Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 224 (2010) 2052–2093
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Positive solutions of nonlinear problems involving the square root of the Laplacian

Xavier Cabré [a,*], Jinggang Tan [b]

[a] *ICREA and Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain*

[b] *Departamento de Matemática, Universidad Técnica Federico Santa María, Avda. España 1680, Valparaíso, Chile*

Received 11 May 2009; accepted 12 January 2010

Available online 20 February 2010

Communicated by Luis Caffarelli

Abstract

We consider nonlinear elliptic problems involving a nonlocal operator: the square root of the Laplacian in a bounded domain with zero Dirichlet boundary conditions. For positive solutions to problems with power nonlinearities, we establish existence and regularity results, as well as a priori estimates of Gidas–Spruck type. In addition, among other results, we prove a symmetry theorem of Gidas–Ni–Nirenberg type.
© 2010 Elsevier Inc. All rights reserved.

Keywords: Fractional Laplacian; Critical exponent; Nonlinear mixed boundary problem; A priori estimates; Nonlinear Liouville theorems; Moving planes method

1. Introduction

This paper is concerned with the study of positive solutions to nonlinear problems involving a nonlocal positive operator: the square root of the Laplacian in a bounded domain with zero Dirichlet boundary conditions. We look for solutions to the nonlinear problem

$$\begin{cases} A_{1/2}u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: xavier.cabre@upc.edu (X. Cabré), jinggang.tan@usm.cl (J. Tan).

0001-8708/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.aim.2010.01.025

where Ω is a smooth bounded domain of $\mathbb{R}^n$ and $A_{1/2}$ stands for the square root of the Laplacian operator $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$.

To define $A_{1/2}$, let $\{\lambda_k, \varphi_k\}_{k=1}^{\infty}$ be the eigenvalues and corresponding eigenfunctions of the Laplacian operator $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$,

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \Omega, \end{cases}$$

normalized by $\|\varphi_k\|_{L^2(\Omega)} = 1$. The square root of the Dirichlet Laplacian, that we denote by $A_{1/2}$, is given by

$$u = \sum_{k=1}^{\infty} c_k\varphi_k \mapsto A_{1/2}u = \sum_{k=1}^{\infty} c_k\lambda_k^{1/2}\varphi_k, \tag{1.2}$$

which clearly maps $H_0^1(\Omega) = \{u = \sum_{k=1}^{\infty} c_k\varphi_k \mid \sum_{k=1}^{\infty} \lambda_k c_k^2 < \infty\}$ into $L^2(\Omega)$.

The fractions of the Laplacian, such as the previous square root $A_{1/2}$, are the infinitesimal generators of Lévy stable diffusion processes and appear in anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, and American options in finance.

Essential to the results in this paper is to realize the nonlocal operator $A_{1/2}$ in the following local manner. Given a function u defined in Ω, we consider its harmonic extension v in the cylinder $\mathcal{C} := \Omega \times (0, \infty)$, with v vanishing on the lateral boundary $\partial_L\mathcal{C} := \partial\Omega \times [0, \infty)$. Then, $A_{1/2}$ is given by the Dirichlet to Neumann map on Ω, $u \mapsto \frac{\partial v}{\partial \nu}|_{\Omega\times\{0\}}$, of such harmonic extension in the cylinder. In this way, we transform problem (1.1) to a local problem in one more dimension. By studying this problem with classical local techniques, we establish existence of positive solutions for problems with subcritical power nonlinearities, regularity and an L^∞-estimate of Brezis–Kato type for weak solutions, a priori estimates of Gidas–Spruck type, and a nonlinear Liouville type result for the square root of the Laplacian in the half-space. We also obtain a symmetry theorem of Gidas–Ni–Nirenberg type.

The analogue problem to (1.1) for the Laplacian has been investigated widely in the last decades. This is the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega; \end{cases} \tag{1.3}$$

see [24] and references therein. Considering the minimization problem $\min\{\|u\|_{H_0^1(\Omega)} \mid \|u\|_{L^{p+1}(\Omega)} = 1\}$, one obtains a positive solution of (1.3) in the case $f(u) = u^p$, $1 < p < \frac{n+2}{n-2}$, since the Sobolev embedding is compact. Ambrosetti and Rabinowitz [1] introduced the mountain pass theorem to study problem (1.3) for more general subcritical nonlinearities. Instead, for $f(u) = u^{\frac{n+2}{n-2}}$, Pohozaev identity leads to nonexistence to (1.3) if Ω is star-shaped. In contrast, Brezis and Nirenberg [4] showed that the nonexistence of solution may be reverted by adding a small linear perturbation to the critical power nonlinearity.

For the square root $A_{1/2}$ of the Laplacian, we derive the following result on existence of positive solutions to problem (1.1).

Theorem 1.1. *Let $n \geqslant 1$ be an integer and $2^\sharp = \frac{2n}{n-1}$ when $n \geqslant 2$. Suppose that Ω is a smooth bounded domain in $\mathbb{R}^n$ and $f(u) = u^p$. Assume that $1 < p < 2^\sharp - 1 = \frac{n+1}{n-1}$ if $n \geqslant 2$, or that $1 < p < \infty$ if $n = 1$.*

Then, problem (1.1) *admits at least one solution. This solution (as well as every weak solution) belongs to $C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.*

As mentioned before, we realize problem (1.1) through a local problem in one more dimension by a Dirichlet to Neumann map. This provides a variational structure to the problem, and we study its corresponding minimization problem. Here the Sobolev trace embedding comes into play, and its critical exponent $2^\sharp = \frac{2n}{n-1}$, $n \geqslant 2$, is the power appearing in Theorem 1.1. We call p critical (respectively, subcritical or supercritical) when $p = 2^\sharp - 1 = \frac{n+1}{n-1}$ (respectively, $p < 2^\sharp - 1$ or $p > 2^\sharp - 1$). In the subcritical case of Theorem 1.1, the compactness of the Sobolev trace embedding in bounded domains leads to the existence of solution. Its regularity will be consequence of further results presented later in this introduction.

Remark 1.2. In [26] the second author J. Tan establishes the nonexistence of classical solutions to (1.1) with $f(u) = u^p$ in star-shaped domains for the critical and supercritical cases. In addition, an existence result of Brezis–Nirenberg type [4] for $f(u) = u^p + \mu u$, $\mu > 0$, is also established.

Gidas and Spruck [14] established a priori estimates for positive solutions of problem (1.3) when $f(u) = u^p$ and $p < \frac{n+2}{n-2}$. Its proof involves the method of blow-up combined with two important ingredients: nonlinear Liouville type results in all space and in a half-space. The proofs of such Liouville theorems are based on the Kelvin transform and the moving planes method or the moving spheres method. Here we establish an analogue: the following a priori estimates of Gidas–Spruck type for solutions of problem (1.1).

Theorem 1.3. *Let $n \geqslant 2$ and $2^\sharp = \frac{2n}{n-1}$. Assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $f(u) = u^p$, $1 < p < 2^\sharp - 1 = \frac{n+1}{n-1}$.*

Then, there exists a constant $C(p, \Omega)$, which depends only on p and Ω, such that every weak solution of (1.1) *satisfies*

$$\|u\|_{L^\infty(\Omega)} \leqslant C(p, \Omega).$$

To prove this result, we combine the blow-up method and two useful ingredients: a nonlinear Liouville theorem for the square root of the Laplacian in all of $\mathbb{R}^n$, and a similar one in the half-space $\mathbb{R}^n_+$ with zero Dirichlet boundary values on $\partial\mathbb{R}^n_+$. The first one in the whole space was proved by Ou [22] using the moving planes method and by Y.Y. Li, M. Zhu and L. Zhang [18,17] using the moving spheres method. Its statement is the following.

