *p* may readily be obtained by describing a square on *bc* instead of *B′C′*, and constructing a triangle on *bc* of the same shape as *A′B′C′*.

Note 2.— The student should be able to extend the construction so as to solve the following more general problem :—

The plans a, b, c are given of three known points A, B, C connected with any plane or solid figure ; complete the plan of the figure.

369. Problem.—**To project a helix of given diameter and pitch.**

Definitions. — A *helix* is traced on a cylinder when a point travels round the surface at the same time advancing axially, the ratio of the two speeds being constant. The advance per revolution is the *pitch*.

If a right-angled triangle ABC having the base AC=circumference, and the height CB=pitch, were made in paper, and wrapped round the cylinder, bringing the points A and C together, the hypothenuse would become a helix. The base angle A is the *pitch angle* of the helix.

To obtain the projection, we proceed as in Prob. 130 for the *sine curve*. The latter may be regarded as the projection of a helix.

Draw the semicircle, centre c', of the given diameter. Project cc of length equal to the given pitch.

Divide the semicircle into six (or other number) of equal parts from $0'$ to $6'$; and divide cc into double the number by the perpendicular lines 0 to 12.

Project from $0'$, $1'$, . . . on the lines 0, 1, . . . Then draw the helical curve through the points as shown.

Note.—A second turn of the helix, with the projection of a helical spring of circular section, is shown. This is determined as the envelope of the projection of a sphere which may be assumed to generate the surface.

370. Problem.—**To project a screw thread and a spiral spring of square section.**

Definition.—If the generating point above be replaced by a line, the helix becomes a *helical surface*. The line or figure tracing the helical surface may be conceived as attached to a screw which turns in a fixed nut.

A line perpendicular to the axis, shown in the various positions, 00, 11, 22 . . . generates one of the helical sides of the screw thread. The spiral spring may be regarded as having been traced by the square s.

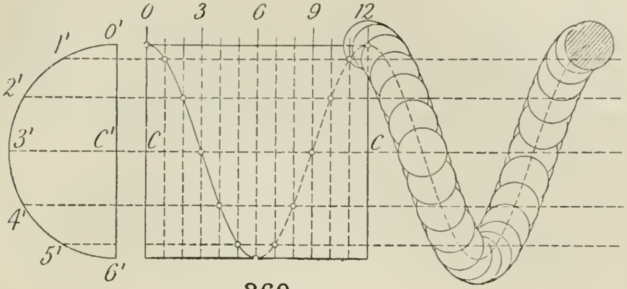
The pitch cc is the same as before, and the divisions 0 to 12 are again made use of in projecting the screw.

371. Problem.—**To project a right-handed V threaded screw and a longitudinal section of the nut.**

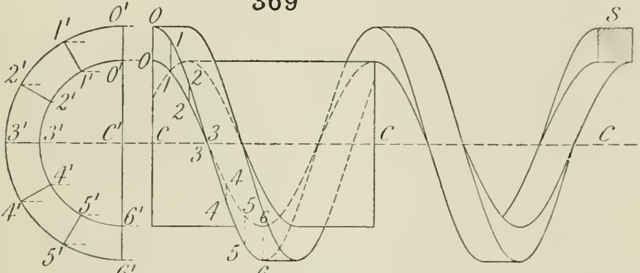
After what has been said above, the construction should be clear from the figure, without description.

The screw is single threaded, and the pitch = pp.

Note that a line such as aa is perpendicular to the axis, since in a half-turn the thread advances half the pitch.


0
3
6
9
12
0'
1'
2'
3'
C'
4'
5'
6'
c
c

369


0'
1'
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3'
3'
C'
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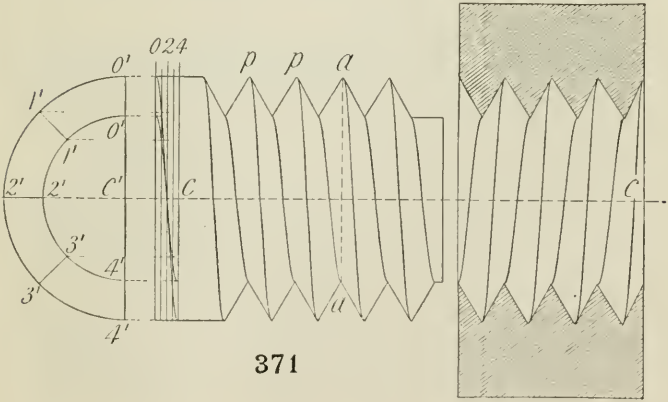
024
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p
a
a
0'
1'
0'
1'
2'
2'
C'
3'
3'
4'
4'
c
c

371

372. Problem. — **A ruled surface is generated by the tangent moving along the given vertical helix ABCD. (a) Draw the horizontal trace of the surface. (b) Obtain the elevation of the section made by the given vertical plane LM. (c) Show the envelope of the tangent in plan and elevation.**

The point *A* is on the ground, and *aoc* is perpendicular to *xy*.

Draw the tangent at *c*, and by Prob. 113 set off *ck* = arc *cb*. Make *ch* = twice *ck* = *cba* = half circumference. Project from *h* to *h'*. Join and produce *h'c'*.

Then *HC* is the tangent to the helix at *C*. See Prob. 369. The base angle α of the right-angled triangle *AHC* is the *pitch angle* of the curve, and all the tangents are inclined at α to the ground. If the triangle *AHC* were wrapped round the cylinder containing the helix, the hypothenuse (produced) would generate the required surface, and the locus of *H* would be the horizontal trace of the surface. The latter is thus seen to be the *involute* of the plan *abcd* of the cylinder; this could be set out as in Prob. 125, or as follows :—

(*a*) Draw the tangent at any point *d* in plan. Project from *d* to *d'*, and draw the horizontal *d'f* to meet *h'c'* in *f*. Draw the vertical *fg*. Mark off *de* equal to *gh'*. Project from *e* to *e'* and join *d'e'*.

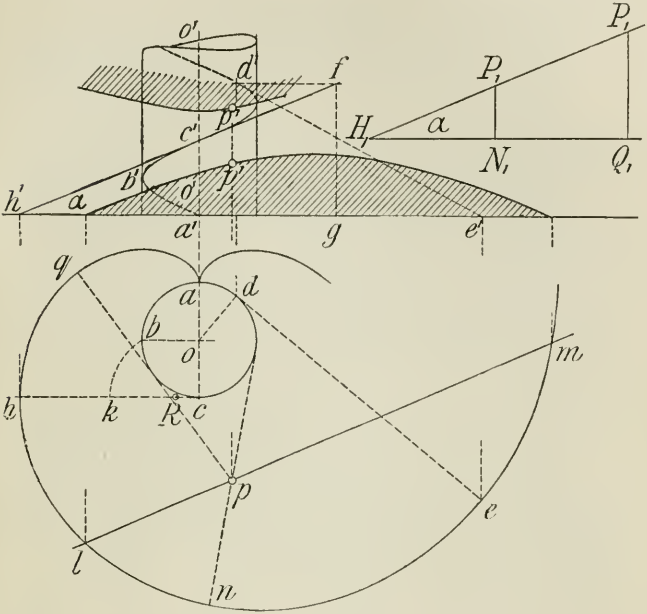
Then *DE* is the tangent to the helix at *D*, and *E* is its horizontal trace. Find similarly other points in the required trace of the surface, and draw the curve *ahe* through them.

(*b*) Obtain the points where the plane *LM* cuts the tangents like *DE* previously found. Or proceed thus :—

Select any point *p* in *lm*. Draw the two tangents *qp*, *np*. Draw the triangle with base angle α, and along its base set off H_1N_1, H_1Q_1 equal to *pn*, *pq*. Erect the perpendiculars N_1P_1, Q_1P_1, and mark off their lengths as heights above *xy* on the projector from *p*, thus obtaining *p'*, *p'*.

Determine other points similar to *p'*, *p'*, and draw the required elevation of the section through them as shown.

(*c*) The envelope in plan is seen to be the circle *abcd*; and in elevation is the curve *a'b'c'd'*.



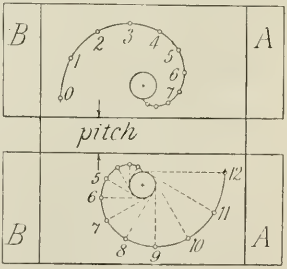
Thus the envelope is the helix itself. The surface has an *edge*, which coincides with the helix.

The figure shows how a model of this surface may be made.

Set out the symmetrical curves on cardboard, and make holes at the points determined by a series of equiangular tangents. Indent and fold along the lines shown, and secure by paper-fasteners through *AA*, *BB*.

Use twine for the generators, laced through the holes.

Take the dimensions of Ex. 27, p. 481.



373. Examples on Problems 359 to 372.

1. Draw the plan of a regular dodecahedron of 1″ edge (*a*) when a face is horizontal; (*b*) when a diagonal of the solid is vertical. Measure the length of the diagonal, the distance between two parallel faces, and the distance between two parallel edges.

2. Draw the plan of a regular icosahedron of 1″ edge (*a*) when a diagonal of the solid is vertical; (*b*) when a face is horizontal. Measure the length of the diagonal, the distance between two parallel faces, and the distance between two parallel edges.

3. Determine the dihedral angles between the faces of a regular dodecahedron and icosahedron. Find also the angles subtended by the sides at the centres of the solids.

4. Determine the spheres inscribed in, and circumscribing, a regular tetrahedron, octahedron, and a cube, each of 2″ edge.

5. Determine the spheres inscribed in, and circumscribing, the solids of Exs. 1 and 2.

6. Find the sizes of the five regular polyhedra inscribed in, and circumscribing, a sphere of 2″ diameter.

7. The lengths of the six edges of an irregular tetrahedron $ABCD$ are respectively $BC = 3''$, $CA = 2\frac{5}{8}''$, $AB = 2\frac{1}{4}''$, $AD = 2\frac{3}{4}''$, $BD = 2\frac{1}{2}''$, $CD = 3\frac{1}{8}''$. Draw the plan of the solid when the face ABC rests on the ground, and index the plan of D.

8. Determine the indexed plans of the spheres which are inscribed in, and which circumscribe, the irregular tetrahedron of Ex. 7.

Solve the following three spherical triangles :—

9. Given $a^\circ = 47^\circ$, $b^\circ = 55^\circ$, $c^\circ = 41^\circ$.
Ans. $A^\circ = 62^\circ$, $B^\circ = 81.5^\circ$, $C^\circ = 52{\cdot}4^\circ$.

10. Given $a^\circ = 75^\circ$, $b^\circ = 59^\circ$, $C^\circ = 50^\circ$.
Ans. $A^\circ = 97.5^\circ$, $B^\circ = 61.6^\circ$, $c^\circ = 48.3^\circ$.

11. Given $a^\circ = 55^\circ$, $b^\circ = 80^\circ$, $A^\circ = 45^\circ$.
Ans. $c^\circ = 67^\circ$, $B^\circ = 58.2^\circ$, $C^\circ = 127^\circ.4$;
or $c^\circ = 39.1^\circ$, $B^\circ = 121.8^\circ$, $C^\circ = 33^\circ$.

12. Taking the answers in the above three examples as data, solve the four triangles.

13. The latitude and longitude of Rome are respectively 41.9° N. and 12.5° E., and of St. Petersburg 60.0° N. and 30.3° E.: find the angle subtended by the two places at the centre of the earth; find also the direction at each place of the great circle connecting the two. *Ans.* 25.05°, 140.8°; 21.2°.

14. Two copper pipes each 10″ diameter meet at right angles to form an elbow joint. Draw the development of the plate near the joint for one of the pipes. Scale $\frac{1}{4}''$.

15. A horizontal range of sheet metal piping, 6″ diameter, is required to have a horizontal 4″ branch fitted at an angle of 60°, the upper surfaces of both pipes being at the same level. Draw the development of the joint for both pipes. Scale $\frac{1}{2}$.

16. Work Prob. 365, the diameter of the sphere being 4″. Take four zones in the hemisphere and develop them all. Let the angle of a lune be $22\frac{1}{2}°$.

17. A square with one corner on the ground projects into a parallelogram, with adjacent sides 3″ and 2″, and included angle 60°. Determine the side of the square, the trace and the inclination of its plane, and the heights of its other corners.

18. Draw a parallelogram *abcd*, taking $ab = .7''$, $ad = 3''$, $bad = 55°$. This figure is the projection of a rectangle whose sides *AD*, *AB* are in the ratio of 3 to 2. Determine the sides and the inclination of the plane of the rectangle.

19. An equilateral triangle of 2″ side is the projection of a right-angled triangle whose three sides are in the ratios $3:4:5$; determine the size and angular position of this latter triangle.

20. A right-angled triangle whose three sides are $1\frac{1}{2}''$, 2″, $2\frac{1}{2}''$ is the projection of an equilateral triangle; determine the size and angular position of the latter.

21. Draw a triangle *oab* making $oa = 2''$, $ob = 1.7''$, $ab = 1.3''$. The point *o* is the plan of the centre, and the line *ab* that of one side of a regular octagon; complete the plan of the figure.

Special solution.—Draw any regular octagon *ABC* . . . centre *O*. Join *AC* to intersect *OB* in *M*. In the given triangle take *m* in *ob* such that $om : mb = OM : MB$; join *am* and produce to *c*, making $mc = am$. The student should be able to complete the solution.

22. In a regular tetrahedron of 3″ edge take points *P*, *Q*, *R* on three edges distant respectively 0.6″, 1.5″, and 2.2″ from one corner. Draw the plan of the solid when the triangle *PQR* projects into an equilateral triangle.

23. Draw a triangle *abe*, have $ab = 3''$, $ae = 2''$, and $be = 1\frac{1}{2}''$; *ab* is the plan of one edge of a regular tetrahedron, and *e* that of the middle point of the edge opposite to *AB*. Complete the plan of the tetrahedron.

24. Work Prob. 369, having given :—diameter = 3″; pitch = 4″; diameter of section of spring = 1″.

25. Work Prob. 370, having given :—diameters = 4″ and 3″; pitch = 4″; width of thread = $\frac{1}{2}''$.

26. Work Prob. 371, having given :—diameter outside = 4″; angle of thread = 60°; pitch = $\frac{3}{4}''$; length of nut = 4″.

27. Work Prob. 372, having given :—diameter of helix = 1″; pitch = $2\frac{1}{2}''$; *lm* makes 15° with *xy* and touches the circular plan *ac*. Make a model of this surface as described in the problem, using stout cardboard.

374. Miscellaneous Examples.

1. A regular tetrahedron of $2''$ side is inscribed in a sphere. Draw the elevation of the solids, the circumscribed sphere resting on the horizontal plane, one face of the tetrahedron horizontal, and one edge of that face inclined at 20° to the vertical plane. (1897)

2. Draw a triangle oab, having $ab = .92''$ and $oa = ob = .58''$. Take o as the plan of the centre of a sphere, radius $\frac{5}{8}''$, and a, b as the plans of two points on its upper surface. Determine the plan of a regular triangular pyramid, circumscribing the sphere, and having two of its sloping faces touching the sphere at A and B respectively.

3. An octahedron of $2''$ edge stands on the horizontal plane. Draw its plan and also that of its inscribed sphere. (1884)

4. Draw the plan of a cube inscribed in a sphere of $3''$ diameter resting on the horizontal plane, one corner of the cube to be at a height of $2.75''$ above the horizontal plane. (1878)

*5. Fig. (*a*). Determine the centre and radius of the sphere circumscribing the irregular pyramid $ABCD$. Unit $= 0.1''$. (1882)

*6. Fig. (*c*). The three given parallel lines are the plans of portions of the edges of a triangular prism. Determine the plan of a sphere inscribed in the prism, and having 16 as the index of its centre. Unit $= 0.1''$. (1887)

*7. Fig. (*d*). The arc bc is part of the plan of the base of a right cone standing on the horizontal plane; v_{34} is the plan of the vertex. The portion $ABCD$ is to be covered with paper. Determine the shape to which the paper must be cut. Unit $= 0.1''$. (1888)

*8. Fig. (*b*). The end elevation and a portion of the plan of two adjacent ridge roofs is given. Determine in plan the shortest distance measured on the roof surface from A to B. (1887)

*9. Fig. (*e*). Determine the shape of the plates used to form the elbow pipe with bell-mouth shown. Draw the figure *four* times the size shown. (1887)

10. Fig. (*f*). Draw the development of the funnel shown, so as to give the shapes of the plates from which it is made. Draw the figure *four* times the size shown.