Theorem 1.4. *(See [18,22,17].) For $n \geqslant 2$ and $1 < p < 2^\sharp - 1 = \frac{n+1}{n-1}$, there exists no weak solution of the problem*

$$\begin{cases} (-\Delta)^{1/2} u = u^p & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1.4}$$

As we will see later, here $(-\Delta)^{1/2}$ is the usual half-Laplacian in all of $\mathbb{R}^n$, and problem (1.4) is equivalent to problem $\Delta v = 0$ and $v > 0$ in $\mathbb{R}^{n+1}_+$, $\partial_\nu v = v^p$ on $\partial\mathbb{R}^{n+1}_+$. The corresponding Liouville theorem for the square root of the Laplacian in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$ was not available and we establish it in this paper for bounded solutions.

Theorem 1.5. *Let $n \geqslant 2$, $2^\sharp = \frac{2n}{n-1}$, and $1 < p \leqslant 2^\sharp - 1 = \frac{n+1}{n-1}$. Then, there exists no bounded solution u of*

$$\begin{cases} A_{1/2}u = u^p & \text{in } \mathbb{R}^n_+, \\ u = 0 & \text{on } \partial\mathbb{R}^n_+, \\ u > 0 & \text{in } \mathbb{R}^n_+, \end{cases} \tag{1.5}$$

where $A_{1/2}$ is the square root of the Laplacian in $\mathbb{R}^n_+ = \{x_n > 0\}$ with zero Dirichlet boundary conditions on $\partial\mathbb{R}^n_+$.

In an equivalent way, let

$$\mathbb{R}^{n+1}_{++} = \big\{z = (x_1, x_2, \ldots, x_n, y) \,\big|\, x_n > 0,\ y > 0\big\}.$$

If $n \geqslant 2$ and $1 < p \leqslant 2^\sharp - 1 = \frac{n+1}{n-1}$, then there exists no bounded solution $v \in C^2(\mathbb{R}^{n+1}_{++}) \cap C(\overline{\mathbb{R}^{n+1}_{++}})$ of

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_{++}, \\ v = 0 & \text{on } \{x_n = 0,\ y > 0\}, \\ \dfrac{\partial v}{\partial \nu} = v^p & \text{on } \{x_n > 0,\ y = 0\}, \\ v > 0 & \text{in } \mathbb{R}^{n+1}_{++}, \end{cases} \tag{1.6}$$

where ν is the unit outer normal to $\mathbb{R}^{n+1}_{++}$ at $\{x_n > 0, y = 0\}$.

The proof of this result combines the Kelvin transform, the moving planes method, and a Hamiltonian identity for the half-Laplacian found by Cabré and Solà-Morales [5]. The result of Theorem 1.5 is still open without the assumption of boundedness of the solution.

Gidas, Ni, and Nirenberg [13] established symmetry properties for solutions to problem (1.3) when f is Lipschitz continuous and Ω has certain symmetries. The proof of these symmetry results uses the maximum principle and the moving planes method. The moving planes method was introduced by Alexandroff to study a geometric problem, while in the framework of problem (1.3) was first used by Serrin. In the improved version of Berestycki and Nirenberg [3], it replaces the use of Hopf's lemma by a maximum principle in domains of small measure.

Here we proceed in a similar manner and obtain the following symmetry result of Gidas–Ni–Nirenberg type for (1.1).

Theorem 1.6. *Assume that Ω is a bounded smooth domain of $\mathbb{R}^n$ which is convex in the x_1 direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let f be Lipschitz continuous and u be a $C^2(\overline{\Omega})$ solution of* (1.1).

Then, u is symmetric with respect to x_1, i.e., $u(-x_1, x') = u(x_1, x')$ for all $(x_1, x') \in \Omega$. In addition, $\frac{\partial u}{\partial x_1} < 0$ for $x_1 > 0$.

In particular, if $\Omega = B_R(0)$ is a ball, then u is radially symmetric, $u = u(|x|) = u(r)$ for $r = |x|$, and it is decreasing, i.e., $u_r < 0$ for $0 < r < R$.

We prove this symmetry result by using the moving planes method combined with the following maximum principle for the square root $A_{1/2}$ of the Laplacian in domains of small measure (see Proposition 4.4 for a more general statement in nonsmooth domains).

Proposition 1.7. *Assume that $u \in C^2(\overline{\Omega})$ satisfies*

$$\begin{cases} A_{1/2}u + c(x)u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $c \in L^\infty(\Omega)$. Then, there exists $\delta > 0$ depending only on n and $\|c^-\|_{L^\infty(\Omega)}$, such that if $|\Omega \cap \{u < 0\}| \leqslant \delta$ then $u \geqslant 0$ in Ω.

The above maximum principle in "small" domains replaces the use of Hopf's lemma to prove symmetry results for $A_{1/2}$ in Lipschitz domains. We point out that Chipot, Chlebík, Fila, and Shafrir [9] studied the related problem:

$$\begin{cases} -\Delta v = g(v) & \text{in } B_R^+ = \{z \in \mathbb{R}^{n+1} \mid |z| \leqslant R,\ z_{n+1} > 0\}, \\ v = 0 & \text{on } \partial B_R^+ \cap \{z_{n+1} > 0\}, \\ \dfrac{\partial v}{\partial \nu} = f(v) & \text{on } \partial B_R^+ \cap \{z_{n+1} = 0\}, \\ v > 0 & \text{in } B_R^+, \end{cases} \tag{1.7}$$

where $f, g \in C^1(\mathbb{R})$ and ν is the unit outer normal. They proved existence, nonexistence, and axial symmetry results for solutions of (1.7). Following one of their proofs, we establish Hopf's lemma for $A_{1/2}$, Lemma 4.3 below. Finally, let us mention that singular solutions and extremal solutions of similar problems to (1.7) have been considered by Davila, Dupaigne, and Montenegro [10,11].

As we mentioned, crucial to our results is that $A_{1/2}$ is a nonlocal operator in Ω but which can be realized through a local problem in $\Omega \times (0, \infty)$. To explain this, let us start with the square root of the Laplacian (or half-Laplacian) in $\mathbb{R}^n$. Let u be a bounded continuous function in all of $\mathbb{R}^n$. There is a unique harmonic extension v of u in the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$. That is,

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+ = \{(x, y) \in \mathbb{R}^n \times (0, \infty)\}, \\ v = u & \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+. \end{cases}$$

Consider the operator $T : u \mapsto -\partial_y v(\cdot, 0)$. Since $\partial_y v$ is still a harmonic function, if we apply the operator T twice, we obtain

$$(T \circ T)u = \partial_{yy} v|_{y=0} = -\Delta_x v|_{y=0} = -\Delta u \quad \text{in } \mathbb{R}^n.$$

Thus, we see that the operator T mapping the Dirichlet data u to the Neumann data $-\partial_y v(\cdot, 0)$ is actually a square root of the Laplacian. Indeed it coincides with the usual half-Laplacian, see [16].

Here we introduce a new analogue extension problem in a cylinder $\mathcal{C} := \Omega \times (0, \infty)$ in one more dimension to realize (1.1) by a local problem in $\mathcal{C}$. More precisely, we look for a function v with $v(\cdot, 0) = u$ in $\mathbb{R}^n$ satisfying the following mixed boundary value problem in a half-cylinder:

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C} = \Omega \times (0, \infty), \\ v = 0 & \text{on } \partial_L \mathcal{C} := \partial\Omega \times [0, \infty), \\ \dfrac{\partial v}{\partial \nu} = f(v) & \text{on } \Omega \times \{0\}, \\ v > 0 & \text{in } \mathcal{C}, \end{cases} \tag{1.8}$$

where ν is the unit outer normal to $\mathcal{C}$ at $\Omega \times \{0\}$. If v satisfies (1.8), then the trace u on $\Omega \times \{0\}$ of v is a solution of problem (1.1). Indeed, since $\partial_y v$ is harmonic and also vanishes on the lateral boundary $\partial\Omega \times [0, \infty)$, we see as before that the Dirichlet to Neumann map $u \mapsto -\partial_y v(\cdot, 0)$ is the unique positive square root $A_{1/2}$ of the Dirichlet Laplacian in Ω.