11. Draw a rectangle $abcd$ ($ab = cd = 3\frac{1}{2}''$; $bc = da = 1\frac{3}{4}''$); ab and cd are the plans of the diagonals of two opposite faces of a cube; bc and da are the plans of two edges. Complete the plan of the cube. (1880)

12. Two equal right cones, height $1\frac{3}{4}''$, diameter of base $2\frac{1}{2}''$, have a common vertex. One rolls upon the other which stands on the ground. Determine the locus of a point on the circumference of the base of the rolling cone. (Honours, 1886)

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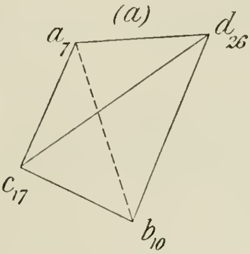
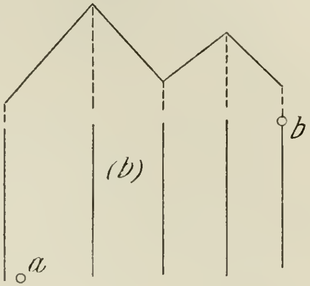
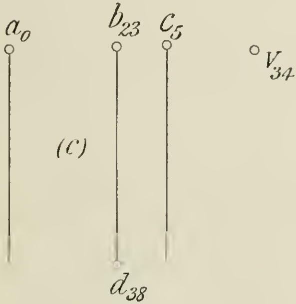
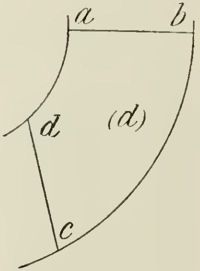
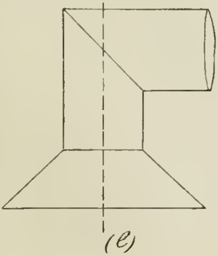
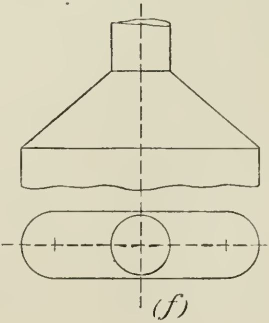












***12.** Fig. (*a*). The area *abcd* on a hill side, the traces of the plane of which are given, is to be levelled; the side slopes are 1 in 1. Complete the plan of the excavation. (1885)

***13.** Fig. (*b*). The plan and section of an embankment are given; the lines *aa*, *bb* represent the sides of a road cut through it. The slopes of the sides of the cutting are 35°. Complete the plan. (1884)

14. On a right circular cylinder of $1\frac{1}{4}''$ diameter, a helix of $3''$ pitch is traced. Draw the elevation of one turn of the helix; draw also the plans and elevations of tangents to the helix at 12 equidistant points, and determine the points in which these tangents meet the horizontal plane through the lowest point of the helix. Axis of cylinder vertical. (1890)

15. A spiral spring, axis vertical, is of the form of a square screw thread. Side of square $\frac{1}{2}''$; external diameter on plan $3''$; pitch $2\frac{1}{4}''$. Draw the elevation of one complete turn of the spring. (1877)

16. A right cone, height $4''$, diameter of base $3''$, stands on the horizontal plane. A point starting from the base of the cone moves round its surface at a uniform speed, and rises half the height of the cone in turning round to cut the generatrix from which it started. Draw the plan and an elevation of the curve traced by the point. (1886)

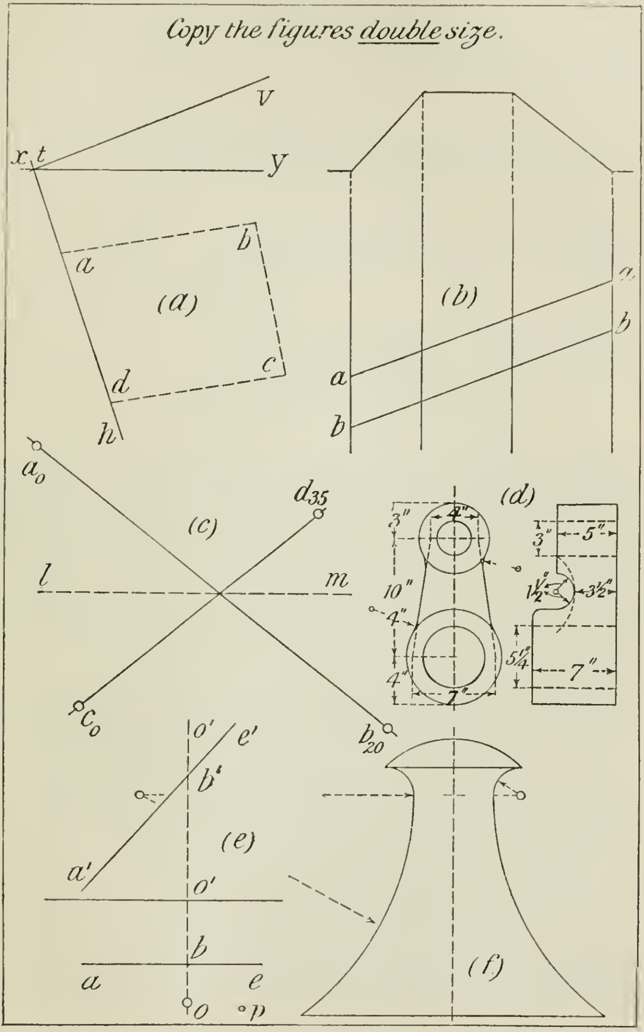
***17.** Fig. (*e*). A twisted surface of revolution is generated by a line *ae*, *a'e'*, revolving round a vertical axis *o*, *o'o'*, to which it is rigidly fixed by the horizontal line *ob* (in elevation *b'*). Draw in plan and elevation the generating line when it has revolved round the axis so as to pass in plan through *p*. (1894)

***18.** Fig. (*c*). Two lines are given by their figured plans *ab*, *cd*. A surface is generated by a line moving parallel to the horizontal plane and always meeting both the given lines. Determine the true form of the section of this surface by a vertical plane *LM*. The section to be carried up to a height of $3''$ above the ground. What is this surface termed, and can it be developed? (1879)

19. Fig. (*d*). Two views of a crank are given. Determine the proper shape of the dotted curve in the right-hand view.

20. A line $3''$ long moves at a uniform speed round a vertical axis, by which it is bisected, and to which it is always at right angles. At the same time it moves along the axis a distance of $1\frac{5}{8}''$ for every complete revolution, thus generating a helical surface. Make a section of this surface by a plane parallel to and $\frac{3}{4}''$ from the fixed axis, the section to show two revolutions of the surface. (1889)

21. Fig. (*f*). A surface of revolution (a capstan) is shown in elevation; determine a sectional elevation on a vertical plane $\frac{1}{2}''$ from the axis.



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SECTION III

GRAPHICS

CHAPTER XX

GRAPHIC ARITHMETIC

375. Graphic representation of magnitude.—In ordinary arithmetic we are concerned with numbers, and with numerical calculations. The former may be *pure* numbers, or they may be *concrete*, that is, may express the *magnitudes* of quantities of some specified kind. In the latter case the number measures the magnitude by telling us how many times the quantity contains an arbitrarily chosen unit of the same kind and of known size.

In graphic arithmetic, numbers and magnitudes are represented by the lengths of straight lines, set out or measured to scale; and calculations are made by drawing.

Thus let it be required to represent the number 7.5.

We may first draw any line, setting off any convenient length on it to represent unity, and then on the same or any other line mark off a segment 7.5 units long.

Or instead of drawing the unit line, we may set out or *specify* some convenient scale, say $\frac{1}{2}''$ to 1 unit, and using this scale mark off the segment 7.5 units long.

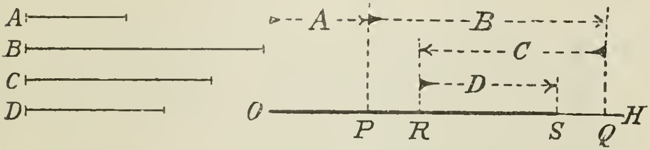
If the number had been concrete, say 7.5 tons, then the unit line would have been labelled 1 *ton*, or the scale would have specified $\frac{1}{2}''$ to 1 ton.

If the number to be represented is comparatively large or small, as for example 750 or .075, then we may replace the unit line by a line which shall represent some convenient known multiple, such as 100 units, or submultiple like .01 unit. The scales are also correspondingly modified; thus we may take as suitable scales for plotting these numbers $\frac{1}{2}''$ to 100 units, or $\frac{1}{2}''$ to .01 unit.

The student should refer to Probs. 7 to 11 on *scales*. Also to Art. 5 for the description of an engine-divided decimal scale, by the use of which we are enabled to pass from the numerical to the graphical system or *vice versa*.

376. Addition and subtraction. — Lines to be added or subtracted must represent quantities of *like kind*, and be drawn all to the same scale.

The process of addition consists in drawing the lines in position, end to end, all in one direction, so as to form a straight line equal to the sum. If lines are to be subtracted, they must be set off in the opposite direction.



Thus in the figure $OP = A$; $OQ = A + B$; $OR = A + B - C$; $OS = A + B - C + D$

Note 1.—Observe that the sum OS would be the same if the terms were added in any other order, say in the order $D + B + A - C$.

Note 2. — The summation is most readily effected by applying the edge of a strip of paper in succession to the lines A, B, C, D, and marking the points O, P, Q, R, S with a sharp pencil.

377. Multiplication and division.—The product of two lines, like that of two numbers, means that one is to be repeated as often as there are units in the other. That is

Unit line : one line : : other line : product.

And in *division* we have the proportion

Divisor : dividend : : unit : quotient.

Thus graphic arithmetic consists mainly of finding fourth proportionals to given lines. The results may be represented *graphically*, or expressed *numerically* by measuring on the unit scale.

378. Problem.—**Having given two lines A, B, and the unit line S ($=\frac{1}{2}''$), to find the product of A and B.**

Let $A \times B = X = X \times 1 = X \times S.$

From which $\frac{A}{S} = \frac{X}{B}$; or $S : A :: B : X.$

We thus find X as a fourth proportional to S, A, B as follows.

Draw two lines intersecting at O.

Along one set off OA equal to A. Along the other set off OB and OS equal to B and S.

Join SA, and draw BX parallel to SA.

Then OX represents the required product *graphically*.

Measuring OX on the scale of S, or $\frac{1}{2}''$, to one unit, the product is 2.94.

379. Problem.—**Having given two lines A, B, and the line S ($=\frac{1}{2}''$), to find the quotient A ÷ B.**

Let $\frac{A}{B} = X = \frac{X}{1} = \frac{X}{S}$. Or $B : A :: S : X.$

So X is a fourth proportional to B, A, S, and may be found thus:

Draw two lines intersecting at O.

Along one set off OA equal to A. Along the other set off OB and OS equal to B and S.

Join BA, and draw SX parallel to BA.

Then OX represents the required quotient *graphically*.

Measuring OX on the scale of S, or $\frac{1}{2}''$, to one unit, the quotient is 2.25.

Note.—With a *different* unit A and B would represent *other* numbers, but their quotient when *measured* would be the *same number*.

380. Problem.—**Having given two lines A, B, and their product X, to find the unit line S.**

This problem is the converse of Prob. 377.

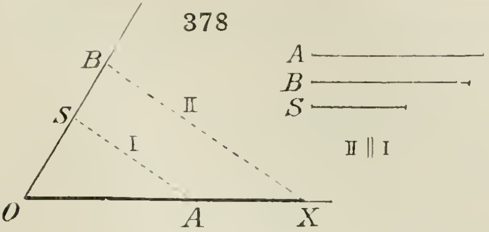
In Fig. 378 set off along one axis OX and OA equal to X and A. Along the other set off OB equal to B.

Join XB, and draw AS parallel to XB.

Then OS is the required unit length.

381. Problem.—**Having given two lines A and B, and X the quotient A ÷ B, to find the unit line S.**

This problem is the converse of Prob. 378.



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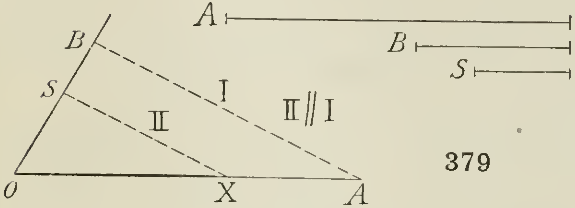


379

In Fig. 379 set off along one axis OX and OA equal to X and A. Along the other set off OB equal to B.

Join AB, and draw XS parallel to AB.

Then OS is the length of the required unit.

Examples.—1. The unit line being 1.34″ long, construct the unit scale, and mark off a length representing 2.27 units.

2. If the line S, Fig. 379, represent unity, what numbers do the lines A and B represent? *Ans.* 3.60; 1.62.

3. In Fig. 379, taking S as the unit of length, find the number of units of area in the rectangle, two adjacent sides of which are equal to A and B. *Ans.* 5.83.

4. Find the line which represents $\frac{5}{7}$ to a unit of $\frac{1}{2}''$.

5. In Fig. 379, if S be the unit, represent graphically $\frac{A+B}{A}$.

6. Draw the lines A and B of Fig. 379 double the length. Then determine the numerical value of $\frac{A-B}{A+B}$. *Ans.* 0.386.

7. In Fig. 379, if S represents the product of A and B, find and measure the unit of length. *Ans.* 2.92″.

8. If the line A, Fig. 379, represent the fraction $\frac{7}{3 \cdot 9}$, determine and measure the unit of length. *Ans.* 1.02″.

382. Problem.— **Having given the lines A, B, C, . . . and the unit line S ($=\frac{1}{2}''$), to find the continued product $\mathbf{A} \times \mathbf{B} \times \mathbf{C} \times \ldots$**

The construction is a repetition of that of Prob. 378.

Draw two lines intersecting at O.

Along one set off OA equal to A. Along the other set off OS, OB, OC, . . . equal to S, B, C, . . .

Join SA, and draw BX_1 parallel to SA. Join SX_1, and draw CX_2 parallel to SX_1; and so on.

Then $OX_1 = A \times B = 1.3$; $OX_2 = A \times B \times C = 2.6$, etc.

383.— **Having given the lines A, B, C, . . . and the unit line S ($=\frac{1}{2}''$), to find the value of $\dfrac{\mathbf{A}}{\mathbf{B} \times \mathbf{C} \times \ldots}$**

The construction is a repetition of that of Prob. 379.

Along one axis set off OA equal to A. Along the other set off OS, OB, OC, . . . equal to S, B, C, . . .

Join AB, and draw SX_1 parallel to AB. Join CX_1, and draw SX_2 parallel to CX_1; and so on.

Then $OX_1 = \dfrac{A}{B} = 1.93$; $OX_2 = \dfrac{A}{B \times C} = 1.05$, etc.

384.—**To find the value of $\mathbf{A} \times \dfrac{\mathbf{B}}{\mathbf{C}} \times \dfrac{\mathbf{D}}{\mathbf{E}} \times \ldots$**

Set off the lengths A, B, C, D, E, . . . from O along the two axes in the manner shown in the figure.

Join CB, and draw AX_1 parallel to CB. Join ED, and draw X_1X_2 parallel to ED. And so on.

Then $OX_1 = A \times \dfrac{B}{C}$. $OX_2 = A \times \dfrac{B}{C} \times \dfrac{D}{E}$, etc.

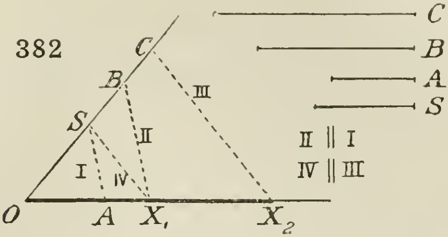
Note.—The Roman numerals in the figures indicate the order in which the dotted lines between the axes are drawn. A reference such as II. || I. means that line II. is to be drawn parallel to line I.

Examples. — **1.** In Fig. 383, if S be the unit of length, find the number of units of volume in the rectangular prism of which A, B, and C are three edges. *Ans.* 19.8.

2. Determine graphically the value of $3.14 \times 1.7^2 \div 0.67$. *Ans.* 13.54.

3. In Fig. 384 determine $A \times \dfrac{C}{B} \times \dfrac{E}{D}$, the unit being $\frac{1}{2}''$. *Ans.* 7.3.

382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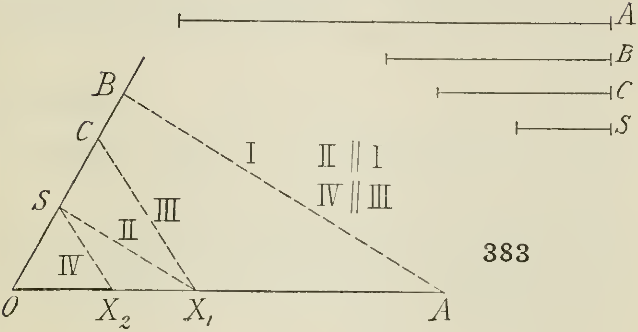
383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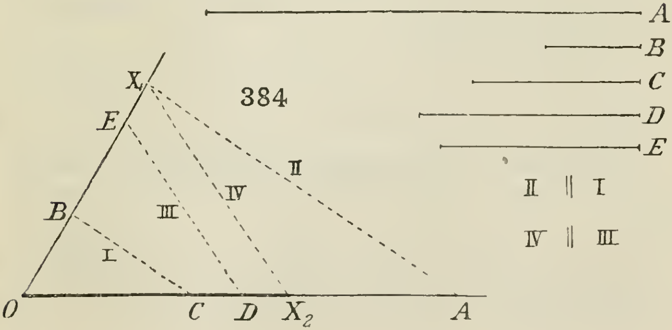
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385. Problem.—To find the value of $\frac{\mathbf{A}}{\mathbf{B}} + \frac{\mathbf{C}}{\mathbf{D}} + \ldots$

The method of solution is to reduce the fractions to a common denominator $K = k$ units, where k is any known number (say 3).

Thus let $\frac{A}{B} = \frac{X_1}{K}$; $\frac{C}{D} = \frac{X_2}{K}$; . . .

Then $X_1 = K \times \frac{A}{B}$; $X_2 = K \times \frac{C}{D}$; . . . and these values may be found as in Prob. 383.

Set off $OK = K$, say $\frac{3}{4}''$, $OA = A$, . . . along any convenient axes in the manner shown.

Determine X_1, X_2, . . . by drawing the pairs of parallel lines between the axes in the order indicated by the Roman numerals.

Then $\frac{A}{B} + \frac{C}{D} + \ldots = OX_1 + OX_2 + \ldots$ measured on the unit scale and *divided* by k; *or* measured directly on the scale on which OK ($\frac{3}{4}''$) is the unit.

386. Problem.—To find the value of $\mathbf{A \cdot B + C \cdot D} + \ldots$

The series of products are first reduced to any convenient common base K ($\frac{3}{4}''$) representing k units, k being a known number (say 3).

Thus let $A \cdot B = X_1 \cdot K$; $C \cdot D = X_2 \cdot K$; . . . .

Then $X_1 = \frac{A \cdot B}{K}$; $X_2 = \frac{C \cdot D}{K}$; . . .

These values may be found as in Prob. 383.

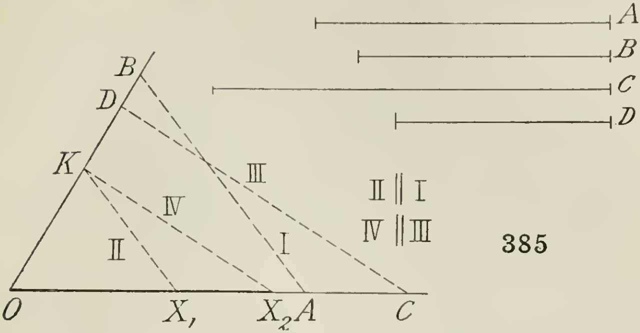
Make the construction which is sufficiently indicated by the notation of the figure.

Then $A \cdot B + C \cdot D + \ldots = OX_1 + OX_2 + \ldots$ measured on the unit scale ($\frac{1}{4}''$) and *multiplied* by k (3); *or* measured directly on the scale on which OK ($\frac{3}{4}''$) $= k^2$ (or 9) units.

387. Problem.—To find the number of units of area in a given irregular polygon, having given the unit of length S.

First reduce the given polygon by Prob. 30 to an equivalent triangle, then find half the product of the base and altitude of the triangle.

In the figure OA = altitude; $OB = \frac{1}{2}$ base; OS = unit length. Then OX measured on the unit scale gives the required area.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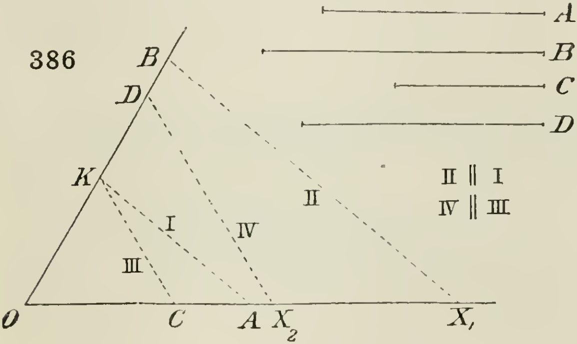
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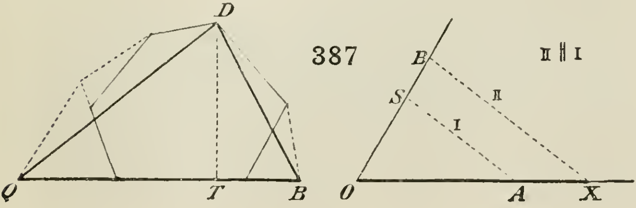
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387

388. Involution and evolution. — *Involution* is the process of raising a number to any integral power. Thus A^n (read A to the nth power) stands for the continued product $A \times A \times \ldots$, where the factor A occurs n times. n is called the index of the power, or simply the *index*. The value of A^n can be found by Prob. 381, putting B, C, . . . each equal to A. It may also be found by the general construction of Prob. 394. In this case, however, the special method given in the next problem is preferable.

Evolution, or the extraction of roots, is the inverse of involution; thus $\sqrt[n]{A}$, the nth root of A, denotes a quantity which being raised to the nth power gives A as the result. The general graphical solution requires the use of a logarithmic curve or spiral, and is given in Prob. 394. The extraction of the square root, and of the fourth, eighth, sixteenth . . . roots may however be effected by simple geometry. See Probs. 390 and 391.

389. Problem.—**Having given a line A, and the unit line S, to find the integral powers of A, viz. A^2, A^3, . . . Also to find the integral powers of the reciprocal of A, viz. $1 \div A$, $1 \div A^2$, . . .**

Draw two perpendicular axes intersecting at O.

Along them set off OA and OS equal to A and S.

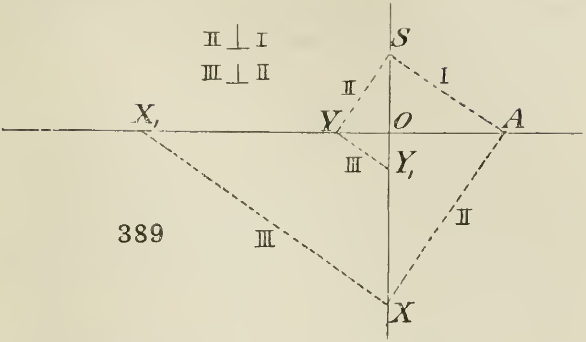
Join SA, and draw the lines AX, XX_1, . . . SY, YY_1, . . . alternately perpendicular and parallel to SA, as shown in the figure.

Then $OX = A^2$; $OX_1 = A_3$; . .

And $OY = \frac{1}{A}$; $OY_1 = \frac{1}{A_2}$; . . .

These results follow from the similarity of all the triangles formed by the axes and the dotted lines.

Note. — If OS had been made equal to s units on the scale for A, then the scale for measuring OX or A^2 would have been OS to s^2 units; and for OX_1 or A^3, OS to s^3 units; . . . And the scales for measuring OY, OY_1, . . . *i.e.* $\frac{1}{A}$, $\frac{1}{A^2}$, . . . would have been OS to $\frac{1}{s}$, $\frac{1}{s^2}$, . . . unit respectively.



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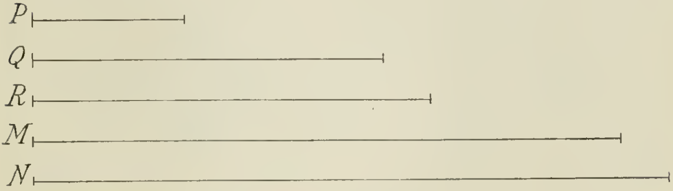


Examples on Problems 384 to 388.

1. Find the value of $\frac{Q}{R}+\frac{Q}{P}+\frac{M}{Q}$. *Ans.* 4.86.
2. Find the value of $P\cdot Q+Q\cdot R+M\cdot N$, the unit being 1″. *Ans.* 15.8.
3. Find the value of $\frac{N}{P}+Q\cdot M$, the unit being R. *Ans.* 5.49.
4. If P be the unit of length, find the values of Q^2, Q^3, $\frac{1}{Q}$, $\frac{1}{Q^2}$, $\frac{1}{Q^3}$. *Ans.* 5.35, 12.37, .432, .187, .0808.
5. Find the line which represents the cube of a line 1.37″ long to a unit of 1″. *Ans.* 2.57″.
6. If the line N represent the square of the number represented by the line R, find the length of the unit. *Ans.* 1.3″.
7. A line 2.9″ long represents the sum of the areas of an equilateral triangle and square, each of 1″ side. Determine the unit of length. *Ans.* .494″.
8. Draw a curve which will give the squares of all quantities from 0 to 6, taking $\frac{1}{8}$″ as the unit.

390. PROBLEM.—**To find the square root of a given line A, having given the unit line S.**

Let $X = \sqrt{A}$; then $X^2 = A = A \times 1 = A \times S$.

That is $\frac{X}{S} = \frac{A}{X}$; $A : X :: X : S$.

Thus the required square root X is a mean proportional between A and S. See Prob. 27.

Draw a straight line, and from any point O in it mark off in opposite directions OA and OS equal to A and S.

Bisect SA in C, and with centre C draw the semicircle on SA. Draw OX perpendicular to SA.

Then $OX\ (= 1.54) = \sqrt{OA \times OS} = \sqrt{A \times 1} = \sqrt{A}$.

If S be not a unit line, OX represents the square root of $A \times S$.

Note 1.—If the construction were repeated on OX, the 4th root OA would be obtained. And by continued repetition the 8th, 16th, . . . , 2^nth roots might be found. See next problem.

Note 2. —If A were a large or small quantity compared with the unit S, an "ill-conditioned" construction might be avoided by making OS equal to s units, where s is any convenient known number. In which case the square root OX would require to be measured on the scale of OS to $\sqrt{s}$ units; the 4th root on the scale of OS to $\sqrt[4]{s}$ units; and so on. See next problem.

391. PROBLEM.—**To find the eighth root of 614.**

To any convenient scale set off OA equal to 614 units, and OS equal to say 2^8, that is 256 units.

Draw OX perpendicular to SA, and describe the semicircles in the order figured, making OB, OC equal to OX, OY.

Then $\sqrt[8]{614} = OZ$ measured on the scale on which $OS = \sqrt{256} = 2$ units.

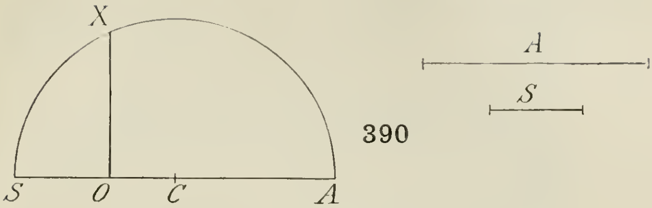
392. PROBLEM.—**To draw a figure which shall give the square roots of the first n natural numbers.**

Draw OA of unit length on any convenient scale.

Draw AB perpendicular and equal to OA. Join OB.

Draw BC perpendicular to OB and equal to OA. Join OC. And so on.

Then $OB = \sqrt{2}$; $OC = \sqrt{3}$; . . . (Euc. I. 47.)



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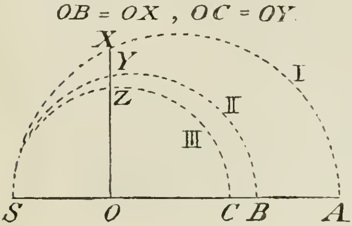

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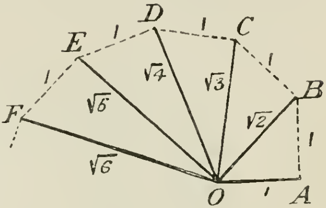

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Examples. — **1.** Determine graphically $\sqrt{5.2}$, $\sqrt{273}$, and $\sqrt{0.035}$. *Ans.* 2.28 ; 16.5 ; .187.

2. Determine $\sqrt[4]{165}$, $\sqrt[4]{0.165}$, $.314^3$, and $.0314^3$. *Ans.* 3.58 ; .637 ; .031 ; .000031.

3. A line 2.6″ long represents $4\frac{1}{2}$ units. Obtain lines which represent 1.3, $\frac{10}{3}$, and $2\sqrt{6}$ units. *Ans.* .75″ ; 1.93″ ; 2.83″.

4. On page 495, taking P as the unit line, determine the line which represents $\sqrt{N} \div Q$. *Ans.* .694″.

5. If the line N on page 495 represent the square root of the line Q, determine the unit line. *Ans.* 6.06″.

6. If a line $2\frac{1}{2}$″ long represent $\sqrt{2}$, determine a line representing $\sqrt{\frac{3}{2}}$. *Ans.* 2.16″.

7. Taking $\frac{1}{4}$″ as unit, obtain lines representing 15, $\sqrt{15}$, $\frac{5}{\sqrt{15}}$. *Ans.* 3.75″ ; .97″ ; .323″.

8. Taking the lines P and Q on page 495, determine $\sqrt{Q^2+P^2}$, $\sqrt{Q^2-P^2}$, and the ratio $\sqrt{\frac{Q^2-P^2}{Q^2+P^2}}$. Unit = 0.5″.

Hint.—Make use of Euclid I. 47. *Ans.* 4.03 ; 3.33 ; 1.21.

393. Examples worked out.

1. *m and n are two given lines, determine* $\sqrt{m/n}$, *the unit being* $0.25''$.

Set off OS, Fig. 1 (*a*), equal to the unit, $0.25''$.

In the opposite direction set off ON, OM equal to m and n, and describe semicircles on SN and SM as diameters.

Draw OM_1 at right angles to OM, then OM_1 is $\sqrt{m}$, and ON_1 is $\sqrt{n}$.

Set off these lengths and OS $(=\frac{1}{4}'')$ along two axes on (*b*) as shown. Join M_1N_1, and draw SX parallel to M_1N_1.

Then OX represents $\sqrt{m/n}$, and measured on the $\frac{1}{4}''$ scale is 1.5.

2. *Find a line which represents* $\dfrac{\sqrt{4\cdot AB\cdot CD}}{\sqrt{3}}$. *Unit* $=1\frac{1}{4}''$. (1896)

$\sqrt{4\cdot AB\cdot CD}\div\sqrt{3}$ is the same as $\sqrt{\frac{4}{3}AB\cdot CD}$. Set off AB and divide it into three equal parts as shown in Fig. 2; make BE equal to one of the parts; then $AE=\frac{4}{3}AB$.

Make $EF=CD$; on AF describe a semicircle, and erect the perpendicular EG; then EG represents $\sqrt{\frac{4}{3}CD\cdot AB}$.

3. *If* $\dfrac{P\times Q}{R\times N}=R$, *find the unit of length.* Fig. 3.

This may be written

$$\frac{P\times Q}{R\times N}=\frac{R}{\text{unit}}\text{; or, unit}=R\times\frac{R}{P}\times\frac{N}{Q}.$$

We may now proceed as in Problem 384.

Or, write,
$$\text{unit}=\frac{R^2}{P}\cdot\frac{N}{Q}.$$

Then set off $OP=P$; and, at right angles, $OR=R$.

Join PR, and draw RX_1 perpendicular to PR to meet PO.

Then OX_1 represents $R^2\div P$; for $OP\cdot OX_1=OR^2$.

Make OQ and ON equal to Q and N; join QN, and draw X_1X_2 parallel to QN.

Then OX_2 represents $\dfrac{R^2}{P}\cdot\dfrac{N}{Q}$, or unit line; it measures $0.76''$.

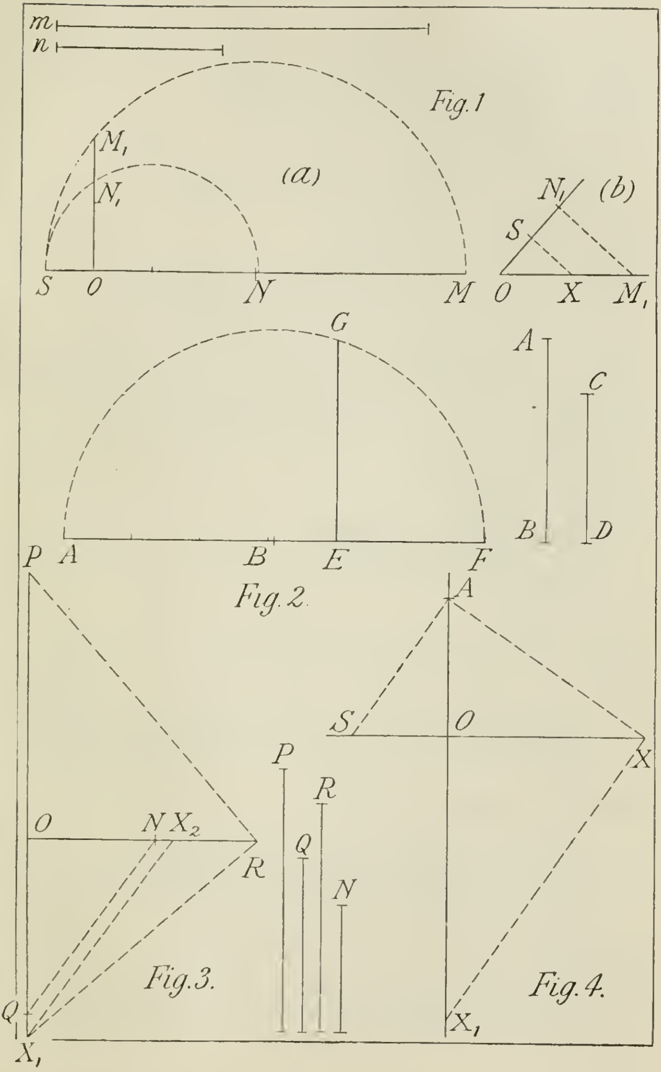
4. *Determine the value of* $(0.289)^3$. Fig. 4.

Set off OA equal to .289 on the scale $\frac{1}{4}''$ to 0.1.

Set off OS equal to $\frac{1}{2}''$, *i.e.* .2 or $\frac{1}{5}$ unit.

Join AS, and draw AX, XX_1 respectively perpendicular and parallel to AS. Then OX_1 represents $(.289)^3$.

Measuring OX_1 on the scale of OS to $(.2)^3$ of a unit, that is, $\frac{1}{2}''$ to .008 unit, or $\frac{1}{16}''$ to .001 unit, the answer is .024.


m
n
Fig. 1
(a)
M,
N,
S
0
N
M
(b)
N,
S
0
X
M,
G
A
B
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F
Fig. 2.
A
B
C
D
P
0
N
X₂
R
Q
X,
Fig. 3.
P
R
Q
N
A
S
0
X
X,
Fig. 4.

394. PROBLEM.—**Having given a line A and the unit line S, to find the value of A^n, where n is any given integer or fraction, positive or negative.**

By the construction of Problem 129 set out on the base *BB* any suitable logarithmic curve *LL*.

Determine *OS*, the ordinate of unit length *S*.

On *OS* set off *OA* equal to *A*. Draw *AP*, *PM* parallel and perpendicular to *BB*.

Mark off $ON = n$ times *OM*; in the same direction as *OM* if *n* is positive, and opposite if *n* is negative.

Draw *NQ*, *QX* perpendicular and parallel to *BB*.

Then *OX* measured on the scale of unit *S* gives A^n.

Note 1.—The logarithmic curve might be used to perform graphic arithmetic in a manner exactly analogous to the way in which a table of logarithms is employed in making arithmetical calculations.

Ordinates such as *PM*, *QN*, represent numbers, and the abscissæ *OM*, *ON*, their logarithms.

Note 2.—The logarithmic curve might be replaced by a logarithmic spiral, see Prob. 128.

395. Miscellaneous Examples.

1. Determine graphically $2.5^{\frac{2}{3}}$, $1.9^{-\frac{3}{7}}$, and $0.72^{-0.6}$. *Ans.* 1.84 ; .76 ; 1.22.

***2.** If $\frac{A \times B}{C \times D} = C$, find the unit of length. *Ans.* $1\frac{7}{8}''$. (1898)

***3.** Find a line which represents $\sqrt{4 \cdot A \cdot B} \div \sqrt{3}$. Unit $= 1\frac{1}{4}''$. (1896)

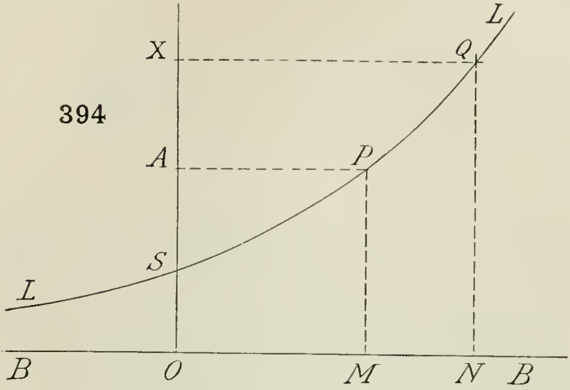
***4.** Copy the figure double size. Then reduce it to an equivalent triangle with its base on *AH* and its vertex at *E* ; and find the length of a line representing the area of the figure, taking for unit 1″. (1894)

***5.** Find the ratio $\frac{C}{A}\sqrt{\frac{A^2 - B^2}{B^2 - C^2}}$, the unit of length being 2″.
Ans. .932″. *Hint.*—Compare Ex. 8, p. 497. (1890)

6. Determine by graphic arithmetic a line representing the contents of a rectangular solid whose dimensions are $3'' \times 1.75'' \times 1.25''$. Unit $= 2.5''$. *Ans.* 1.05″ long. (1897)

7. Find a line to represent x^3 when $x = 2\sqrt{2} - \sqrt{3}$. Unit $= 1''$. *Ans.* 1.32″. (1895)

8. A line 3″ long represents the sum of the areas of a pentagon, square, and equilateral triangle of 1″ side. Determine the unit of length. *Ans.* 1.025″. (1891)



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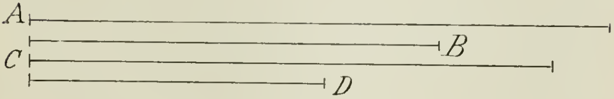

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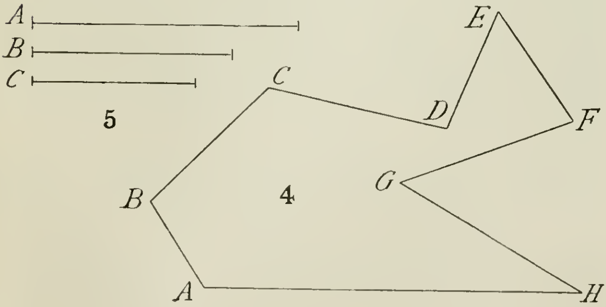

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CHAPTER XXI

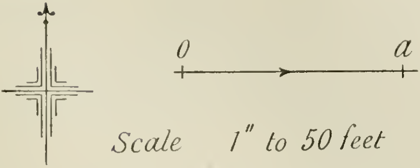
GRAPHIC STATICS

396. Directed quantities.—In arithmetic and algebra we are concerned with number and magnitude. A second important quality possessed by many objects and phenomena, and which can be made the subject of calculation, is that of having a definite *direction* in space. Graphical processes are uniquely fitted to deal with this property.