The generators of Lévy symmetric stable diffusion processes are the fractional powers of the Laplacian $(-\Delta)^s$ in all of $\mathbb{R}^n$, $0 < s < 1$. Fractional Laplacians attract nowadays much interest in physics, biology, finance, as well as in mathematical nonlinear analysis (see [2]). One of the few nonlinear results which is not recent is due to Sugitani [25], who proved blow up results for solutions of heat equations $\partial_t u + (-\Delta)^s u = f(u)$ in $\mathbb{R}^n$, for all $0 < s < 1$. It is important to note that the fundamental solution of the fractional heat equation has power decay (or heavy) tails, in contrast with the exponential decay in case of the classical heat equation. Lévy processes have also been applied to model American options [2]. As recent nonlinear works for fractional diffusions, let us mention the following. Caffarelli and Silvestre [7] have given a new local realization of the fractional Laplacian $(-\Delta)^s$, for all $0 < s < 1$, through the Dirichlet–Neumann map of an appropriate degenerate elliptic operator. The regularity of the obstacle problem for the fractional powers of the Laplacian operator was proved by Silvestre [23]. The optimal regularity for such Signorini problem was improved in [6]. Moreover, the operator $(-\Delta)^s$ plays an important role in the study of the quasi-geostrophic equations in geophysical fluid dynamics; see the important recent paper [8] by Caffarelli and Vasseur. Cabré and Solà-Morales [5] studied layer solutions (solutions which are monotone with respect to one variable) of $(-\Delta)^{1/2} u = f(u)$ in $\mathbb{R}^n$, where f is of balanced bistable type.

To prove Theorem 1.1, in view of (1.8) being a local realization of (1.1), we consider the Sobolev space

$$H^1_{0,L}(\mathcal{C}) = \left\{ v \in H^1(\mathcal{C}) \;\middle|\; v = 0 \text{ a.e. on } \partial_L \mathcal{C} = \partial\Omega \times [0, \infty) \right\},$$

equipped with the norm $\|v\| = (\int_{\mathcal{C}} |\nabla v|^2 \, dx \, dy)^{1/2}$. Since problem (1.8) has variational structure, we consider its corresponding minimization problem

$$I_0 = \inf \left\{ \int_{\mathcal{C}} \left|\nabla v(x, y)\right|^2 dx \, dy \;\middle|\; v \in H^1_{0,L}(\mathcal{C}),\ \int_{\Omega} \left|v(x, 0)\right|^{p+1} dx = 1 \right\}.$$

We will prove that, for subcritical powers, there is a minimizer for this problem. Its trace on $\Omega \times \{0\}$ will provide with a weak solution of (1.1).

Thus, it is important to characterize the space $\mathcal{V}_0(\Omega)$ of all traces on $\Omega \times \{0\}$ of functions in $H^1_{0,L}(\mathcal{C})$. This is stated in the following result—which corresponds to Proposition 2.1 of the next section.

Proposition 1.8. *Let $\mathcal{V}_0(\Omega)$ be the space of all traces on $\Omega \times \{0\}$ of functions in $H^1_{0,L}(\mathcal{C})$. Then, we have*

$$\begin{aligned}\mathcal{V}_0(\Omega) &:= \{u = \mathrm{tr}_\Omega\, v \mid v \in H^1_{0,L}(\mathcal{C})\} \\ &= \left\{u \in H^{1/2}(\Omega) \,\Big|\, \int_\Omega \frac{u^2(x)}{d(x)}\,dx < +\infty\right\} \\ &= \left\{u \in L^2(\Omega) \,\Big|\, u = \sum_{k=1}^\infty b_k\varphi_k \text{ satisfying } \sum_{k=1}^\infty b_k^2\lambda_k^{1/2} < +\infty\right\},\end{aligned}$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$, and $\{\lambda_k, \varphi_k\}$ is the Dirichlet spectral decomposition of $-\Delta$ in Ω as above, with $\{\varphi_k\}$ an orthonormal basis of $L^2(\Omega)$.

Furthermore, $\mathcal{V}_0(\Omega)$ equipped with the norm

$$\|u\|_{\mathcal{V}_0(\Omega)} = \left\{\|u\|^2_{H^{1/2}(\Omega)} + \int_\Omega \frac{u^2}{d}\right\}^{1/2} \tag{1.9}$$

is a Banach space.

The fact that $d^{-1/2}u \in L^2(\Omega)$ if u is the trace of a function in $H^1_{0,L}(\mathcal{C})$ follows from a trace boundary Hardy inequality, originally due to Nekvinda [21]; see Lemma 2.6 in the next section for a simple proof. Thus, in the next section we need to consider the operator $A_{1/2}$ defined as in (1.2) but now mapping $A_{1/2} : \mathcal{V}_0(\Omega) \to \mathcal{V}_0^*(\Omega)$, where $\mathcal{V}_0^*(\Omega)$ is the dual space of $\mathcal{V}_0(\Omega)$. For $u = \sum_{k=1}^\infty b_k\varphi_k \in \mathcal{V}_0(\Omega)$, we will have $A_{1/2}(\sum_{k=1}^\infty b_k\varphi_k) = \sum_{k=1}^\infty b_k\lambda_k^{1/2}\varphi_k$. Moreover, there will be a unique harmonic extension $v \in H^1_{0,L}(\mathcal{C})$ in $\mathcal{C}$ of u, and it is given by the expression

$$v(x, y) = \sum_{k=1}^\infty b_k\varphi_k(x)\exp\bigl(-\lambda_k^{1/2}y\bigr) \quad \text{for all } (x, y) \in \mathcal{C}.$$

Thus, the operator $A_{1/2} : \mathcal{V}_0(\Omega) \to \mathcal{V}_0^*(\Omega)$ is given by the Dirichlet–Neumann map

$$A_{1/2}u := \frac{\partial v}{\partial \nu}\bigg|_{\Omega\times\{0\}} = \sum_{k=1}^\infty b_k\lambda_k^{1/2}\varphi_k.$$

Note that $A_{1/2} \circ A_{1/2}$ is equal to $-\Delta$ in Ω with zero Dirichlet boundary value on $\partial\Omega$. More precisely, we will have that the inverse $B_{1/2} = A_{1/2}^{-1}$—which maps $\mathcal{V}_0^*(\Omega)$ into itself, and also $L^2(\Omega)$ into itself—is the unique square root of the inverse Laplacian $(-\Delta)^{-1}$ in Ω with zero Dirichlet boundary values on $\partial\Omega$; see the next section for details.

To establish the regularity of weak solutions to (1.1) obtained by the previous minimization technique, we establish the following results of Calderón–Zygmund and of Schauder type for the linear problem

$$\begin{cases} A_{1/2}u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{1.10}$$

see Proposition 3.1 for more details.

Theorem 1.9. *Let $u \in \mathcal{V}_0(\Omega)$ be a weak solution of* (1.10)*, where $g \in \mathcal{V}_0^*(\Omega)$ and Ω is a $C^{2,\alpha}$ bounded domain in $\mathbb{R}^n$, for some $0 < \alpha < 1$.*

If $g \in L^2(\Omega)$, then $u \in H_0^1(\Omega)$.

If $g \in H_0^1(\Omega)$, then $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

If $g \in L^\infty(\Omega)$, then $u \in C^\alpha(\overline{\Omega})$.

If $g \in C^\alpha(\overline{\Omega})$ and $g|_{\partial\Omega} \equiv 0$, then $u \in C^{1,\alpha}(\overline{\Omega})$.

If $g \in C^{1,\alpha}(\overline{\Omega})$ and $g|_{\partial\Omega} \equiv 0$, then $u \in C^{2,\alpha}(\overline{\Omega})$.

In this paper we will give full—and rather simple—proofs of these regularity results, specially since we could only find references for some of them and, besides, in close statements to ours but not precisely ours. Our proof of Theorem 1.9 uses the extension problem in $\Omega \times (0, \infty)$ related to (1.10), and transforms it to a problem with zero Dirichlet boundary in $\Omega \times \{0\}$ by using an auxiliary function introduced in [5]. Then, by making certain reflections and using classical interior regularity theory for the Laplacian, we prove Hölder regularity for u and its derivatives.

To apply the previous Hölder regularity linear results to our nonlinear problem (1.1), we first need to prove that $g := f(u)$ is bounded, i.e., u is bounded. We will see that boundedness of weak solutions holds for subcritical and critical nonlinearities; we establish this result in Section 5. We will follow the Brezis–Kato approach using the bootstrap method. In this way, we establish the following (see Theorem 5.2).

Theorem 1.10. *Assume that g_0 is a Carathéodory function in $\Omega \times \mathbb{R}$ satisfying*

$$\left|g_0(x,s)\right| \leqslant C\left(1 + |s|^p\right) \quad \textit{for all } (x,s) \in \Omega \times \mathbb{R},$$

for some constant C, $1 \leqslant p \leqslant \frac{n+1}{n-1}$ if $n \geqslant 2$, or $1 \leqslant p < \infty$ if $n = 1$, where Ω is a smooth bounded domain in $\mathbb{R}^n$. Let $u \in \mathcal{V}_0(\Omega)$ be a weak solution of

$$\begin{cases} A_{1/2}u = g_0(x,u) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Then, $u \in L^\infty(\Omega)$.

The paper is organized as follows. In Section 2, we study the appropriate function spaces $H_{0,L}^1(\mathcal{C})$ and $\mathcal{V}_0(\Omega)$, and we give the proof of Proposition 1.8 and other related results. The regularity results of Theorem 1.9 can be found in Section 3. Maximum principles, Hopf's lemma, and the maximum principle in "small" domains of Proposition 1.7 are proved in Section 4. The complete proof of Theorem 1.1 is given in Section 5 by studying the minimization problem and applying the previous results on regularity and maximum principles. We prove Theorem 1.10 also in Section 5, while Theorems 1.3 and 1.5 are established in Section 6, and Theorem 1.6 in Section 7.

2. Preliminaries: function spaces and the operator $A_{1/2}$

In this section we collect preliminary facts for future reference. First of all, let us set the standard notations to be used in the paper. We denote the upper half-space in $\mathbb{R}^{n+1}$ by

$$\mathbb{R}^{n+1}_{+} = \{z = (x, y) = (x_1, \dots, x_n, y) \in \mathbb{R}^{n+1} \mid y > 0\}.$$

Denote by $H^s(U) = W^{s,2}(U)$ the Sobolev space in a domain U of $\mathbb{R}^n$ or of $\mathbb{R}^{n+1}_{+}$. Letting $U \subset \mathbb{R}^n$ and $s > 0$, $H^s(U)$ is a Banach space with the norm

$$\|u\|_{H^s(U)} = \left\{ \int_U \int_U \frac{|u(x) - u(\bar{x})|^2}{|x - \bar{x}|^{n+2s}}\, dx\, d\bar{x} + \int_U |u(x)|^2\, dx \right\}^{1/2}.$$

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Denote the half-cylinder with base Ω by

$$\mathcal{C} = \Omega \times (0, \infty)$$

and its lateral boundary by

$$\partial_L \mathcal{C} = \partial\Omega \times [0, \infty).$$

To treat the nonlocal problem (1.1), we will study a corresponding extension problem in one more dimension, which allows us to investigate (1.1) by studying a local problem via classical nonlinear variational methods. We consider the Sobolev space of functions in $H^1(\mathcal{C})$ whose traces vanish on $\partial_L \mathcal{C}$:

$$H^1_{0,L}(\mathcal{C}) = \{v \in H^1(\mathcal{C}) \mid v = 0 \text{ a.e. on } \partial_L \mathcal{C}\}, \tag{2.1}$$

equipped with the norm

$$\|v\| = \left(\int_{\mathcal{C}} |\nabla v|^2\, dx\, dy \right)^{1/2}. \tag{2.2}$$

We denote by tr_Ω the trace operator on $\Omega \times \{0\}$ for functions in $H^1_{0,L}(\mathcal{C})$:

$$\mathrm{tr}_\Omega\, v := v(\cdot, 0), \quad \text{for } v \in H^1_{0,L}(\mathcal{C}).$$

We have that $\mathrm{tr}_\Omega\, v \in H^{1/2}(\Omega)$, since it is well known that traces of H^1 functions are $H^{1/2}$ functions on the boundary.

Recall the well known spectral theory of the Laplacian $-\Delta$ in a smooth bounded domain Ω with zero Dirichlet boundary values. We repeat each eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary conditions according to its (finite) multiplicity:

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \to \infty, \quad \text{as } k \to \infty,$$

and we denote by $\varphi_k \in H_0^1(\Omega)$ an eigenfunction corresponding to λ_k for $k = 1, 2, \ldots$. Namely,

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \Omega. \end{cases} \tag{2.3}$$

We can take them to form an orthonormal basis $\{\varphi_k\}$ of $L^2(\Omega)$, in particular,

$$\int_\Omega \varphi_k^2 \, dx = 1,$$

and to belong to $C^2(\overline{\Omega})$ by regularity theory.

Now we can state the main results which we prove in this section.

Proposition 2.1. *Let $\mathcal{V}_0(\Omega)$ be the space of all traces on $\Omega \times \{0\}$ of functions in $H_{0,L}^1(\mathcal{C})$. Then, we have*

$$\begin{aligned} \mathcal{V}_0(\Omega) &:= \left\{u = \operatorname{tr}_\Omega v \;\middle|\; v \in H_{0,L}^1(\mathcal{C})\right\} \\ &= \left\{u \in H^{1/2}(\Omega) \;\middle|\; \int_\Omega \frac{u^2(x)}{d(x)} \, dx < +\infty\right\} \\ &= \left\{u \in L^2(\Omega) \;\middle|\; u = \sum_{k=1}^\infty b_k\varphi_k \text{ satisfying } \sum_{k=1}^\infty b_k^2\lambda_k^{1/2} < +\infty\right\}, \end{aligned}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$, and $\{\lambda_k, \varphi_k\}$ is the Dirichlet spectral decomposition of $-\Delta$ in Ω as above, with $\{\varphi_k\}$ an orthonormal basis of $L^2(\Omega)$.

Furthermore, $\mathcal{V}_0(\Omega)$ equipped with the norm

$$\|u\|_{\mathcal{V}_0(\Omega)} = \left\{\|u\|_{H^{1/2}(\Omega)}^2 + \int_\Omega \frac{u^2}{d}\right\}^{1/2} \tag{2.4}$$

is a Banach space.

Proposition 2.2. *If $u \in \mathcal{V}_0(\Omega)$, then there exists a unique harmonic extension v in $\mathcal{C}$ of u such that $v \in H_{0,L}^1(\mathcal{C})$. In particular, if the expansion of u is written by $u = \sum_{k=1}^\infty b_k\varphi_k \in \mathcal{V}_0(\Omega)$, then*

$$v(x, y) = \sum_{k=1}^\infty b_k\varphi_k(x) \exp\bigl(-\lambda_k^{1/2} y\bigr) \quad \text{for all } (x, y) \in \mathcal{C},$$

where $\{\lambda_k, \varphi_k\}$ is the Dirichlet spectral decomposition of $-\Delta$ in Ω as above, with $\{\varphi_k\}$ an orthonormal basis of $L^2(\Omega)$.