Every one is acquainted with such quantities, and would, for instance, distinguish between a length measured to the north, and the same measured, say, to the east.

The student will readily call to mind other instances in which direction is associated with magnitude. He may think of a force of so many pounds weight, and acting along some definite line; of a top spinning at a definite speed about an axis, perhaps inclined; or of a field of force, which at any point has intensity and direction. The word **vector** is a general term, used to denote all such *directed quantities.*

Some kinds of quantities like volume, time, mass, temperature, energy, are essentially *non-directive;* others, like forces, velocities, which have direction, may yet be treated arithmetically as to magnitude only. But in the class of problems we are now about to investigate, it is a cardinal feature that the directions of the quantities which enter into the case, as well as their magnitudes, shall have influence on the result.



397. Graphic representation of a vector.—The *relative positions* of points and their *changes of position* are amongst the simplest examples of directed quantities. Such quantities may evidently be represented with great convenience and directness on paper by directed lines.

For example, a displacement of say 50 feet to the east might be exhibited thus: Let the direction to the north be indicated; then (1) draw a line running east and west; (2) on this line mark off a segment representing 50 feet to a suitable scale, and *specify the scale;* and (3) to the segment affix an *arrow-head* pointing eastwards. In the figure above, *oa* measures 58 feet.

The change of position is thus represented completely and without ambiguity, it being understood that the displacement is a horizontal one. The arrow-head indicates what is called the *sense* of the directed line; it distinguishes an eastward from a westward movement. Sometimes it is convenient to indicate the sense in another way. The ends of the line are marked and named in the *order* or *sequence* corresponding to the beginning and end of the displacement. Thus *oa* would indicate a displacement in the sense of *o* to *a*, or eastwards. If the ends of the line were named in the sequence *ao*, this would be understood to indicate the opposite or westward direction, from *a* towards *o*.

Example.—Mortlake is 2 miles west of Putney, and Wimbledon is 3 miles south of Putney. Plot the relative positions of these places to a scale of 1 inch to the mile, and measure the distance and direction of Wimbledon to Mortlake.

Ans. 3.6 miles, 33.7° west of north.

398. Resultant and components. Rectangular components.—When a series of displacements are given to a body, the change of position represented by the straight line which joins the initial to the final position is called the *resultant* displacement, and the several displacements are called *components.*

Any displacement may be imagined to be made up of components, and in an indefinite number of ways, just as we may walk from one place to another by an indefinite number of routes. But there is only one resultant which corresponds to a given set of components.

We are said to *compound* a set of displacements when we combine them to obtain their resultant, and in the reverse operation we *resolve* a vector into components.

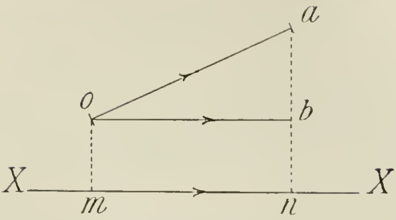
As just stated, a vector can be resolved into components in an indefinite number of ways. For the case, however, of *two* components which are parallel to given lines, there is only one solution. If the two lines are *perpendicular* to each other, the components are said to be *rectangular.*

Suppose a person to walk from one corner of a room for a distance of 10 feet, in a direction making 30° with a side wall. He would reach the same spot by walking 8.67 feet along the side, and then walking 5 feet parallel to the end wall; or he might first walk 5 feet along the end wall, and then 8·67 feet parallel to the side wall. The *rectangular components* of his change of position parallel to the side and end walls are respectively 8.67 and 5 feet.

Thus *the rectangular component of a vector in any direction is obtained by projecting the vector on a line parallel to the direction.* This is often called simply the component in that direction, rectangular being understood.

For example, the component of the given vector oa parallel to the given direction XX, is equal to ob, or mn, obtained by drawing om, an perpendicular to XX, and ob parallel to XX.

It is important that the student should be familiar with the idea of resolving a vector in a given direction.



Examples on Articles 398 and 399.

1. A person journeys 1 mile eastwards and $\frac{3}{4}$ mile to the north-east, find his resultant change of position. Find also the northerly and easterly components of the resultant. *Ans.* 1.62 mile, 19.1° N. of E.; .53 mile, 1.53 mile.
2. A football is kicked a distance of 52 yards in a direction making 35° with the side lines of the field. Find the component displacements of the ball forwards and sidewise respectively. *Ans.* 42.6 yards, 29.6 yards.
3. Find the component of a vector 3.7 units long, in a direction making 53.6° with the direction of the vector. *Ans.* 2.18 miles.
4. The two rectangular components respectively southwards and westwards of a horizontal vector are 31.3 feet and 20.7 feet. Determine the magnitude and direction of the vector. *Ans.* 37.1 feet, 56.5° S. of W.
5. A boy on a raft walks 9 feet from a point A to a point B, and during this time the raft moves (without rotation) a distance of 14 feet in a direction making 115° with AB. Find the resultant motion of the boy. *Ans.* 13.1 feet; 76.4° with AB.
6. A ship at sea sails 8.7 miles through the water apparently to the east, but an ocean current simultaneously carries the vessel 3.4 miles to the south-west. Find the resultant movement of the ship, *i.e.* its displacement as regards the bottom of the sea. *Ans.* 20.9° S. of E.; 5.88 miles.
7. A body receives component displacements of 7.38 and 5.16 feet in two directions which make 71.3° with one another. Find the resultant displacement. *Ans.* 9.58 feet, 47° with second vector.
8. Two places are 1.29 miles apart in the direction north and south. A person journeying from one to the other first walks to the north-north-east, and then to the north-west. Find the lengths of the two stages of the journey. *Ans.* .987 mile, .534 mile.
9. If a person walk between the two places of the preceding example in two straight paths, one of 1 mile, the other of $\frac{1}{2}$ mile, find the directions of the paths. *Ans.* 20.6°, 44.8°.

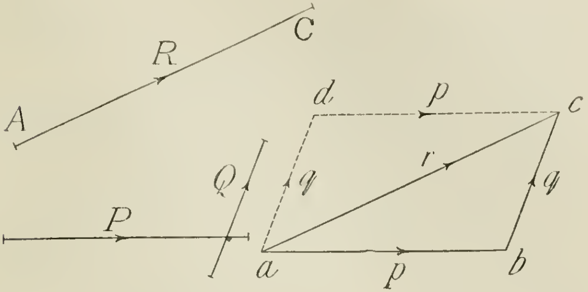
399. The triangle and parallelogram vectors.—Suppose an object to be placed on a table in a room. Let the position of the object be changed on the table in the manner represented by the directed line P, and let the position of the table in the room be also changed as represented by Q. (The table must remain parallel to itself.) It is required to find the resultant displacement of the object in the room.

From any point a draw ab equal and parallel to P and similarly directed; and from b draw bc equal and parallel to Q and similarly directed; mark ac with an arrow-head pointing from a to c; then the directed line ac represents the resultant displacement. The triangle abc with the arrow-heads or their equivalent is called the **triangle of displacements.** It is a vector triangle.

If the movement of the object on the table had preceded the movement of the table, the path of the object in the room would have been similar to ab, bc; if the displacement of the table had been effected first, the path would have been similar to ad, dc, where $abcd$ is a parallelogram; if the two movements had occurred simultaneously, each at a *constant rate*, beginning and ending together, the path would have been a straight line similar to ac. Any path, stepped or curved, could be effected by suitably timing and adjusting the two component displacements; but in all cases, when the component changes directed by P and Q are fully effected, the *resultant* is the same, viz. ac. In the problems which follow, we are seldom concerned with the *actual* paths.

Let A be the initial position of the object in the room; draw AC parallel and equal to ac, and similarly directed, then C is the final position of the object.

The law of the triangle is of fundamental importance; two vector quantities of any like kind are compounded by the same rule. The student should pay great attention to the rule. Observe that, for compounding P and Q, two triangles, which are similar, may be drawn; that the two



p's are similarly directed in the triangle, and also the two *q*'s; that these directions are *with* the motion of a point which travels round either triangle *against* the resultant *ac*. This last property may be expressed by saying that in either triangle the components *p* and *q* are *circuital*, and the resultant *r* is *non-circuital* (with *p* and *q*).

Note also that in the *parallelogram abcd* the three vectors *p*, *q*, *r* which meet at *a* are all directed *away* from *a*, and those that meet at *c* are all directed *towards c*.

The resultant of *P* and *Q* might therefore be found as the diagonal *r* of a parallelogram of which *p* and *q* are adjacent sides; *p*, *q* and *r* being directed *all towards*, or *all away from*, the common point where the three meet. This construction is known as the *parallelogram of displacements;* it is exactly equivalent to the triangle of displacements.

We thus have the following equivalent rules for obtaining the resultant of two given vectors:—

Rule 1.—*Place the two given vector lines end to end circuitally, and obtain the non-circuital closing side of the vector triangle.*

Rule 2.—*Obtain the diagonal of a parallelogram which has the two given vector lines as adjacent sides, the three lines being all directed away from, or towards, their common end point.*

400. The vector polygon.—Let P, Q, R, S represent any component displacements which a body receives; it is required to find the resultant displacement.

Starting from any point a, Fig. (1), draw ab, bc, cd, de respectively, equal, parallel, and similarly directed to P, Q, R, S; join the first point a to the last point e, and direct it from a to e; then ae represents the resultant.

For, by the rule of the triangle, ac is the resultant of ab and bc; ad is the resultant of ac and cd, and therefore of ab, bc, cd; and ae is the resultant of ad and de, and therefore of ab, bc, cd, de. The vector polygon $abcde$ is called the *polygon of displacements*, and ae is its *closing side.*

In drawing the polygon the components may be joined in any sequence; it will be found that in all the polygons which are thus possible the closing sides will be *vectorally equal* (*i.e.* equal, parallel, and similarly directed). Fig. (2) shows the polygon when the sequence is S, Q, R, P; the closing side of (2) is seen to give the same resultant as the closing side of (1).

Observe that in both polygons the component sides p, q, r, s are directed circuitally, and the resultant side v non-circuitally with the others.

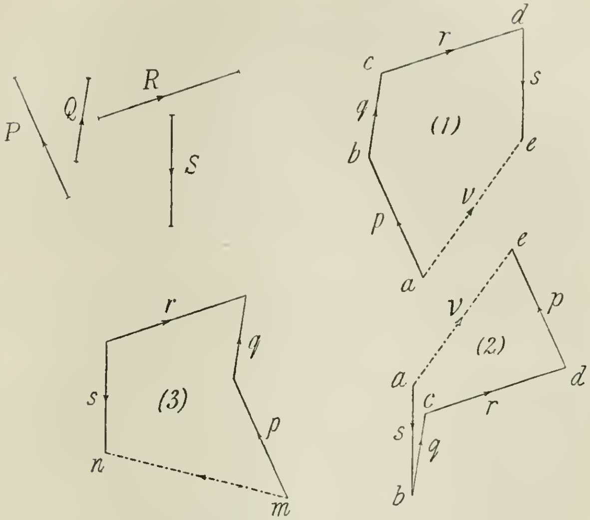
The polygon (3) has been drawn to show the student that if the component sides are not *all* circuital, the answer is incorrect; the closing side does *not* then give the resultant.

The vector polygon may be applied to find the resultant of any number of components.

If the end point e had coincided with the starting-point a the resultant would have been zero, and all the sides would have been circuital. Thus a body which receives component displacements, represented by the circuital sides of any closed polygon, has no resultant displacement.

The rule for finding the resultant of a given system of vectors may be stated as follows:—

Rule.—*Place the given vector lines end to end circuitally, and obtain the non-circuital closing side of the vector polygon.*



Examples.—**1.** A body receives component displacements of 2, 5, and 7 units in directions parallel respectively to the sides of an equilateral triangle taken circuitally. Find the resultant displacement. *Ans.* 4.3. 83.4°, 36.6° with 3 and 5.

2. Draw a quadrilateral *ABCD* having given $AB=3.7$, $BC=2.7$, $CD=5.2$, $DA=3.4$, $AC=5.4$. Suppose a body to be moved 4 feet parallel respectively to each of the directions *AB*, *BC*, *CD*, *AD*, *AC*; find the resultant movement. Show, by drawing a figure, that if the displacements had taken place in any other sequence, say the sequence *BC*, *AD*, *AB*, *AC*, and *CD*, the resultant displacement would have been the same both in magnitude and direction. *Ans.* 11.4 feet, 60° with *AB*.

3. A person walks 29 steps east, 51 steps south-south-east, and 13 steps south-west. Find how many steps respectively west and north will bring him back to his original position.
Ans. 39.3; 56.3.

4. A body is moved 10 feet parallel to a line *OX*, and 15, 20, and 25 feet each in directions making respectively to 30°, 90°, and 135° with *OX*. Find the component displacements parallel and perpendicular to *OX*, and the resultant displacement.
Ans. 5.31 feet, 45.2 feet, 45.5 feet, 83.3°.

401. Concurrent systems of forces.—All vectors have magnitude, direction, and sense. Some, like linear velocities and couples, have no particular location in space; they are completely represented by a directed segment of a line which is free to be moved parallel to itself either laterally or longitudinally or in both ways. Others, such as rotations and forces, take place along definite lines; they are represented by a directed segment of the line; the segment may have any position in the line, but may not be moved laterally out of the line.

In the problems which follow, the vectors will be confined to forces. To specify a force we must define its magnitude, direction, sense, and any one point in its line of action.

When there are several forces acting on a body, we speak of them collectively as a *system* of forces. If the lines of action of all the forces *pass through a common point*, we have a *concurrent* system. In the first instance we shall direct attention to such systems.

The general propositions on unlocalised vectors already given apply to forces, with the restriction that forces are vectors localised in lines.

Thus to compound a system of concurrent forces, we first construct a *polygon of forces*, that is a vector polygon drawn as if the forces were unlocalised. The resultant is then represented by a line drawn through the common point of the system, and equal, parallel, and similarly directed to the non-circuital closing side of the force polygon. If the force polygon close, *i.e.* if the first and last points coincide, the resultant is zero. In this case the system is in *equilibrium*, the components *balancing* one another.

A force which being added to a system of forces produces equilibrium is called the *equilibrant* of the system. It is equal in magnitude to the resultant, and acts in the opposite direction along the same line.

We now state some theorems relating to concurrent forces.

Theorem 1.—*The resultant of a concurrent system of forces is represented by a line through the common point, equal, parallel, and similarly directed to the non-circuital closing side of a force polygon.*

Theorem 2.—*It is necessary and sufficient for the equilibrium of a concurrent system of forces that a force polygon shall close.*

Theorem 3.—*Two forces which are equal in magnitude and which act in opposite directions along the same line balance one another.*

Theorem 4.—*Two forces which are in equilibrium must be equal in magnitude, and must act in opposite directions along the same line.*

Theorem 5.—*The resultant of two forces passes through their intersection, and its magnitude, direction, and sense may be obtained by constructing the triangle or the parallelogram of forces as described in Rules* 1 *and* 2, *Art.* 399.

Theorem 6.—*Three concurrent forces will be in equilibrium if their magnitudes and senses can be represented by the circuital sides of a triangle of forces.*

Theorem 7.—*In order that three forces may be in equilibrium they must be concurrent (unless they are parallel), and their magnitudes and senses must be represented by the circuital sides of the triangle of forces.*

Theorem 8.—*If forces which balance one another be added to or subtracted from any system of forces, the resultant of the system is unaltered.*

Theorem 9.—*If a system of forces is in equilibrium, any force reversed in sense is the resultant of the others.*

Theorem 10.—*The algebraical sum of the components of a system of forces in any direction is equal to the component of the resultant in the same direction.*

These theorems are collected for easy reference, but the beginner will only gradually come to understand them as he applies them to problems. He should consult them from time to time as he works the succeeding examples of this chapter.

402. Problem.—**Forces of 5 and 2 units act respectively to the right and upwards along the given lines OM, ON which include an angle of 70°. Required the resultant.**

First Method.—By the parallelogram of forces.

Let *O* be the intersection of the given lines.

To any convenient scale mark off from *O*, to the right along *OM*, and upwards along *ON*, the lengths *OP*, *OQ* of 5 and 2 units respectively.

Complete the parallelogram by drawing *PR*, *QR* parallel to *OQ*, *OP*. Draw the diagonal *OR*, and direct it from *O* to *R*.

Then *OR* represents the required resultant. It measures 6 units and makes 18.5° with *OM* and 51.5° with *ON*.

Second Method.—By the triangle of forces.

Draw any line *ab* parallel to *OM*, directed to the right and 5 units long. From *b* draw *bc* parallel to *ON*, directed upwards, and 2 units long. Join *ac*, and direct it from *a* to *c*, *i.e.* non-circuitally.

From *O* draw *OR vectorally equal* to *ac*, that is parallel, equal, and similarly directed. Then *OR* represents the resultant as before.

403. Problem.—**A given force is represented by OR. Required its components along the given lines OM and ON.**

Draw *RQ* and *RP* parallel to *MO* and *NO*. Direct *OP* and *OQ* away from *O*, like *OR* is directed.

Then *OP* and *OQ* represent the required components.

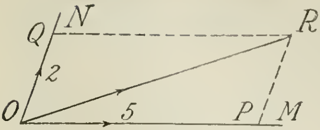
404. Problem.—**A given force of 34 lbs. acts along OG. Required its component parallel to LL, which makes 35° with OG.**

Rectangular component is understood.

Select a convenient scale, say $\frac{1}{4}''$ to 10 lbs., and mark off *OG* 34 units long. Direct it from *O* to *G*.

Draw *OH*, *GK* parallel to *LL*, and *OK*, *GH* perpendicular to *LL*. Direct *OH* and *OK* like *OG*, away from *O*.

Then *OH* represents the required component parallel to *LL*, and measures 27.9 lbs. The perpendicular component *OK* measures 19.5 lbs.



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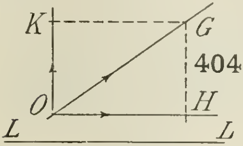

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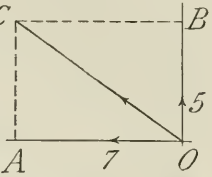

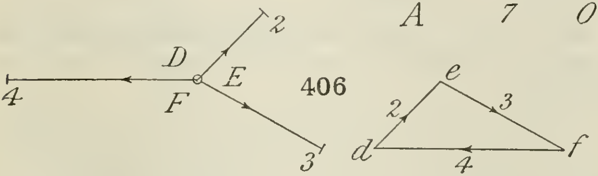

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405. Problem.—**The rectangular components of a force are 7 and 5 lbs. Find the force.**

Draw any two perpendicular lines, along which, to a convenient scale, mark off from their intersection *O*, *OA* and *OB* to represent 7 and 5 lbs.

Complete the rectangle contained by *OA*, *OB*, and draw the diagonal *OC*. Then *OC* represents the required force; it measures 8.60 lbs. and makes 35.5° with *OA* and 54.5° with *OB*.

406. Problem.—**A small ring is at rest under pulls of 2, 3, and 4 lbs. exerted through strings. Find the angles between the strings.**

Draw a triangle of forces *def* with its sides 2, 3, and 4 units long and circuitally directed.

From the centre of the ring draw lines *vectorally equal* to the sides of the triangle.

These represent the three pulls, and the angles *D*, *E*, *F* between them are found to measure 133.4°, 75.6°, and 151°.

407. Problem. — **A load of 2 tons is suspended from a pin A, which is maintained in position by the pull of a horizontal tie AL, and the thrust of a strut MA inclined at 25°. Find the pull and thrust.**

Draw *ab* 2 units long and directed vertically downwards to represent the given load. From the ends *a* and *b* draw two lines parallel to the tie and strut to intersect in *c*, and direct the sides *bc*, *ca* circuitally with *ab*.

Then *abc* is the triangle of forces representing the equilibrium of the pin *A*. The pull of the tie is found by measuring *ca* to be 4.28 tons. And the thrust of the strut *MA* is 4.72 tons, as given by *bc*.

408. Problem.—**Four given pulls and thrusts K, L, M, N act on a point as defined in the figure. Find their equilibrant.**

Construct a vector polygon thus: Select a convenient scale (say $\frac{1}{4}''$ to 10 units), and from any point *a* draw *ab*, *bc*, *cd*, *de* respectively parallel and similarly directed to *K*, *L*, *M*, *N*, and 72, 40, 105, and 58 units long.

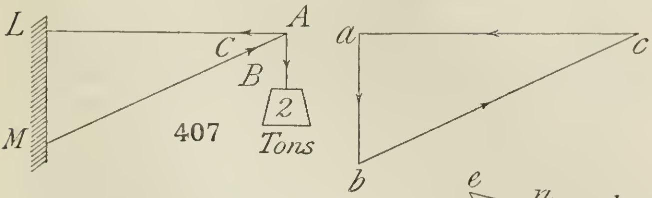
Join *ea* and direct it circuitally, *i.e.* in from *e* to *a*. A line *R* drawn through the common point vectorally equal to *ea* represents the required balancing force.

Measuring *ea*, the magnitude of *R* is found to be 88.5 units. The angles *E* and *A* measure 45° and 121°.

Observe the two systems of notation. In one, the forces are denoted by *K*, *L*, *M*, *N*, *R*, and the corresponding sides of the force polygon by *k*, *l*, *m*, *n*, *r*.

In the other, known as **Bow's notation**, the *angles* round the point are lettered *A*, *B*, *C*, *D*, *E*, and the corresponding *corners* of the force polygon *a*, *b*, *c*, *d*, *e*. In this system a force is referred to by naming the two letters in the angular spaces on each side of it. Thus the force "*BC*" (or *CB*) would mean the force "*L*," and the corresponding letters *b*, *c* would appear at the ends of the side *l*.

On going round the point (clockwise), the sequence *A*, *B*, *C*, *D*, *E* of the angles is seen to be the same as the circuital sequence *a*, *b*, *c*, *d*, *e* of the corners of the vector polygon. If we agree that the two letters which denote any force shall be named in the sequence derived from the rotation, *e.g.* *BC* for *L*, then the same sequence, *bc*, agrees with the circuital arrow. This convention will be used later.



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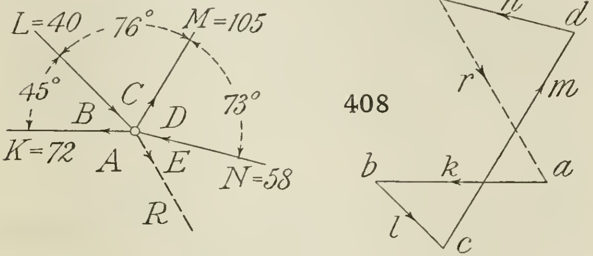

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Examples.—1. Find the force whose horizontal and vertical components are 72.2 lbs. and 45.6 lbs. *Ans.* 85.3 lbs. inclined at 32.3°.

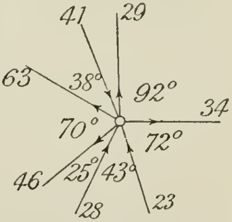
2. A force of 100 lbs. acts at an angle of 53.7° with the horizontal: find its horizontal and vertical components. *Ans.* 59.2 lbs.; 80.6 lbs.

3. Two forces of 3.5 and 6.7 lbs. act in directions which include an angle of 75°; find their resultant. *Ans.* 8.32 lbs., making 24° with the larger force.

4. The lines of action of two forces include an angle of 120°; one of the forces is 10 lbs., and their resultant is 9 lbs. Find the other force. *Ans.* 7.4 or 2.62 lbs.

5. A horizontal string 8 feet long is attached to two walls, and from a point 3 feet from one end a load is hung which is increased until at 10 lbs. the string breaks, the deflection when this happens being 6″. If a piece of the same string were to hang vertically, what load would it carry? *Ans.* 37.9 lbs.

6. Determine the resultant of the concurrent system of forces defined by the figure. *Ans.* A pull of 54.4 making 20.4° with the 41 force, and 17.6° with the 63 force.



409. Problem.—**Forces which act at a point are in equilibrium; all except two, M and N, are known completely, but only the lines of action of these latter are given. To find the magnitudes and senses of M and N.**

The problem is solved by applying Theorem 2, Art. 401.

We shall again illustrate Bow's notation, and to be able to apply this the figure has been first prepared by drawing the lines of action of the forces each with one end at the common point, so that there shall be the same number of angular spaces as there are forces. It has also been arranged that M and N shall be adjacent so as to bound one of the spaces; this is always possible, since a force may be represented either as a *pull* from one side or as an equal *push* from the other.

The spaces are then labelled A, B, C, D, X. Clockwise rotation being adopted, the construction may be described as follows:

Draw ab, bc, cd the three sides of the force polygon corresponding to the three given forces AB, BC, CD.

Close the polygon by drawing the two remaining sides dx, ax parallel to DX, AX, intersecting in x.

The required forces M and N are then given by the sides dx, xa taken circuitally, or as determined by the clockwise sequence DX, XA. M is seen to be a pull, and N a thrust.

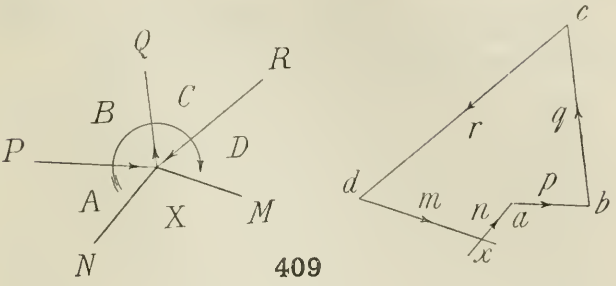
410. Problem.—**A load W is suspended from a jib crane; to find the thrust in the jib CA, and the pull in the tie AB.**

First consider the equilibrium of the tie. At the end A it is in contact with the pin, and between the two surfaces there is a force action consisting of equal and opposite forces, one on the pin, the other on the bar. If the joint is supposed frictionless, the line of action of these forces must pass through the centre of the pin.

A similar force occurs at the other end B of the tie. Neglecting its weight, these are the only two forces acting on the bar, since it is nowhere else in contact with anything. And as the two forces balance they must be equal and opposite. The bar must therefore be subject to a *direct pull* or a *direct thrust*. The tie is shown separately under forces T, T indicating a pull.

So for the jib. And generally, *in a hinged frame, loaded at the joints only, the forces at the ends of any bar must be equal and opposite, and must act along the line joining the centres of the pins.*

To obtain these forces, and to distinguish pulls from thrusts, we must draw the closed polygons of forces for the pins which hold the frame together at the joints.



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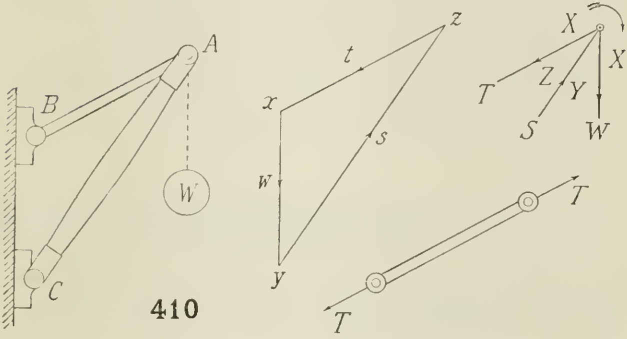

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Consider the forces on the pin A. There is the vertical force W, being the load suspended from it. There is the force which the tie exerts on it acting in the line of the tie. And there is the force exerted by the jib along the line of the jib. The pin is shown separately with the lines of action of these three forces, the angular spaces round the pin being labelled X, Y, Z.

To draw the triangle of forces for the pin A. Adopting clockwise rotation, the sequence of letters for W, the known force, is XY. So set off xy vertically downwards to represent W. Draw xz, yz parallel to XZ, YZ. By watch-hand rotation about the pin obtain the sequences YZ, ZX for jib and tie, and direct those lines near the pin A to agree with the sequences yz, zx of the triangle of forces.

YZ or S is directed *towards* the pin A, indicating a *thrust* in the jib; and ZX or T is directed *away* from the pin, denoting a *pull* in the tie. The magnitudes are obtained by measuring yz and zx.

411. Problem.—A hinged triangular frame is in equilibrium under given forces P, Q, R applied at its corners. To find the forces in the three bars of the frame.

Since the forces *P*, *Q*, *R* maintain equilibrium, their lines must all pass through one point (not shown), and their magnitudes and senses must be given by the circuital sides of the triangle of forces *abc*.

To employ Bow's notation, the lines of the applied forces *P*, *Q*, *R* are all drawn *outside* the frame, and with their ends at the joints. The external and internal spaces are then lettered *A*, *B*, *C*, *D*.

To obtain the forces in the bars, we may draw the triangles of forces for the three joint pins. Clockwise rotation is adopted.

The upper joint. The spaces round this joint named in watch-hand sequence are *A*, *B*, *D*. The triangle of forces named circuitally is *abd*. The forces *BD*, *DA*, directed in the senses *bd*, *da*, both act towards the joint, and indicate thrusts in the two inclined bars.

The left-hand joint. The spaces in clockwise sequence are *C*, *A*, *D*. The triangle of forces is *cad*. The forces *AD*, *DC*, directed as *ad*, *dc*, act the first towards, the second away from the joint. The horizontal bar of the frame is thus in tension.

The right-hand joint. The spaces are *B*, *C*, *D*, and the triangle of forces is *bcd*.

The force diagram. The four triangles of forces may be superposed on one another so as to form the figure *abcd*. The six lines of the latter measured to scale give the magnitudes of *P*, *Q*, *R* and of the forces in the bars. It is called the *force diagram* for the frame. The arrow-heads would conflict and are omitted, but the sense of any force may be obtained by the convention of Bow's notation, rotation about the joints being always clockwise.

412. Problem.—A string has its ends fixed, and four forces P, Q, R, S act on it along the given lines. The form taken by the string is shown in the figure. If P is 3 lbs., find the other forces Q, R, S, and the tensions in the five segments of the string.

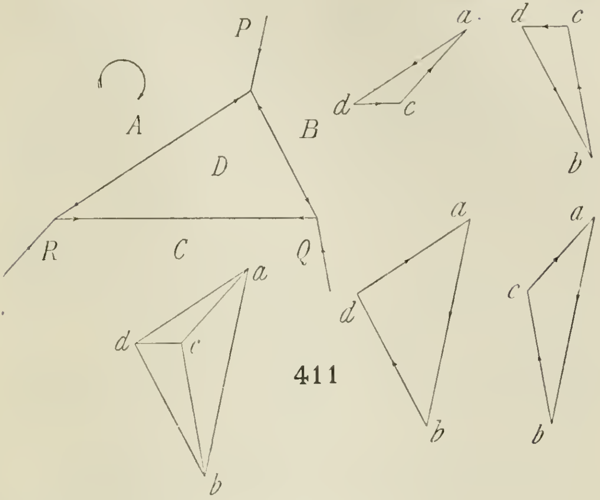
Bow's notation can be applied. The spaces are lettered *A*, *B*, *C*, *D*, *E*, *O*. We shall again adopt clockwise rotation.

Draw *ab* 3 units long to represent *P* or *AB*. And draw *ao*, *bo* parallel to *AO*, *BO* to intersect in *o*. Then *abo* is the triangle of forces for the joint *ABO*.

Draw *bc*, *oc* parallel to *BC*, *OC*, to intersect in *c*.

Draw *cd*, *od* parallel to *CD*, *OD*; and *de*, *oe* parallel to *DE*, *OE*.

Then *oabcde* is the force diagram. The forces *P*, *Q*, *R*, *S* can be found by measuring *ab*, *bc*, *cd*, *de* to a scale of $\frac{1}{8}''$ to 1 lb.; and the ten-



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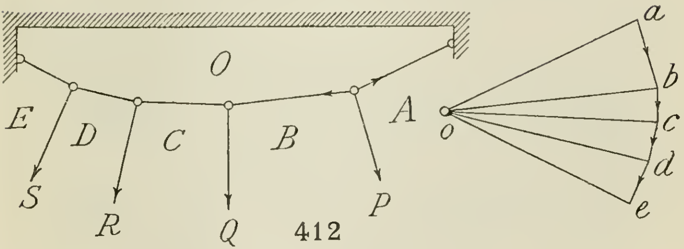

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sions in the segments A, B, C, D, E of the string by measuring oa, ob, oc, od, oe.

Definition.— The line of the string is called a *funicular polygon* or *link polygon.* The point o is its *pole.* The several lengths of the string may be supposed to be replaced by weightless rigid links, connected by hinged joints.

An indefinite number of such link polygons having o for their pole, or having new poles, could be found, all in equilibrium under the action of the same forces P, Q, R, S; in some of these the links might be in compression.

The link polygon plays an important part in graphic statics, as the succeeding problem will show.

413. Moments of forces. Couples. — The constructions hitherto given are not sufficient in themselves to solve completely the general problem on *non-concurrent* forces in one plane. It may be shown that the magnitude, direction, and sense of the resultant of such a system is given, exactly as before, by the non-circuital closing side of the vector force polygon; but we still require to find the line in which the resultant is located; this we can do by obtaining in addition to the above any one point in its line of action.

Forces produce, or tend to produce, rotations in bodies as well as motions of simple translation, and we now require to measure such a tendency. This is done by finding the *moment* as defined in the next paragraph.

Definition 1. — *The moment of a force about any point is the product of the magnitude of the force and the perpendicular distance from the point to the line of the force.*

Thus in Fig. (a) the moment of P about $K = P \times KM$, where P is to be measured on the force scale, and KM on the linear scale.

It is seen that P tends to cause *watch-hand* rotation about K. This we may agree to consider as *positive*: the opposite tendency is then *negative*.

Definition 2.—*A pair of forces equal in magnitude, opposed in sense, and acting in parallel lines, is called a* **couple**.

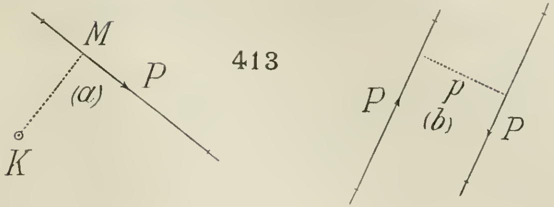
Definition 3. — *The moment of a couple is the moment of either force about any point in the line of the other.*

Thus in Fig. (b) the forces P, P constitute a couple. The moment is $P \times p$ and is positive because the tendency is to rotate clockwise.

Theorem.—In any system of forces the algebraical sum of the moments of the components about any point is equal to the moment of the resultant about the point.

This important theorem is proved in books on mechanics. The resultant moment may be called the moment of the system.

For example, two downward vertical forces, 8 ft. apart, each of 10 lbs., have moments of 60 and -20 ft.-lbs. about an intermediate point K, distant 6 ft. and 2 ft. from their lines. The algebraical sum is $60 - 20 = 40$ ft.-lbs. Now the resultant is 20 lbs. acting 2 ft. from K, and its moment is $20 \times 2 = 40$ ft.-lbs., the same as before.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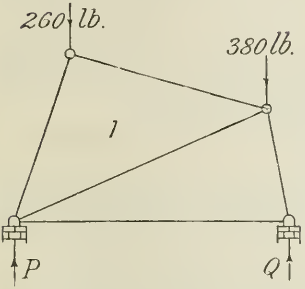
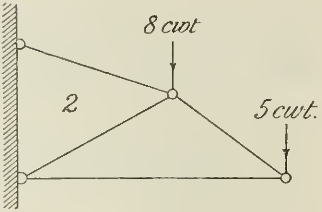




Examples on Problems 409 to 413.

***1.** Copy the jointed frame double size with the supports placed at the same level. The loads at the two upper joints, and the supporting forces P and Q, are all vertical. Draw the force diagram for the frame to a scale of $\frac{1}{2}''$ to 100 lbs., and measure P and Q. *Ans.* 235 lbs., 405 lbs.

***2.** Copy the braced cantilever double size. Then draw the force diagram to a scale of $\frac{1}{4}''$ to 1 cwt., and measure the forces in the four bars of the frame, distinguishing pulls from thrusts.
Ans. Pulls: 8.37, 20.1; thrusts: 6.68, 14.2 cwt.