The operator $A_{1/2} : \mathcal{V}_0(\Omega) \to \mathcal{V}_0^*(\Omega)$ is given by

$$A_{1/2}u := \frac{\partial v}{\partial \nu}\bigg|_{\Omega \times \{0\}},$$

where $\mathcal{V}_0^*(\Omega)$ is the dual space of $\mathcal{V}_0(\Omega)$. We have that

$$A_{1/2}u = \sum_{k=1}^{\infty} b_k \lambda_k^{1/2} \varphi_k,$$

and that $A_{1/2} \circ A_{1/2}$ (when $A_{1/2}$ is acting, for instance, on smooth functions with compact support in Ω) is equal to $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$. More precisely, the inverse $B_{1/2} := A_{1/2}^{-1}$ is the unique positive square root of the inverse Laplacian $(-\Delta)^{-1}$ in Ω with zero Dirichlet boundary values on $\partial\Omega$.

The proofs of these two propositions need the development of several tools. First let us give some properties of the space $H_{0,L}^1(\mathcal{C})$. Denote by $\mathcal{D}^{1,2}(\mathbb{R}_+^{n+1})$ the closure of the set of smooth functions compactly supported in $\overline{\mathbb{R}_+^{n+1}}$ with respect to the norm of $\|w\|_{\mathcal{D}^{1,2}(\mathbb{R}_+^{n+1})} = (\int_{\mathbb{R}_+^{n+1}} |\nabla w|^2\, dx\, dy)^{1/2}$. We recall the well known Sobolev trace inequality. For $w \in \mathcal{D}^{1,2}(\mathbb{R}_+^{n+1})$, we have

$$\left(\int_{\mathbb{R}^n} \left| w(x,0) \right|^{2n/(n-1)} dx \right)^{(n-1)/2n} \leqslant C \left(\int_{\mathbb{R}_+^{n+1}} \left| \nabla w(x,y) \right|^2 dx\, dy \right)^{1/2}, \tag{2.5}$$

where C depends only on n.

Denote for $n \geqslant 2$,

$$2^\sharp = \frac{2n}{n-1} \quad \text{and} \quad 2^\sharp - 1 = \frac{n+1}{n-1}.$$

We say that p is subcritical if $1 < p < 2^\sharp - 1 = \frac{n+1}{n-1}$ for $n \geqslant 2$, and $1 < p < \infty$ for $n = 1$. We also say that p is critical if $p = 2^\sharp - 1 = \frac{n+1}{n-1}$ for $n \geqslant 2$, and that p is supercritical if $p > 2^\sharp - 1 = \frac{n+1}{n-1}$ for $n \geqslant 2$.

Lions [19] showed that

$$S_0 = \inf \left\{ \frac{\int_{\mathbb{R}_+^{n+1}} |\nabla w(x,y)|^2\, dx\, dy}{(\int_{\mathbb{R}^n} |w(x,0)|^{2^\sharp}\, dx)^{2/2^\sharp}} \;\middle|\; w \in \mathcal{D}^{1,2}\left(\mathbb{R}_+^{n+1}\right) \right\} \tag{2.6}$$

is achieved. Escobar [12] proved that the extremal functions have all the form

$$U_\varepsilon(x,y) = \frac{\varepsilon^{(n-1)/2}}{|(x - x_0, y + \varepsilon)|^{n-1}}, \tag{2.7}$$

where $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ are arbitrary. In addition, the best constant is

$$S_0 = \frac{(n-1)\sigma_n^{1/n}}{2},$$

where σ_n denotes the volume of n-dimensional sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

The Sobolev trace inequality leads directly to the next three lemmas. For $v \in H^1_{0,L}(\mathcal{C})$, its extension by zero in $\mathbb{R}^{n+1}_+ \setminus \mathcal{C}$ can be approximated by functions compactly supported in $\overline{\mathbb{R}^{n+1}_+}$. Thus the Sobolev trace inequality (2.5) leads to:

Lemma 2.3. *Let $n \geqslant 2$ and $2^\sharp = \frac{2n}{n-1}$. Then there exists a constant C, depending only on n, such that, for all $v \in H^1_{0,L}(\mathcal{C})$,*

$$\left(\int_\Omega \left|v(x,0)\right|^{2^\sharp} dx\right)^{1/2^\sharp} \leqslant C\left(\int_{\mathcal{C}} \left|\nabla v(x,y)\right|^2 dx\,dy\right)^{1/2}. \tag{2.8}$$

By Hölder's inequality, since Ω is bounded, the above lemma leads to:

Lemma 2.4. *Let $1 \leqslant q \leqslant 2^\sharp$ for $n \geqslant 2$. Then, we have that for all $v \in H^1_{0,L}(\mathcal{C})$,*

$$\left(\int_\Omega \left|v(x,0)\right|^{q} dx\right)^{1/q} \leqslant C\left(\int_{\mathcal{C}} \left|\nabla v(x,y)\right|^2 dx\,dy\right)^{1/2}, \tag{2.9}$$

where C depends only on n, q, and the measure of Ω. Moreover, (2.9) *also holds for $1 \leqslant q < \infty$ if $n = 1$.*

This lemma states that $\mathrm{tr}_\Omega(H^1_{0,L}(\mathcal{C})) \subset L^q(\Omega)$, where $1 \leqslant q \leqslant 2^\sharp$ for $n \geqslant 2$ and $1 \leqslant q < \infty$ for $n = 1$ (see the proof of Lemma 2.5 for the case $n = 1$). In addition, we also have the following compact embedding.

Lemma 2.5. *Let $1 \leqslant q < 2^\sharp = \frac{2n}{n-1}$ for $n \geqslant 2$ and $1 \leqslant q < \infty$ for $n = 1$. Then $\mathrm{tr}_\Omega(H^1_{0,L}(\mathcal{C}))$ is compactly embedded in $L^q(\Omega)$.*

Proof. It is well known that $\mathrm{tr}_\Omega(H^1_{0,L}(\mathcal{C})) \subset H^{1/2}(\Omega)$ and that $H^{1/2}(\Omega) \subset\subset L^q(\Omega)$ when $1 \leqslant q < 2^\sharp = \frac{2n}{n-1}$ for $n \geqslant 2$ and $1 \leqslant q < \infty$ for $n = 1$. Here $\subset\subset$ denotes compact embedding. This completes the proof of the lemma. However, if one wants to avoid the use of the fractional Sobolev space $H^{1/2}(\Omega)$, the following is an alternative simple proof.