3. Draw a semicircle of $2\frac{1}{4}''$ radius with the four equal chords. These chords and the diameter form a link polygon. Five forces perpendicular to the diameter act at the joints and maintain equilibrium, causing a thrust of 10 lbs. in the long link. Draw the force diagram, and determine the forces at the joints and the tensions in the four short links.

4. In Figs. (*a*) and (*b*) above suppose the linear scale to be $\frac{1}{4}''$ to $10''$, and the force scale $\frac{1}{2}''$ to 100 lbs., measure the distances KM and p, and the forces P, P, and calculate the moment of P about K in (*a*), and the moment of the couple in (*b*).
Ans. 4800 lb.-ft.; 3880 lb.-ft.

414. PROBLEM.—**To find the moment about K of the given force P of 7 lbs., the linear scale for the figure being $\frac{1}{4}''$ to 10″.**

Adopting Bow's notation, let the force be known as AB.

Selecting a suitable force scale ($\frac{1}{8}''$ to 1 lb. in the figure) draw ab 7 units long to represent the given force AB.

Select any point o as pole, at a distance os from ab representing unit force, or a force expressed by a simple number. In the figure os measures 5 lbs. on the force scale.

Join oa, ob. Through any point p on AB draw the two lines, or links, parallel to oa, ob and of indefinite length. Those links may be known as links A and B.

Through K draw the line parallel to ab, to intersect the two links A and B in A_0, B_0.

Then $A_0B_0 \times os$ gives the moment of AB about K, these lines being measured by the two scales, one as a length, the other as a force.

Proof.—Since the triangles pA_0B_0, oab are similar, $\frac{A_0B_0}{pm} = \frac{ab}{os}$; or $A_0B_0 \times os = ab \times pm = P \times KM =$ moment of P about K.

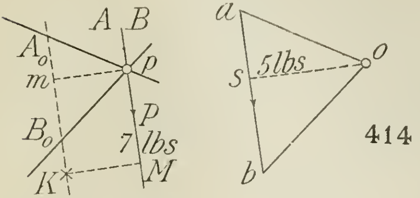
Moment scale.—A moment scale may be found on which A_0B_0 can be directly measured. Thus if A_0B_0 were $\frac{1}{4}''$, it would measure 10″ on the linear scale, and would represent a moment of $10'' \times os$ lbs., that is $10'' \times 5$ lbs. or 50 inch-lbs. The moment scale is therefore $\frac{1}{4}''$ to $10'' \times os$ lbs., that is $\frac{1}{4}''$ to 50 inch-lbs.; or more conveniently, $\frac{1}{2}''$ to 100 inch-lbs.

Measuring A_0B_0 on this scale, the required moment is found to be 110 inch-lbs.

Note 1.—If K were any other point, the new intercept A_0B_0, measured on the moment scale, would give the moment of P about the new K. The figure is thus a *diagram of moments*.

Note 2.—The diagram is a link polygon for the force P with respect to the pole o. See Definition, Prob. 412.

If the links were secured each at a point, and hinged together at p, they would form two bars of a frame, for which oab would be the force diagram. The forces in the bars due to the load P at the joint p would be found by measuring oa, ob on the force scale.



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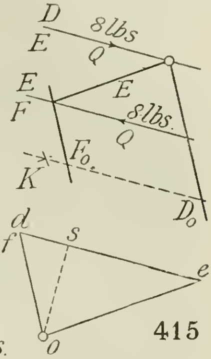
Linear Scale		¼" to 10"
Force "		⅛" to 1 lb.
Moment "		¼" to 10"× os lbs.
	i.e	¼" to 10"× 5 lbs.
	or	½" to 100 inch-lbs.



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415. Problem.—**To find the moment about K of the given couple 8 lbs., 8 lbs., the linear scale being given.**

The linear and force scales of Prob. 414 are used.

Draw the two sides *de*, *ef* of the force polygon, to represent the given forces *Q*, *Q*, or *DE*, *EF* after Bow.

Take the pole *o* so that *os* is $\frac{1}{2}''$, representing 4 lbs., thus giving a moment scale of $\frac{1}{16}''$ to 10 inch-lbs.

Join *oe*, *od*, *of*. Draw any link *E* parallel to *oe*, and where this link meets the lines *DE*, *EF*, draw the two links *D*, *F* of indefinite length, parallel to *od*, *of*.

Draw through *K* the line parallel to *de* to cut the two links *D* and *F* in D_0, F_0.

Then D_0F_0 gives the moment of the couple about *K*. Measured on the scale of $\frac{1}{16}''$ to 10 inch-lbs., the moment is seen to be 117 inch-lbs.

Note 1.—As before, the link polygon is a diagram of moments. The links *D* and *F* are parallel, and the intercept D_0F_0 is of constant length for all positions of *K*, illustrating the well-known theorem that a couple exerts the same turning moment about any point.

Note 2.—Owing to the limited space the figures in the book are necessarily small. But the student, with a sheet of drawing-paper at his disposal, should, where possible, select scales which give diagrams of ample size.

416. PROBLEM. **To find the resultant of a given non-concurrent system of forces in one plane.**

Let P, Q, M, N or AB, BC, CD, DE be the forces.

First, to find the magnitude, direction, and sense of the resultant. Draw the four circuital sides ab, bc, cd, de of the force polygon to correspond with the given forces. Join ae and direct it non-circuitally. Then the vector ae, when localised, will represent the resultant.

Next, to find a point in the line of the resultant. Choose any pole o, and join oa, ob, oc, od, oe.

Start from any point p on AB. Through p draw the link pr or A of indefinite length parallel to oa; draw also the link pq or B between AB, BC, and parallel to ob. Draw the link qm or C, between BC, CD, and parallel to oc. Draw the link mn or D, between CD, DE, and parallel to od. And draw the *closing* link nr or E parallel to oe to meet the first link A in r.

Then r is the required point. A line through r, vectorally equal to ae, represents the resultant completely.

Reason.—If the link polygon were a hinged frame with the forces P, Q, M, N, and R *reversed in sense*, acting at the joints, it will be found on examination that $oabcdeo$ would be its force diagram, and that each of the joints, and therefore the whole frame, would be in equilibrium.

Note 1.—*Rule for drawing the links.* For a balanced system of forces, or for a system to which the resultant has been added, observe that in Bow's modified notation there are the same number of letters as forces, and that each letter is associated with two, and only two, forces. And in any complete or *closed* link polygon the links A, B, . . . are parallel to oa, ob, . . . and have their ends on the pairs of forces which have A, B . . . common.

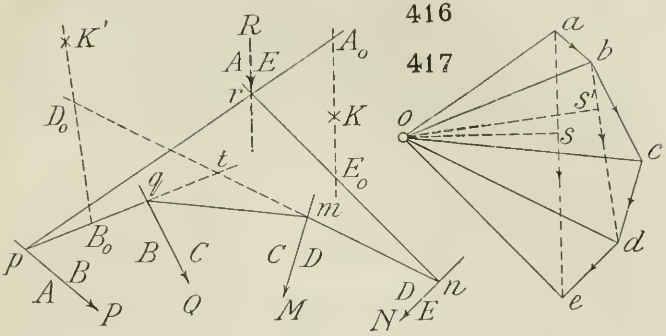
For example, *the link B is parallel to ob and terminated by AB, BC* which have B common. Attention to this rule will prevent mistakes.

417. **Properties of the link polygon. Conditions of equilibrium of coplanar forces.**

(*a*) *Partial resultant.* The resultant of any portion of the system for which the vectors are consecutive in the force polygon can readily be found.

Thus the resultant of BC, CD is represented by a vector equal to bd, localised through the point t where the links B, D intersect.

Observe that B, D are the first and last letters in the sequence BC, CD.


416

417

(*b*) *Moment of the system about any point K.* Through K draw a line parallel to *ae* to meet the links A, E in A_0, E_0. Draw *os* perpendicular to *ae*.

Then the required moment $= A_0E_0 \times os$, the scales being determined as in Prob. 414.

(*c*) *Partial moment.* The moment of any part of the system for which the vectors are added in sequence as in (*a*) is easily found.

Thus to obtain the moment of BC, CD about K', draw a line through K' parallel to *bd*, and draw *os'* perpendicular to *bd*.

Then the required moment $= B_0D_0 \times os'$.

(*d*) *Parallel forces.* If the forces are all parallel the vector polygon becomes a line; the lengths of *os*, *os'* become equal; and the lines through K, K' are parallel to the system.

Thus for parallel forces the link polygon is a very useful diagram of moments, the scale being the same for all the intercepts.

(*e*) *Conditions of equilibrium of a system of forces in one plane.*

The necessary and sufficient conditions are two, viz.

1. **The force polygon must close.**
2. **The link polygon must close.**

The first ensures that there shall be no resultant force, though there might be a couple.

The second ensures that there shall be no couple, since when this condition holds, the moment of the system about any point is zero.

By the closing of the link polygon is meant that the last link shall intersect the first link on the first force.

We shall now give some simple problems illustrating the application of the above general principles to special cases.

418. Problem.—**Three parallel forces L, M, N of given magnitudes act as shown; (a) find the parallel forces X and Y along the given lines which will balance them; (b) find the resultant of L, M, N; (c) find the moment of the given forces about K; (d) find the moment of Y and L about K.**

The scales are given in the figure, the moment scale being derived as explained in Prob. 414.

Employing Bow's modified notation, the letters *AB*, *BC*, *CD*, *DE*, *EA*, forming a cycle, are appended to the five balanced forces *LMNXY*.

(*a*) *To find X and Y*. Draw the three sides *ab*, *bc*, *cd* of the force polygon to correspond with the known forces *AB*, *BC*, *CD*.

Select a pole *o*, and join *oa*, *ob*, *oc*, *od*.

Draw the links *yl*, *lm*, *mn*, *nx* (in Bow's notation named *A*, *B*, *C*, *D*) parallel to *oa*, *ob*, *oc*, *od*.

Draw *xy* (*i.e.* the *closing link E*). And draw *oe* parallel to *xy*, thus determining the *closing sides de*, *ea* of the force polygon.

The forces *X* and *Y*, that is *DE*, *EA*, are now found by measuring *de*, *ea* on the force scale. They are 8 lbs. and 38 lbs., in the senses given by the sequences *de*, *ea*, that is they both act upwards.

(*b*) *To find the resultant of AB*, *BC*, *CD*. The first and last letters of this sequence are *A* and *D*.

The resultant acts through *r* where the links *A* and *D* intersect. And it is found by measuring *ad* to be a *downward* force of 46 lbs.

(*c*) *To find the moment of AB*, *BC*, *CD about K*. Again observe that the first and last letters of this sequence are *A* and *D*.

Draw through *K* a line parallel to *ad*, to meet the links *A* and *D* in A_0, D_0.

Then A_0D_0 represents the required moment, which measured on the moment scale gives the value 61 ft.-lbs.

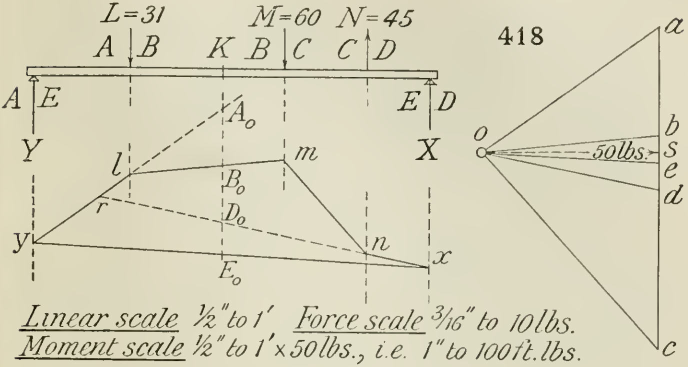
Note 1.—*os* was taken 50 lbs. on the force scale, which in conjunction with the linear scale leads to the easy moment scale of 1″ to 100 ft.-lbs. See Prob. 414.

(*d*) *To find the moment of EA*, *AB about K*. The first and last letters are *E* and *B*.

Draw through *K* a line parallel to *eb*, to cut the links *E* and *B* in E_0B_0.

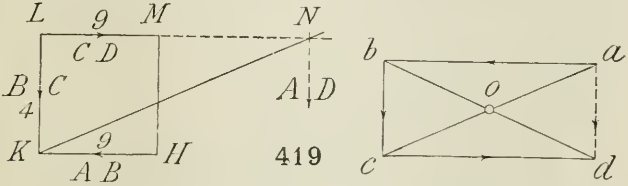
Measuring E_0B_0 on the moment scale, the required answer is 47 ft.-lbs.

Note 2.—The student of applied mechanics will recognise in Fig. 418 a diagram of bending moments for a beam supported at the ends and loaded at intermediate points. See also Fig. 420.



418

Linear scale ½" to 1′ Force scale 3/16" to 10 lbs.
Moment scale ½" to 1′ × 50 lbs., i.e. 1" to 100 ft. lbs.



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419. Problem. — **HKLM is a square of $2\frac{1}{2}''$ side. Two forces of 9 units each act from H to K, and from L to M, forming a couple. A third force of 4 units acts from L to K. Find their resultant.**

The scales for the figure are $\frac{1}{4}''$ to $1''$ and $\frac{1}{8}''$ to 1 unit of force.

The given forces are lettered *AB*, *BC*, *CD*. The required resultant will be *AD*.

Draw the three circuital sides *ab*, *bc*, *cd* of the force polygon, and the non-circuital closing side *ad*.

The pole *o* is chosen at the intersection of *ab* and *cd*.

The link polygon is begun at *K* on *AB*. The first link *A* is a line through *K* parallel to *oa*. The second link *B* is the point *K*. The third link *C* is the line *KN* parallel to *oc*. The closing link *D* is a line drawn through *N* parallel to *od* to meet the first link, which it does in *N*; so the link *D* reduces to the point *N*.

Thus the required resultant *AD* passes through *N*, a point on *LM* produced, distant by measurement 3.12" from *M*, and is a downward force of 4 units as given by *ad*.

420. A uniform horizontal rod HK 6 ft. long and weighing $1\frac{1}{2}$ lbs. is hinged at H, and a downward vertical force of $3\frac{1}{2}$ lbs. is applied 2 feet from the hinge. Where must an upward vertical force of 2 lbs. be applied to maintain equilibrium; and what will be the pressure on the hinge?

The scales for the figure are $\frac{1}{4}''$ to 1′ and $\frac{1}{6}''$ to 1 lb.

The weight of the rod may be supposed to act at its middle point.

Let *AB*, *BC*, *CD*, *DA* denote the four forces as in the figure, of which the first two are given completely, and we require to find the line of the third and the magnitude of the fourth.

Draw the three circuital sides *ab*, *bc*, *cd* of the force polygon, to represent the three known magnitudes. The non-circuital closing side *ad* gives a *downward* force of 3 lbs. for the pressure on the hinge, this being the *resultant* of the other three forces.

Take any pole *o* and join *oa*, *ob*, *oc*, *od*.

Draw the links *A*, *B* parallel to *oa*, *ob*, and the two closing links *C*, *D* parallel to *oc*, *od* to meet at *t*.

The vertical line *tL* through *t* is the required line of action of the supporting force, and *HL* measures 5.75 feet.

Examples on Problems 414 to 420.

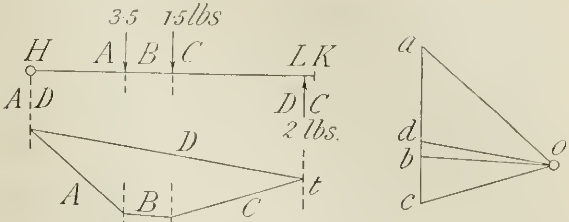
***1.** Copy the diagram double size, and let the linear scale then be 1″ to the foot. Take for the force scale $\frac{1}{2}''$ to 10 lbs., and let the polar distance *os* be 2″, representing 40 lbs.

- (*a*) Find the resultant of *M*, *P*, *Q*, *N*. *Ans.* 18 lbs. upwards 6.6 feet from *Y*.
- (*b*) Find the moment of *M* about *X*. *Ans.* – 54 lb.-ft.
- (*c*) Find the moment of the couple *P*, *Q* about *X*. *Ans.* 80.6 lb.-ft.
- (*d*) Find the moment of *M*, *P*, *Q*, *N* about *X*. *Ans.* – 82.8 lb.-ft.
- (*e*) Find the forces *X*, *Y* which balance *M*, *P*, *Q*, *N*. *Ans.* 36.9 lbs. down; 18.9 lbs. up.

***2.** The jib *HJ* of a 3-ton crane is inclined at 57° to the horizontal, and the tie rod *HI* at an angle of 27°. Find the thrust in the jib and the pull in the tie. *Ans.* 5.35 tons, 3.27 tons.

If a back stay *IK* be added inclined at 45°, and attached to the end of a horizontal strut *JK*, find the counter-balance weight *W* required at *K* to balance the load on the crane about *J*. Find also the tension in the back stay and the thrust in the crane post *IJ*. *Ans.* 2.91 tons; 4.12 tons; 1.43 tons.

Note. — The counter balance weight *W* should be found by two methods; first by drawing the force diagram for the frame; next by drawing a link polygon for the three external forces *W*, the load, and the vertical supporting force at *J*.



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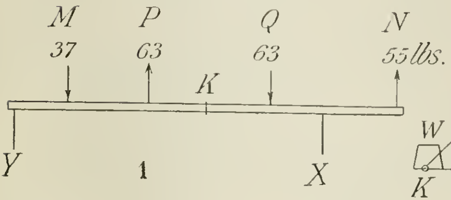

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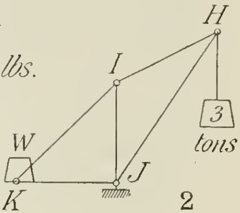

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3. On a horizontal line OX mark off towards X the lengths $OA = 0.5''$, $OB = 1.1''$, $OC = 2.0''$, $OD = 2.5''$, and above the line set off the angles $OAP = 35°$, $OBQ = 70°$, $OCM = 110°$, and $ODN = 120°$. Suppose forces of 320, 145, 570, and 416 lbs. to act respectively along PA, QB, MC, and ND. Find the resultant force, and the resultant moment about O, if the linear scale is $\frac{1}{4}$. Measure and write down (*a*) the magnitude of the resultant; (*b*) the angle which its line of action makes with OX; (*c*) the distance from O (to scale) of the point where the resultant cuts OX; (*d*) the resultant moment of the system about O.

Ans. (*a*) 1220 lbs., (*b*) 94.3°, (*c*) 7.3″, (*d*) 8860 inch-lbs.

4. Draw a rectangle $ABCD$, making $AB = 3.6''$, $BC = 1.4''$. Forces each of 6.5 lbs. act along AB and CD thus constituting a couple. Determine graphically and measure the moment of this couple. *Ans.* 9.1 inch-lbs.

5. In Ex. 4 determine graphically the forces of a couple which, acting along the short sides of the rectangle, shall balance the given couple. *Ans.* 2.53 lbs. in the senses AD and CB.

6. In Ex. 4 let two additional forces each of 3.7 lbs. act along CB and DA. Determine graphically the resultant of the four forces, and find the moment of the system about a point K inside the rectangle, distant 0.9″ from both AB and CD.

421. Centre of gravity.—Two examples are now given of the determination of the *centre of area* of a plane figure, often termed its centre of gravity.

The first case represents the cross section of a cast-iron beam, symmetrical about a vertical axis YY. The figure is divided into three rectangles, the areas of which are calculated from the data, supposed given.

Now through the centres of these rectangles draw parallel vectors AB, BC, CD in any direction other than that of YY (but at right angles to YY for convenience), and let the magnitudes of the vectors be proportional to the calculated areas. Draw the vector polygon $abcd$, and a link polygon to any pole o, and thus determine the resultant vector AD, passing through the intersection of the closing links.

The *centre of area* required is the point G, where the resultant vector AD intersects YY, the axis of symmetry.

In the second example, that of a section of angle iron, there being no axis of symmetry, it is necessary to draw two link polygons, the parallel vectors, AB, BC, of one having a direction differing from those, $A'B'$, $B'C'$, of the other. The intersection of the resultants AC, $A'C'$ gives G, the required centre of area of the section.

Examples.—1. In the upper figure opposite let the dimensions of the top flange be 4″ by $1\frac{1}{8}''$; of the bottom flange 9″ by $1\frac{1}{2}''$; and of the vertical web 10″ by 1″. Determine the distance of the centre of area from the top. *Ans.* 8″.

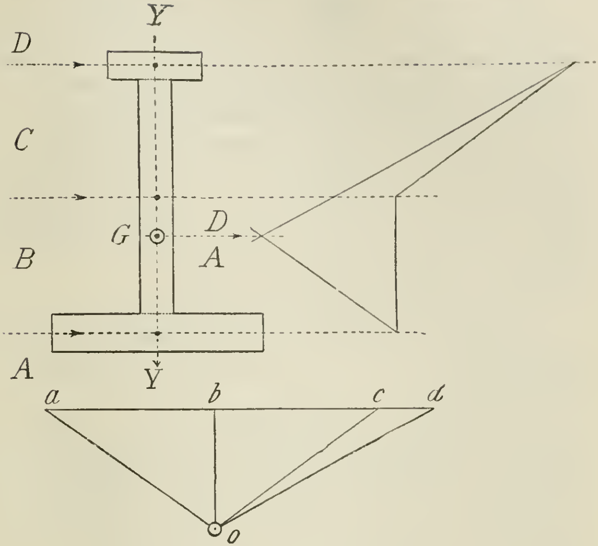
2. In the lower figure let the angle iron be 4″ by $2\frac{1}{2}''$ outside, and the thickness $\frac{1}{2}''$. Find the centre of area. *Ans.* 1.42″ and 0.67″ from the top and left sides.

3. A uniform straight wire $4\frac{1}{2}''$ long is bent at right angles at a point $1\frac{1}{2}''$ from one end; find its centre of gravity. *Ans.* $\frac{1}{4}''$ and 1″ from the long and short sides.

4. A straight wire 6″ long is bent at points distant 1.2″ and 3.2″ from end, forming three straight lengths, the first and last being parallel, and both making 60° with the middle segments. Find the centre of gravity of the wire.

5. A uniform wire 6″ long is bent into the form of a semicircle with the diameter. Find its centre of gravity.

6. Find the centre of area of the segment of a circle of 2″ radius, the chord being $3\frac{1}{2}''$ long.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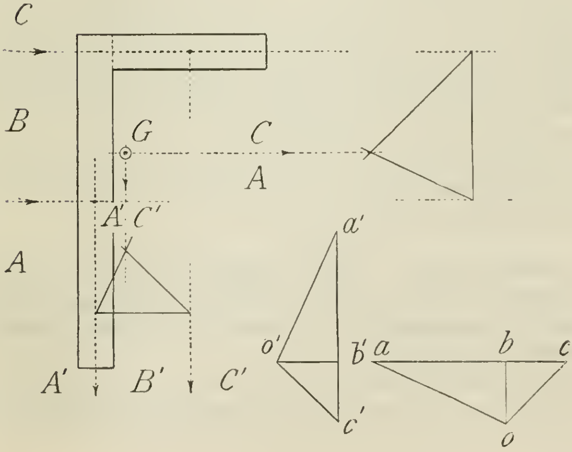
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422. Problem. — **Known forces are balanced by two others, the line of action of one of which, and a point in the line of action of the other, are given. To find the magnitude and sense of the first balancing force, and the magnitude, line of action, and sense of the other.**

Let AB, BC, CD be the known forces; and let YY be the given line of action of one of the balancing forces, which call DE, and P the given point in the line of action of the other; this latter, on Bow's system, must be labelled EA. To find the magnitude and sense of DE, and the magnitude, line of action, and sense of EA.

Draw the force polygon so far as the data will admit, viz. the sides ab, bc, cd, and an indefinite side through d parallel to DE. The solution consists in finding the point e in this line.

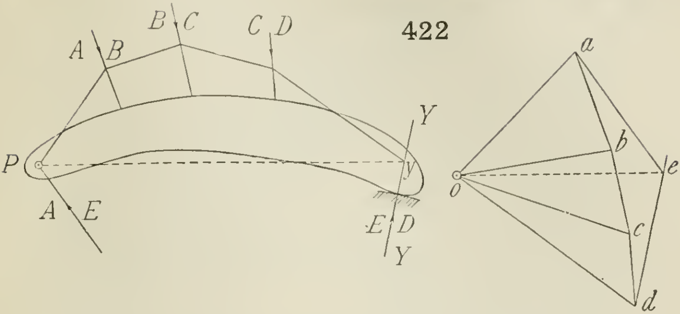
Take any pole o, and join oa, ob, oc, od. Draw the link polygon with respect to this pole, *beginning at the given point P.* Although the line of action of the force at P is unknown, yet P is a point on it, and we may therefore begin the link polygon at P. Call y the point where the fourth link D meets YY. Join Py in order to *close the link polygon.* Draw oe parallel to Py, and join ea to *close the force polygon.* Then de gives the required magnitude and sense of the balancing force DE along YY; and ea gives the magnitude, line of action, and sense of EA, the balancing force which acts through P.

This problem occurs in connection with roof trusses provided with expansion rollers at one end, and subjected to the pressure of the wind.

423. Problem. — **Any number of known forces are balanced by three others which act along given lines. To find the magnitudes and senses of the three balancing forces or reactions.**

Let AB, BC, CD be the given forces, and XX, YY, ZZ the given lines along which the balancing forces act. To find the magnitudes and senses of the latter.

422



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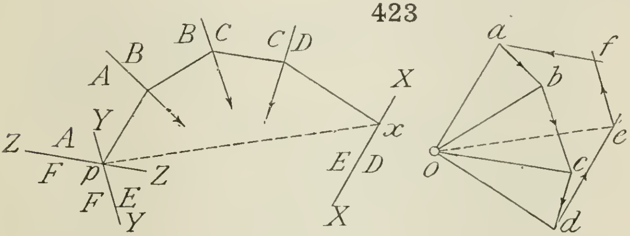


Denote the required forces by DE, EF, FA as shown.

Draw the force polygon so far as the data allows; that is draw ab, bc, cd, and the indefinite sides through d and a parallel to DE and AF. The problem is solved when we have found e and f on these two lines.

Take any pole o, and join oa, ob, oc, od. Begin the link polygon at one of the intersections of the lines X, Y, Z, say at p. The link F then reduces to a point. Draw the links A, B, C, D, terminating at x on XX.

Close the link polygon by drawing px which is the link E. Then *close the force polygon* by drawing oe parallel to px or E, and ef parallel to EF.

Then de, ef, fa taken circuitally give the required magnitudes and senses of the balancing forces DE, EF, FA acting along XX, YY, ZZ.

424. Miscellaneous Examples.

***1.** The figure shows a buttress subject to the forces of its own weight and a thrust of 2800 lbs. at its upper end. Draw the "line of resistance" for the structure.

Hint. — This is a link polygon for the forces, beginning at the point of intersection of the forces 2800 lbs. and 400 lbs. Compare Prob. 412.

***2.** A heavy uniform wire is bent into the form $ABCD$, and is suspended by a string attached to the point A. Draw the direction of the string. (1892)

***3.** The directions and magnitudes in lbs. of five unequal forces acting at a point a are given. Determine the direction and magnitude of their resultant. (1884)

***4.** Five given forces act as shown at a point. Determine by construction two forces acting along the given dotted lines which will keep the point in equilibrium. Write down the magnitudes and indicate the directions of these forces. Use a scale of $1'' = 100$ lbs. (1887)

***5.** Resolve each of the given forces P and Q ($P = 7$ lbs., $Q = 9$ lbs.) along and perpendicular to the given line AB, and write down the resultant force in each direction. (1885)

***6.** A uniform beam AB, weight 44 lbs., is suspended by two equal strings from its extremities to the point O. A weight P ($= 13$ lbs.) is hung on the beam in the given position. Determine the position of equilibrium of the system.; (1891)

***7.** A weight of 5 tons is suspended from a, the apex of a triangle formed of two bars ba, ca fixed in a vertical wall. Determine and write down the stresses in the bars ba, ca. (1893)

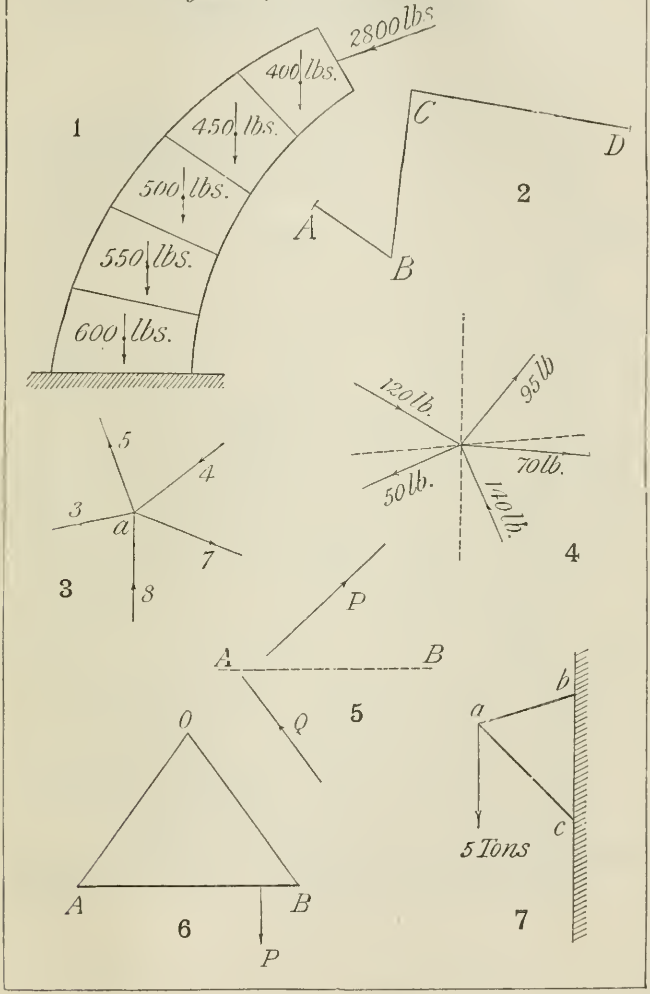
8. The wire passing round the top of a telegraph pole is horizontal, and the two directions make an angle of 110° with one another. The pole is supported by a wire stay inclined at 45° to the horizon. Given the tension of the telegraph wire to be 200 lbs., find that of the stay. *Ans.* 324 lbs. (1896)

9. $ABCD$ is a square, the angular points being lettered in order. Two forces, of 10 units each, act from A to B and from C to D, forming a couple. A third force of 15 units acts from C to A. Find their resultant. (1892)

10. Draw six lines oa, ob, oc, od, oe, of, radiating from a point o, any two adjacent lines including an angle of 60°. These six lines are respectively the lines of action of forces of 80, 100, 90, 60, 120, and 50 lbs. all acting towards o except those along oa and of, which act away from o. Determine and write down the magnitude and direction of the resultant of the forces.

Ans. A pull of 115 lbs., making 4.3° and 55.7° with oc and od.

Copy the figures double size.



***11.** Determine the line of action, and write down the magnitude of the resultant of the five given parallel forces acting in one plane in the directions shown by the arrows. (1888)

***12.** Four vertical forces act downwards as follows :—

At *A* 10 lbs., at *B* 18 lbs., at *C* 16 lbs., and at *D* 12 lbs. Determine the position of their resultant.

Supposing the two forces at *B* and *C* only to act downwards, determine the values of two vertical forces acting upwards through *A* and *D* so as to make equilibrium with those through *B* and *C*. Employ for the force polygon the scale 0.1″ to 2 lbs. (1895)

***13.** Find (and write down) the moment in foot-tons of the resultant of the pairs of parallel forces, *A* and *B*, *C* and *D*, with regard to the point *E*. Scale of forces, 0.25″ per ton ; scale of distances, 0.1″ per foot. (1894)

***14.** A force of $14\frac{1}{2}$ lbs. acts along the given line *ab*. Determine by construction what force acting along *cd* will have the same moment about *P*. (1886)

***15.** A uniform rod *AB*, weighing 53 lbs., is pivoted at *A*. If a force *P* of 32 lbs. is applied at *C*, where must a parallel force of 41 lbs. be applied to maintain equilibrium ?

***16.** Obtain by construction a line representing the moment of the resultant of the two given forces *P*, *Q*, about the point *O*, using a scale of $\frac{1}{8}'' = 1$ lb., and linear scale full size. (1888)

***17.** $P_1 = 150$ lbs., $P_2 = 200$ lbs., $P_3 = 120$ lbs., are three forces acting in the direction indicated by the arrow-heads. Find by the funicular polygon the resultant moment of the three forces acting round the point *S*.

Scale of forces, 100 lbs. = 1 inch. Scale of lengths, 50 feet = 1 inch. (1897)

18. A bar of uniform thickness inclined at an angle of 30° to the horizontal, with one end against a wall, rests across a rail at a point 2 feet away from that end. Find the length of the bar if the rail and wall are both smooth. *Ans.* 5 feet 4 inches.

Hint. — The three forces which act on the bar are (1) the force from the wall, which is in a direction at right angles to the wall, (2) the force from the rail which is at right angles to the bar, (3) the weight of the bar which acts vertically through its middle point. Now use Theorem 7, Art. 401.

19. Three forces of 11, $19\frac{1}{2}$, and 26 lbs. act at a point *P* in such directions that their resultant is nil. Draw lines representing the forces in direction and magnitude. (1886)

Copy the figures double size.

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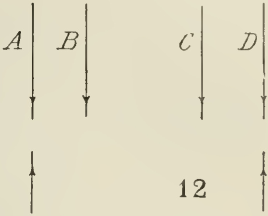

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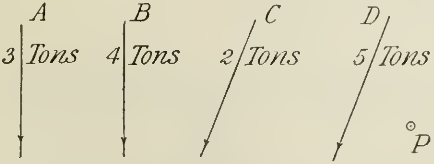

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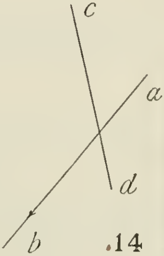

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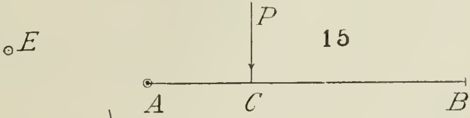

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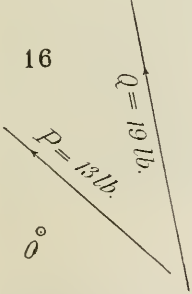

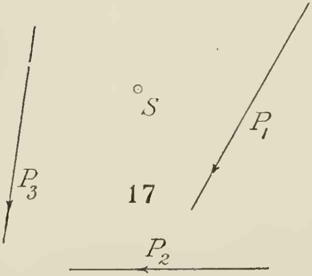

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APPENDIX I

SCIENCE AND ART DEPARTMENT EXAMINATION
May 1899

Advanced Stage

INSTRUCTIONS

Only *eight* questions are to be attempted.

Plane Geometry.

***21.** C is one vertex of a triangle; Q is the centre of the circumscribed circle; O the centre of the inscribed circle. Draw the triangle. (22)

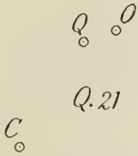
22. Draw an indefinite line BC. At any point in it, N, draw a line perpendicular to BC, and set off from N on the perpendicular, above and below BC, lengths NP, NP_2, each equal to 2.4 inches. BC is the axis of a parabola, and P, P_2 are points on the curve. Calling A the vertex of the parabola, draw the curve such that the area bounded by the double ordinate PP_2 and the portion of the curve PAP_2 shall be 4 square inches in area. (20)

***23.** Two equal elliptic wheels, A and B, are in contact at O. Their foci are: those of A, F_1, and F_2; those of B, f_1, and f_2. The wheel A is driven by the wheel B; and the wheels rotate round fixed axes at the foci F_1 and f_1 respectively. A pin is attached to A at its focus F_2. Draw a diagram representing the vertical heights of the pin above the line CD during a complete revolution of the wheels, the pin at the commencement of the revolution being at the point F_2 on the diagram. Take for abscissæ $\frac{1}{4}$ inch to represent $\frac{1}{16}$th of a complete revolution; and for ordinates the actual heights of the pin corresponding, above CD. The position should be shown for, at least, every $\frac{1}{16}$th of a complete revolution. (22)

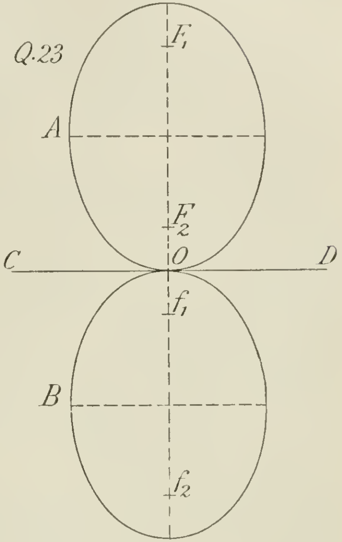
Solid Geometry.