Considering the restriction of functions in $\mathcal{C}$ to $\Omega \times (0,1)$, it suffices to show that the embedding is compact with $\mathcal{C}$ replaced by $\Omega \times (0,1)$. To prove this, let $v_m \in H^1_{0,L}(\Omega \times (0,1)) := \{v \in H^1(\Omega \times (0,1)) \mid v = 0$ a.e. on $\partial\Omega \times (0,1)\}$ such that $v_m \rightharpoonup 0$ weakly in $H^1_{0,L}(\Omega \times (0,1))$, as $m \to \infty$. We may assume by the classical Rellich's theorem in $\Omega \times (0,1)$ that $v_m \to 0$ strongly in $L^2(\Omega \times (0,1))$, as $m \to \infty$. We introduce the function $w_m = (1-y)v_m$. It is clear that

$$w_m|_{\Omega\times\{0\}} = v_m, \qquad w_m|_{\Omega\times\{1\}} = 0.$$

By direct computations we have

$$\int_{\Omega} \left|v_m(x,0)\right|^2 dx = \int_{\Omega} \left|w_m(x,0)\right|^2 dx = -\int_0^1 \int_{\Omega} \partial_y\bigl(w_m^2(x,y)\bigr)\,dx\,dy$$
$$\leqslant 2\left(\int_0^1 \int_{\Omega} w_m^2(x,y)\,dx\,dy\right)^{1/2} \left(\int_0^1 \int_{\Omega} \left|\nabla w_m(x,y)\right|^2 dx\,dy\right)^{1/2}.$$

Therefore, since $w_m = (1-y)v_m$ is bounded in $H^1(\Omega \times (0,1))$ and $w_m \to 0$ strongly in $L^2(\Omega \times (0,1))$, we find that, as $m \to \infty$,

$$v_m(x,0) \to 0 \quad \text{strongly in } L^2(\Omega) \text{ and hence also in } L^1(\Omega).$$

On the other hand, since q is subcritical, the following interpolation inequality,

$$\left\|v_m(\cdot,0)\right\|_{L^q(\Omega)} \leqslant \left\|v_m(\cdot,0)\right\|_{L^1(\Omega)}^{\theta} \left\|v_m(\cdot,0)\right\|_{L^{2^\sharp}(\Omega)}^{1-\theta}$$

for some $0<\theta<1$ completes the proof since we already know that v_m converges strongly to zero in $L^1(\Omega)$. $\square$

We also need to establish a trace boundary Hardy inequality, which already appeared in a work of Nekvinda [21].

Lemma 2.6. *We have that*

$$\operatorname{tr}_{\Omega}\bigl(H^1_{0,L}(\mathcal{C})\bigr) \subset H^{1/2}(\Omega)$$

is a continuous injection. In addition, for every $v \in H^1_{0,L}(\mathcal{C})$,

$$\int_{\Omega} \frac{|v(x,0)|^2}{d(x)}\,dx \leqslant C \int_{\mathcal{C}} \left|\nabla v(x,y)\right|^2 dx\,dy, \tag{2.10}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$ and the constant C depends only on Ω.

Proof. The first statement is clear since the traces of $H^1(\mathcal{C})$ functions belong to $H^{1/2}(\partial\mathcal{C})$. Regarding the second statement, we prove it in two steps.

Step 1. Assume first that $n=1$ and $\Omega = (0,1)$. For $0 < x_0 < 1/2$, consider the segment from $(0,x_0)$ to $(x_0,0)$ in $\mathcal{C} = (0,1) \times (0,\infty)$. We have

$$v(x_0,0) = v(t, x_0 - t)\big|_{t=0}^{x_0} = \int_0^{x_0} (\partial_x v - \partial_y v)(t, x_0 - t)\,dt.$$

Then

$$\left|v(x_0,0)\right|^2 \leqslant x_0 \int_0^{x_0} 2\left|\nabla v(t,x_0-t)\right|^2 dt.$$

Dividing this inequality by x_0 and integrating in x_0 over $(0,1/2)$, and making the change of variables $x=t$, $y=x_0-t$, we deduce

$$\int_0^{1/2} \frac{|v(x_0,0)|^2}{x_0}\,dx_0 \leqslant 2\int_0^{1/2} dx \int_0^{1/2} dy |\nabla v|^2 \leqslant 2\int_{\mathcal{C}} |\nabla v|^2\,dx\,dy.$$

Doing the same on $(1/2,1)$, this establishes inequality (2.10) of the lemma.

Step 2. In the general case, after straightening a piece of the boundary $\partial\Omega$ and rescaling the new variables, we can consider the inequality in a domain $D=\{x=(x',x_n)\mid |x'|<1,\ 0<x_n<1/2\}$ and assume that $v=0$ on $\{x_n=0,\ |x'|<1\}\times(0,\infty)$, since the flatting procedure possesses equivalent norms. By the argument in Step 1 above, we have

$$\int_0^{1/2} \frac{|v(x,0)|^2}{x_n}\,dx_n \leqslant C\int_0^{1/2}\int_0^{\infty} |\nabla v|^2\,dx_n\,dy,$$

for all x' with $|x'|<1$. From this, integrating in x' we have

$$\begin{aligned}\int_D \frac{|v(x,0)|^2}{x_n}\,dx &= \int_D \int_0^{1/2} \frac{|v(x,0)|^2}{x_n}\,dx'\,dx_n \\ &\leqslant C \int_{D\times(0,\infty)} |\nabla v|^2\,dx\,dy.\end{aligned}$$

Since after flattening of $\partial\Omega$, x_n is comparable to $d(x)=\operatorname{dist}(x,\partial\Omega)$, this is the desired inequality (2.10). $\square$

Recall that the fractional Sobolev space $H^{1/2}(\Omega)$ is a Banach space with the norm

$$\|u\|^2_{H^{1/2}(\Omega)} = \int_\Omega\int_\Omega \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+1}}\,dx\,d\bar{x} + \int_\Omega \left|u(x)\right|^2 dx. \tag{2.11}$$

Note that the closure $H_0^{1/2}(\Omega)$ of smooth functions with compact support, $C_c^\infty(\Omega)$, in $H^{1/2}(\Omega)$ is all the space $H^{1/2}(\Omega)$ (see Theorem 11.1 in [20]). That is, $C_c^\infty(\Omega)$ is dense in $H^{1/2}(\Omega)$. However, in contrast with this, the traces in Ω of functions in $H^1_{0,L}(\mathcal{C})$ "vanish" on $\partial\Omega$ in the sense given by (2.10).

Recall that we have denoted by $\mathcal{V}_0(\Omega)$ the space of traces on $\Omega\times\{0\}$ of functions in $H^1_{0,L}(\mathcal{C})$:

$$\mathcal{V}_0(\Omega) := \left\{u=\operatorname{tr}_\Omega v \;\middle|\; v\in H^1_{0,L}(\mathcal{C})\right\} \subset H^{1/2}(\Omega), \tag{2.12}$$

endowed with the norm (2.4) in Proposition 2.1. The dual space of $\mathcal{V}_0(\Omega)$ is denoted by $\mathcal{V}_0^*(\Omega)$, equipped with the norm

$$\|g\|_{\mathcal{V}_0^*(\Omega)} = \sup\bigl\{\langle u, g\rangle \bigm| u \in \mathcal{V}_0(\Omega),\ \|u\|_{\mathcal{V}_0(\Omega)} \leqslant 1\bigr\}.$$

Next we give the first characterization of the space $\mathcal{V}_0(\Omega)$:

Lemma 2.7. *Let $\mathcal{V}_0(\Omega)$ be the space of traces on $\Omega \times \{0\}$ of functions in $H^1_{0,L}(\mathcal{C})$, as in* (2.12). *Then, we have*

$$\mathcal{V}_0(\Omega) = \left\{u \in H^{1/2}(\Omega) \Bigm| \int_\Omega \frac{u^2(x)}{d(x)}\,dx < +\infty\right\},$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Proof. The inclusion $\subset$ follows from Lemma 2.6. Next we show the other inclusion. Let $u \in H^{1/2}(\Omega)$ satisfy $\int_\Omega u^2/d < \infty$. Let $\tilde u$ be the extension of u in all of $\mathbb{R}^n$ assigning $\tilde u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. The quantity

$$\|\tilde u\|^2_{H^{1/2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\tilde u(x) - \tilde u(\bar x)|^2}{|x - \bar x|^{n+1}}\,dx\,d\bar x + \int_{\mathbb{R}^n} \bigl|\tilde u(x)\bigr|^2\,dx$$

can be bounded—using $\tilde u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$—by a constant times

$$\left\{\|u\|^2_{H^{1/2}(\Omega)} + \int_\Omega \frac{u^2(x)}{d(x)}\,dx\right\}^{1/2},$$

that we assume to be finite. Hence, $\tilde u \in H^{1/2}(\mathbb{R}^n)$ and thus $\tilde u$ is the trace in $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$ of a function $\tilde v \in H^1(\mathbb{R}^{n+1}_+)$.