***24.** ab, $a'b'$ are the projections of a line AB; $l'h$, hg are the traces of a plane. Draw the projections of a line in the plane, meeting AB, and making with it an angle of 35°. (24)

The diagrams (except №26) to be pricked off or accurately transferred to the paper. *(1899)*



Q.21



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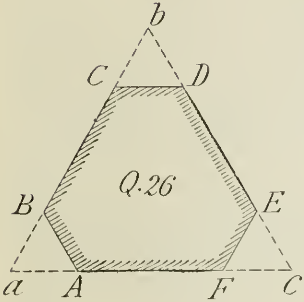

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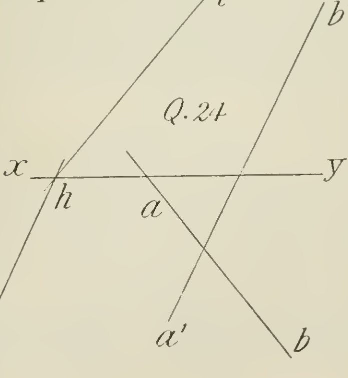


Q.24

25. A cube, inscribed in a sphere of $1\frac{1}{2}$ inches radius, has two adjacent faces inclined at 30° and 70° respectively, to the horizontal plane. Draw the solids in plan and elevation. (24)

***26.** Draw an equilateral triangle *abc* (*see diagram, which is not drawn to scale*) of 4.2 inches side, and set off lengths *aA*, *aB*, *cF*, etc., 1 inch distant from each vertex, along the sides. The figure *ABCDEF*, so formed, is a horizontal section of a regular octahedron, lying with one face on the horizontal plane. Draw the octahedron, showing on it the outline of the section in plan, and its height above the horizontal plane in elevation. (26)

***27.** *A* and *B* are the scales of slope of two planes. Draw the plan of a sphere of 1 inch radius, resting on the horizontal plane, and touching the two planes, both of which pass over the sphere. Show the points of contact with the planes, and write down their figured heights. Unit = 0.1 inch. (24)

***28.** *aa′*, *bb′*, *cc′* are the projections of three points *A*, *B*, and *C*. Find the projections of a fourth point *D*, distant $2\frac{1}{4}$ inches from each of the points *A*, *B*, and *C*. Complete the tetrahedron formed by joining the four points. (22)

29. The vertex of a cone is 1.5 inches above the horizontal plane, and its axis is inclined at 45°. Its generating lines make an angle of 30° with the axis. Determine the scale of slope of a plane tangent to the cone, and inclined at 60°. Show the line of contact on the cone in plan. Unit = 0.1 inch. (24)

***30.** The diagram represents a cone *ABV*, lying on a block *DEFG* whose thickness is DD_2. Draw on the plan the outline of shadow thrown by the solids, one on the other, and on the horizontal plane of projection. Show also on the plan the limit of light and shade on the cone. The arrows indicate the direction of the parallel rays of light, inclined at 45° to the *xy* line in plan and elevation. (30)

***31.** Draw the solids of Question 30 in isometric projection. The vertical isometric planes to be taken parallel to the planes *DE* and *DG*, and the vertical through *D* nearest to the observer. An isometric scale must be employed. (24)

Graphic Statics.

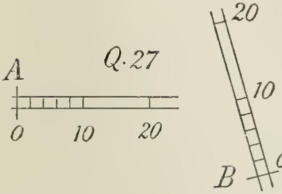
Alternative and Optional.

32. A right truncated prism has for base an equilateral triangle of 2 inches side. The three edges perpendicular to the base are respectively 2 inches, 1.75 inches, and 1.25 inches in length. Find, by graphic construction, a line representing the cubic contents of the solid, to a unit of 1 inch. (20)

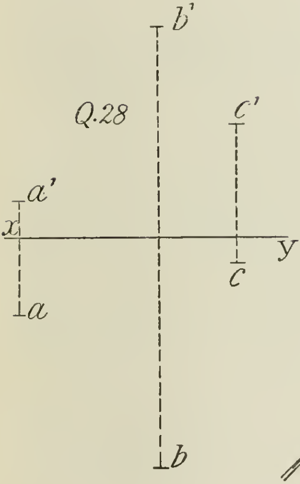
***33.** *ABCD* is a horizontal rigid bar, hinged at *A*, loaded at *B* with 40 lbs., at *C* with 50 lbs., and retained in place by a cord *DE* (passing over a pulley) attached to a pin at *D*. Find the stress in the cord *DE*. Employ the funicular polygon, using, for the scale of loads, $\frac{1}{4}$ inch to represent 10 lbs. (24)

(1899)

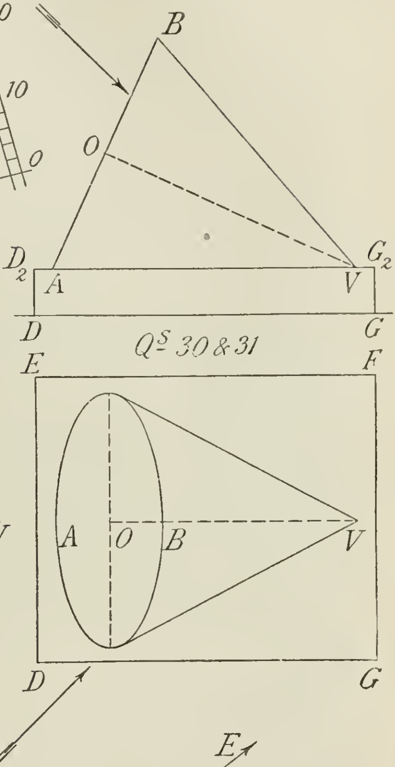
Q. 27



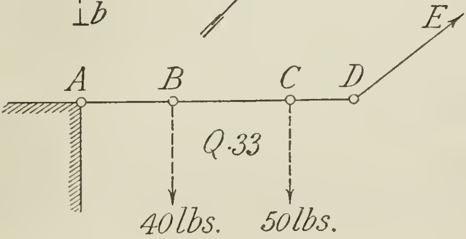
Q. 28



Qs 30 & 31



Q. 33



Note.—The figures are reproduced half size.

APPENDIX II

DEFINITIONS AND THEOREMS OF PURE SOLID GEOMETRY

Definitions

Definition 1. A *plane* is a surface such that any two points being taken in it, the straight line joining them lies wholly in the surface.

Definition 2. *Parallel planes* are such as do not meet each other, though produced.

Definition 3. A straight line is *parallel to a plane* when the two do not meet each other, though produced.

Definition 4. A straight line is *perpendicular to a plane*, when it is perpendicular to every straight line which meets it in that plane.

Definition 5. Two planes are *perpendicular to each other* when any straight line drawn in one, perpendicular to the intersection of the planes, is perpendicular to the other plane.

Definition 6. The *orthographic projection of a point* on a plane is the foot of the perpendicular from the point to the plane. The perpendicular is called a *projector*, and the plane is called the *plane of projection*.

Definition 7. The orthographic projection of a given *line* on a plane is the line generated by the foot of a perpendicular to the plane, which perpendicular moves so as always to intersect the given line.

The surface generated by the moving perpendicular is called the *projecting surface*. When the line which is projected is straight, the projecting surface is called the *projecting plane*.

Definition 8. The *inclination of a straight line to a plane* is the angle between the line and its orthographic projection on the plane.

Definition 9. The *angle between two planes*, called a *dihedral* angle, is measured by the angle between two straight lines, drawn from a point in their intersection, each perpendicular to the intersection, and lying one in each plane.

Definition 10. The inclination to each other of *two straight lines in space which do not intersect* is measured by the angle between two lines from any point, respectively parallel to those lines.

Definition 11. A *solid* or *polyhedral angle* is formed when three or more planes meet in a point. It consists of as many plane angles, and also of as many dihedral angles as there are planes.

Definition 12. A *solid* is that which has length, breadth, and thickness. It is completely bounded by a surface, or by surfaces, which may be plane or curved.

Definition 13. A *polyhedron* is a solid bounded by plane surfaces, called the *faces*, which meet in straight lines called the *edges*.

Definition 14. A *prism* is a polyhedron of which the side faces are parallelograms, and the two end faces, or *bases*, are similar and equal polygons in parallel planes. The line joining the centres of the bases is called the *axis* of the prism.

If the axis be perpendicular to the base, the prism is said to be a *right* prism ; if not, it is said to be *oblique*.

The perpendicular distance between the bases is called the *altitude*.

Definition 15. A *pyramid* is a polyhedron, one face of which, called the *base*, is a polygon, the other faces being triangles which have a common vertex.

The common vertex of the triangles is called the *vertex* of the pyramid, and the line joining the vertex to the centre of the base is called the *axis* of the pyramid. If the axis be perpendicular to the base, the pyramid is said to be a *right* pyramid ; if not, it is said to be *oblique*. The perpendicular distance from the vertex to the base is called the *altitude*.

Definition 16. A pyramid having a triangular base is called a *tetrahedron*.

Definition 17. A *polyhedron* is said to be *regular* when its faces are similar, equal, and regular polygons, and all its dihedral angles are equal to one another.

Definition 18. A *regular tetrahedron* is a solid having four equal and equilateral triangles for its faces.

Definition 19. A *cube* is a solid having six equal squares for its faces.

Definition 20. A *regular octahedron* is a solid having eight equal and equilateral triangles for its faces.

Definition 21. A *regular dodecahedron* is a solid having twelve equal and regular pentagons for its faces.

Definition 22. A *regular icosahedron* is a solid having twenty equal and equilateral triangles for its faces, and all its dihedral angles equal.

Definition 23. A *surface of revolution* is the surface generated by the rotation of a line (straight or curved), about a fixed straight line, to which it is supposed to be rigidly connected.

The fixed straight line is called the *axis*, and the rotating line the *generator*.

Definition 24. A *conical surface* is that generated by a straight line which moves so as to always pass through a fixed point, and to intersect a fixed curve in space.

The fixed point is called the *vertex*. The fixed curve is called the *directing* curve.

Definition 25. A *conical surface of revolution* is the surface generated by one of two intersecting straight lines, rotating about the other as axis, the lines being supposed rigidly connected together.

The point of intersection of the lines is called the *vertex*, and the complete surface consists of two portions extending indefinitely, one on each side of the vertex.

Definition 26. A *right circular cone* is the solid generated by the revolution of a right-angled triangle about one of the sides containing the right angle as axis. The circle generated by the other of the sides containing the right angle is called the *base* of the cone.

Definition 27. A *cylindrical surface* is that generated by a straight line which moves so as to be always parallel to a fixed straight line, and to intersect a fixed curve.

Definition 28. A *right circular cylinder* is the solid generated by the revolution of a rectangle about one side as axis.

The two sides of the rectangle which are perpendicular to the axis generate circles, each of which is called a *base* of the cylinder.

Definition 29. A *sphere* is the solid generated by the revolution of a semicircle about its diameter as axis. The *centre* of the sphere is the point which is equidistant from all points in the surface.

Theorems

Theorem 1. *The plane which contains two parallel straight lines will also contain any third straight line which intersects them.*

Theorem 2. *A plane can be found which shall contain two intersecting straight lines.*

Cor.—*A plane is determined by three intersecting straight lines AB, BC, CA not concurrent, or by three points A, B, C not collinear.*

Theorem 3. *If two planes cut each other, their intersection is a straight line.*

Theorem 4. *If a straight line be perpendicular to each of two other straight lines at their point of intersection, it will be perpendicular to the plane containing the two lines.*

Theorem 5. *If one of two parallel straight lines be perpendicular to a plane, the other line will also be perpendicular to the plane.*

Theorem 6. *If a straight line be perpendicular to a plane, every plane containing the line will be perpendicular to that plane.*

Theorem 7. *The orthographic projection of a finite straight line on a plane is the straight line joining the projections of its ends.*

Theorem 8. *If three or more straight lines which meet at a point are each perpendicular to the same straight line, they are in one plane.*

Theorem 9. *If two or more straight lines are perpendicular to the same plane, they are parallel to one another.*

Theorem 10. *If two or more straight lines are parallel to the same straight line, they are parallel to one another.*

Theorem 11. *If two straight lines which meet are respectively parallel to two others which meet, but are not in the same plane, the first two and the other two will contain equal angles. Also the plane containing the first two is parallel to the plane containing the other two.*

Theorem 12. *Planes which are perpendicular to the same straight line are parallel to one another.*

Theorem 13. *If two or more parallel planes be cut by another plane, the lines of intersection are parallel to one another.*

Theorem 14. *If two or more straight lines be cut by parallel planes, they will be cut in the same ratio.*

Theorem 15. *If two intersecting planes be each perpendicular to a third plane, their intersection will also be perpendicular to that plane.*

Theorem 16. *If two planes which meet be cut by a third plane which is perpendicular to their line of intersection, the lines in which the third plane cuts the other two will both be perpendicular to the intersection of the other two.*

Theorem 17. *If two or more straight lines be parallel to one another, their projections on any plane will also be parallel to one another.*

Theorem 18. *If a finite straight line be parallel to a plane, the lengths of the line and its projection on the plane are equal to each other. If the line be inclined to the plane, the length of the projection is less than the length of the line. If the line be perpendicular to the plane, its projection is a point.*

Theorem 19. *If a straight line be divided into two or more segments, their projections on any plane (not perpendicular to the line) are in the same ratio as the segments themselves.*

Theorem 20. *If two planes intersect, and a straight line perpendicular to one be projected on the other, the projection is perpendicular to the intersection.*

Theorem 21. *If two straight lines be perpendicular to each other, their projections on any plane parallel to one of them will also be perpendicular to each other.*

Theorem 22. *If the projections of two straight lines on any plane be perpendicular to each other, and one of the lines is parallel to the plane of projection, the two lines are perpendicular to each other.*

The Sphere and the circular Cone and Cylinder.

Theorem 23. *The projection of a sphere on any plane is a circle of diameter equal to the diameter of the sphere, the centre of the circle being the projection of the centre of the sphere.*

Theorem 24. *Any plane section of a sphere is a circle, the centre C of which is the foot of the perpendicular from the centre O of the sphere.*

Note.—If the plane contain O, the section is called a *great circle;* other sections are *small circles.*

Theorem 25. *The intersection of two spheres is a circle whose plane is perpendicular to the line joining the centres of the spheres, the centre of the circle being in this line.*

Theorem 27. *The tangent plane to a sphere, centre O, at any point P on its surface is perpendicular to OP.*

Theorem 28. *A line tangential to a sphere, centre O, at any point P on its surface is perpendicular to OP.*

Theorem 29. *If two spheres touch, the point of contact is in the line joining their centres.*

Theorem 30. *The section of a cone or cylinder by a plane perpendicular to the axis is a circle whose centre is in the axis.*

Theorem 31. *The intersection of a cone or cylinder by a sphere whose centre is in the axis consists of two circles with their planes perpendicular to the axis and their centres in the axis.*

Theorem 32. *In a cone or cylinder a sphere can be inscribed which has any given point O on the axis as centre, or which passes through any given point P on the surface. The curve of contact is a circle whose centre is in the axis and whose plane is perpendicular thereto.*

Theorem 33. *A cone of indefinite length may be defined by any two inscribed spheres, or by one such sphere and the vertex.*

Theorem 34. *A cylinder of indefinite length may be defined by any two inscribed spheres, or by one such sphere and the axis.*

Theorem 35. *The projection of a cone or cylinder of indefinite length on any plane consists of two lines which touch the projections of any two inscribed spheres.*

Note. — For the cylinder, if the plane be *perpendicular to the axis*, the projection is a *circle* which is an *edge view* of the surface. For the cone, if the projection of one inscribed sphere *fall within that of the other*, the surface has *no definite form of projection.* For illustrations of Theorem 35, see the figure on page 321.

Theorem 36. *The projection of a cone or cylinder (or of any other surface of revolution) may be determined as the envelope of the projections of its inscribed spheres.*

Theorem 37. *Any plane containing the axis of a cone or cylinder cuts the surface in a pair of generating lines.*

Theorem 38. *The tangent plane to a cone or cylinder at any point P on its surface is perpendicular to the plane which contains P and the axis. The tangent plane has line contact with the surface, this line being the generator through P. The axial plane through P cuts the surface in the line of contact.*

Theorem 39. *Any tangent plane to a cone or cylinder touches all inscribed spheres.*

Theorem 40. *Any plane which contains the vertex of a cone and which touches an inscribed sphere also touches the cone.*

Theorem 41. *Any plane which is parallel to the axis of a cylinder and which touches an inscribed sphere also touches the cylinder.*

Theorem 42. *If two cones, or two cylinders, or a cone and cylinder circumscribe the same sphere, the intersection of their surfaces is a pair of ellipses whose planes are perpendicular to the axes of the surfaces.*

This is found to be a very useful theorem.

Theorem 43. *A common tangent plane to two cylinders can in general be found (a) when their axes are parallel; (b) when they circumscribe the same sphere.*

Note.—The tangent plane may be determined as the plane which is *parallel to the axes*, and which in (*a*) *touches any two spheres inscribed one in each cylinder;* and in (*b*) *touches the common sphere.*

Theorem 44. *A common tangent plane to two cones can in general be found (a) when their axes are parallel and their vertical angles equal; (b) when they have a common vertex: and (c) when they both circumscribe the same sphere.*

Note.—The tangent plane may be determined as the plane which in (*a*) *contains the two vertices and touches any inscribed sphere;* in (*b*) *contains the common vertex and touches two spheres* inscribed one in each cone; and which in (*c*) *contains the vertices and touches the common sphere.*

Theorem 45. *A common tangent plane to a cone and cylinder can be found when they circumscribe the same sphere.*

Note 1.—The tangent plane may be determined as the plane which *contains the vertex of the cone;* is *parallel to the axis of the cylinder*, and which *touches the common sphere.*

Note 2.—In theorems 43, 44, and 45 there may be impossible cases. Moreover the conditions stated are not exhaustive.

Theorem 46. *The normal from any point to the surface of a cone or cylinder lies in the plane determined by the point and the axis, and is perpendicular to one of the generators in which this axial plane cuts the surface.*

Theorem 47. *If a sphere touch a cone or cylinder, the point of contact lies in the generator determined by the axial plane through the centre of the sphere.*

Theorem 48. *Two cones, two cylinders, or a cone and cylinder which touch may have line contact or point contact.*

If they have line contact their axes must intersect or be parallel; and the line of contact is the generator determined by the common axial plane.

If they have point contact at P, the common tangent plane at P contains the two generators through P.

Theorem 49. *The shortest path on a sphere between two points is the smaller arc of the great circle through them.*

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