Next, we use a partition of the unity, and local bi-Lipschitz maps (defined below) sending $\overline{\mathbb{R}^{n+1}_+}$ into $\overline{\Omega} \times [0, \infty) = \bar{\mathcal{C}}$ being the identity on $\Omega \times \{0\}$ and mapping $\mathbb{R}^n \setminus \Omega = (\partial\mathbb{R}^{n+1}_+) \setminus \Omega$ into $\partial\Omega \times [0, \infty)$. By composing these maps with the function $\tilde v$ (cut off with the partition of unity), we obtain an $H^1_{0,L}(\mathcal{C})$ function with u as trace on $\Omega \times \{0\}$, as desired.

Finally we give a concrete expression for one such bi-Lipschitz map. First, consider the one-dimensional case $\Omega = (0, \infty)$. Then simply take the bi-Lipschitz map

$$\begin{aligned}(x, y) \ &\in\ (0, \infty) \times (0, \infty) = \Omega \times (0, \infty)\\ &\mapsto \left(\frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \frac{2xy}{\sqrt{x^2 + y^2}}\right) \in \mathbb{R} \times (0, \infty),\end{aligned}$$

whose Jacobian can be checked to be identically 2. In the general case, we can flatten the boundary $\partial\Omega$ and use locally the previous map. $\square$

Next we consider, for a given function $u \in \mathcal{V}_0(\Omega)$, the minimizing problem:

$$\inf\left\{ \int_{\mathcal{C}} |\nabla v|^2 \, dx \, dy \;\middle|\; v \in H^1_{0,L}(\mathcal{C}),\ v(\cdot,0) = u \text{ in } \Omega \right\}. \tag{2.13}$$

By the definition of $\mathcal{V}_0(\Omega)$, the set of functions v where we minimize is nonempty. By lower weak semi-continuity and by Lemma 2.5, we see that there exists a minimizer v. We will prove next that this minimizer v is unique. We call v a *weak solution* of the problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C}, \\ v = 0 & \text{on } \partial_L \mathcal{C}, \\ v = u & \text{on } \Omega \times \{0\}. \end{cases} \tag{2.14}$$

That is, we have

Lemma 2.8. *For $u \in \mathcal{V}_0(\Omega)$, there exists a unique minimizer v of* (2.13). *The function $v \in H^1_{0,L}(\mathcal{C})$ is the harmonic extension of u (in the weak sense) to $\mathcal{C}$ vanishing on $\partial_L \mathcal{C}$.*

Proof. By the definition of $\mathcal{V}_0(\Omega)$, we have that, for every $u \in \mathcal{V}_0(\Omega)$, there exists at least one $w \in H^1_{0,L}(\mathcal{C})$ such that $\mathrm{tr}_\Omega(w) = u$. Then the standard minimization argument gives (using lower semi-continuity and Lemma 2.5) the existence of a minimizer. The uniqueness of minimizer follows automatically from the identity of the parallelogram used for two possible minimizers v_1 and v_2,

$$0 \leqslant J\left(\frac{v_1 - v_2}{2}\right) = \frac{1}{2} J(v_1) + \frac{1}{2} J(v_2) - J\left(\frac{v_1 + v_2}{2}\right) \leqslant 0,$$

where $J(v) = \int_{\mathcal{C}} |\nabla v|^2 \, dx \, dy$, which leads to $v_1 = v_2$. $\square$

By Lemma 2.8, there exists a unique function $v \in H^1_{0,L}(\mathcal{C})$ which is the harmonic extension of u in $\mathcal{C}$ vanishing on $\partial_L \mathcal{C}$, and that we denote by

$$v := \text{h-ext}(u).$$

It is easy to see that for every $\eta \in C^\infty(\bar{\mathcal{C}}) \cap H^1(\mathcal{C})$ and $\eta \equiv 0$ on $\partial_L \mathcal{C}$,

$$\int_{\mathcal{C}} \nabla v \nabla \eta \, dx \, dy = \int_{\Omega} \frac{\partial v}{\partial \nu} \eta \, dx. \tag{2.15}$$

By Lemma 2.6, there exists a constant C such that for every $u \in \mathcal{V}_0(\Omega)$,

$$\|u\|_{\mathcal{V}_0(\Omega)} \leqslant C \left\| \text{h-ext}(u) \right\|_{H^1_{0,L}(\mathcal{C})}. \tag{2.16}$$

Next, note that the h-ext operator is bijective from $\mathcal{V}_0(\Omega)$ to the subspace $\mathcal{H}$ of $H^1_{0,L}(\mathcal{C})$ formed by all harmonic functions in $H^1_{0,L}(\mathcal{C})$. Since both $\mathcal{V}_0(\Omega)$ and $\mathcal{H}$ are Banach spaces, the open mapping theorem gives that we also have the reverse inequality to (2.16), i.e., there exists a constant C such that

$$\left\| \text{h-ext}(u) \right\|_{H^1_{0,L}(\mathcal{C})} \leqslant C \|u\|_{\mathcal{V}_0(\Omega)}, \tag{2.17}$$

for all $u \in \mathcal{V}_0(\Omega)$. From this we deduce the following. Given a smooth $\xi \in \mathcal{V}_0(\Omega)$, consider the h-ext$(\xi)$ and call it η. Now, we use (2.15) and (2.17) (for u and ξ) to obtain $|\int_\Omega \frac{\partial v}{\partial \nu} \xi \, dx| \leqslant C\|u\|_{\mathcal{V}_0(\Omega)} \|\xi\|_{\mathcal{V}_0(\Omega)}$. That is, $\frac{\partial v}{\partial \nu}|_\Omega \in \mathcal{V}_0^*(\Omega)$ and there is the bound:

$$\left\| \frac{\partial}{\partial \nu} \text{h-ext}(u) \right\|_{\mathcal{V}_0^*(\Omega)} \leqslant C \|u\|_{\mathcal{V}_0(\Omega)}.$$

Hence we have

Lemma 2.9. *The operator $A_{1/2} : \mathcal{V}_0(\Omega) \to \mathcal{V}_0^*(\Omega)$ defined by*

$$A_{1/2} u := \left. \frac{\partial v}{\partial \nu} \right|_{\Omega \times \{0\}}, \tag{2.18}$$

where $v = \text{h-ext}(u) \in H^1_{0,L}(\mathcal{C})$ is the harmonic extension of u in $\mathcal{C}$ vanishing on $\partial_L \mathcal{C}$, is linear and bounded from $\mathcal{V}_0(\Omega)$ to $\mathcal{V}_0^*(\Omega)$.

We now give the spectral representation of $A_{1/2}$ and the corresponding structure of the space $\mathcal{V}_0(\Omega)$.

Lemma 2.10.

(i) *Let $\{\varphi_k\}$ be an orthonormal basis of $L^2(\Omega)$ forming a spectral decomposition of $-\Delta$ in Ω with Dirichlet boundary conditions as in* (2.3)*, with $\{\lambda_k\}$ the corresponding Dirichlet eigenvalues of $-\Delta$ in Ω. Then, we have*

$$\mathcal{V}_0(\Omega) = \left\{ u = \sum_{k=1}^{\infty} b_k \varphi_k \in L^2(\Omega) \;\middle|\; \sum_{k=1}^{\infty} b_k^2 \lambda_k^{1/2} < +\infty \right\}.$$

(ii) *Let $u \in \mathcal{V}_0(\Omega)$. Then we have, if $u = \sum_{k=1}^{\infty} b_k \varphi_k$,*

$$A_{1/2} u = \sum_{k=1}^{\infty} b_k \lambda_k^{1/2} \varphi_k \in \mathcal{V}_0^*(\Omega).$$

Proof. Let $u \in \mathcal{V}_0(\Omega)$, which is contained in $L^2(\Omega)$. Let its expansion be written by $u(x) = \sum_{k=1}^{\infty} b_k \varphi_k(x)$. Consider the function

$$v(x, y) = \sum_{k=1}^{\infty} b_k \varphi_k(x) \exp\bigl(-\lambda_k^{1/2} y\bigr), \tag{2.19}$$

which is clearly smooth for $y > 0$. Observe that $v(x, 0) = u(x)$ in Ω and, for $y > 0$,

$$\Delta v(x,y)=\sum_{k=1}^{\infty} b_k\bigl\{-\lambda_k\varphi_k(x)\exp\bigl(-\lambda_k^{1/2}y\bigr)+\lambda_k\varphi_k(x)\exp\bigl(-\lambda_k^{1/2}y\bigr)\bigr\}=0.$$

Thus, v is a harmonic extension of u. We will have that $v=\text{h-ext}(u)$, by uniqueness, once we find the condition on $\{b_k\}$ for v to belong to $H^1_{0,L}(\mathcal{C})$. But such condition is simple. Using (2.19) and that $\{\varphi_k\}$ are eigenfunctions of $-\Delta$ and orthonormal in $L^2(\Omega)$, we have

$$\begin{aligned}\int_0^{\infty}\int_{\Omega}|\nabla v|^2\,dx\,dy&=\int_0^{\infty}\int_{\Omega}\bigl\{|\nabla_x v|^2+|\partial_y v|^2\bigr\}\,dx\,dy\\&=2\sum_{k=1}^{\infty} b_k^2\lambda_k\int_0^{\infty}\exp\bigl(-2\lambda_k^{1/2}y\bigr)\,dy\\&=2\sum_{k=1}^{\infty} b_k^2\lambda_k\frac{1}{2\lambda_k^{1/2}}=\sum_{k=1}^{\infty} b_k^2\lambda_k^{1/2}.\end{aligned}$$

This means that $v\in H^1_{0,L}(\mathcal{C})$ if and only if $\sum_{k=1}^{\infty} b_k^2\lambda_k^{1/2}<\infty$. Therefore, this condition on $\{b_k\}$ is equivalent to $u\in\mathcal{V}_0(\Omega)$.

Assertion (ii) follows from the direct computation of $-\frac{\partial v}{\partial y}|_{y=0}$ using (2.19). □

In functional analysis, the classical spectral decomposition holds for self-adjoint compact operators, such as the Dirichlet inverse Laplacian $(-\Delta)^{-1}: L^2(\Omega)\to L^2(\Omega)$. This is the reason why we now define, with the aid of the Lax–Milgram theorem, a compact operator $B_{1/2}$ which will be the inverse of $A_{1/2}$.

Definition 2.11. Define the operator $B_{1/2}:\mathcal{V}_0^*(\Omega)\to\mathcal{V}_0(\Omega)$, by $g\mapsto \operatorname{tr}_\Omega v$, where v is found by solving the problem:

$$\begin{cases}\Delta v=0 & \text{in } \mathcal{C},\\ v=0 & \text{on } \partial_L\mathcal{C},\\ \dfrac{\partial v}{\partial \nu}=g(x) & \text{on } \Omega\times\{0\},\end{cases}\tag{2.20}$$

as we indicate next.

We say that v is a weak solution of (2.20) whenever $v\in H^1_{0,L}(\mathcal{C})$ and

$$\int_{\mathcal{C}}\nabla v\nabla\xi\,dx\,dy=\bigl\langle g,\xi(\cdot,0)\bigr\rangle\tag{2.21}$$

for all $\xi\in H^1_{0,L}(\mathcal{C})$. We see that there exists a unique weak solution of (2.20) by the Lax–Milgram theorem, via studying the corresponding functional in $H^1_{0,L}(\mathcal{C})$:

$$I(v) = \frac{1}{2} \int_{\mathcal{C}} |\nabla v|^2 \, dx \, dy - \langle g, v(\cdot, 0) \rangle,$$

where $g \in \mathcal{V}_0^*(\Omega)$ is given. Observe that the operator $B_{1/2}$ is clearly the inverse of the operator $A_{1/2}$.

On the other hand, let us compute $B_{1/2} \circ B_{1/2} |_{L^2(\Omega)}$. Here note that since $\mathcal{V}_0(\Omega) \subset L^2(\Omega)$, we have $L^2(\Omega) \subset \mathcal{V}_0^*(\Omega)$. For a given $g \in L^2(\Omega)$, let $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of Poisson's problem for the Laplacian

$$\begin{cases} -\Delta \varphi = g & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $H_0^1(\Omega) \subset \mathcal{V}_0(\Omega)$ (for instance, by Lemma 2.10), there is a unique harmonic extension $\psi \in H_{0,L}^1(\mathcal{C})$ of φ in $\mathcal{C}$ such that

$$\begin{cases} \Delta \psi = 0 & \text{in } \mathcal{C}, \\ \psi = 0 & \text{on } \partial_L \mathcal{C}, \\ \psi = \varphi & \text{on } \Omega \times \{0\}. \end{cases}$$

Moreover, $\tilde{\psi}(x, y) := \psi(x, y) - \varphi(x)$ solves

$$\begin{cases} -\Delta \tilde{\psi} = \Delta \varphi = -g(x) & \text{in } \mathcal{C}, \\ \tilde{\psi} = 0 & \text{on } \partial_L \mathcal{C}, \\ \tilde{\psi} = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Considering the odd reflection $\tilde{\psi}_{od}$ of $\tilde{\psi}$ across $\Omega \times \{0\}$, and the function

$$g_{od}(x, y) = \begin{cases} g(x), & y \geqslant 0, \\ -g(x), & y < 0, \end{cases}$$

we have

$$\begin{cases} -\Delta \tilde{\psi}_{od} = -g_{od} & \text{in } \Omega \times \mathbb{R}, \\ \tilde{\psi}_{od} = 0 & \text{on } \partial\Omega \times \mathbb{R}. \end{cases}$$

Therefore, since $g_{od} \in L^2(\Omega \times (-2, 2))$, we deduce $\tilde{\psi}_{od} \in H^2(\Omega \times (-1, 1))$ and hence $\psi \in H^2(\Omega \times (0, 1))$. We deduce, by the smoothness of the harmonic function ψ for $y > 0$ and by its exponential decay in y—see (2.19)—that $\psi \in H_{0,L}^1(\mathcal{C}) \cap H^2(\mathcal{C})$.

It follows that $-\partial_y \psi \in H_{0,L}^1(\mathcal{C})$ solves

$$\begin{cases} \Delta(-\partial_y \psi) = 0 & \text{in } \mathcal{C}, \\ -\partial_y \psi = 0 & \text{on } \partial_L \mathcal{C}, \end{cases}$$

and

$$\frac{\partial}{\partial \nu}(-\partial_y \psi) = \partial_{yy} \psi = -\Delta_x \psi = -\Delta \varphi = g \quad \text{on } \Omega \times \{0\}.$$

Since $\mathcal{V}_0(\Omega) \subset L^2(\Omega)$, we have that $g \in L^2(\Omega) \cong L^2(\Omega)^* \subset \mathcal{V}_0^*(\Omega)$, and we deduce that the solution $v \in \mathcal{V}_0(\Omega)$ of (2.20) is $v = -\partial_y \psi$, because of the uniqueness of $H^1_{0,L}(\mathcal{C})$ solution of (2.20). In particular, $B_{1/2} g = v(\cdot, 0) = -\partial_y \psi(\cdot, 0)$. On the other hand, since $\psi \in H^1_{0,L}(\mathcal{C})$ solves

$$\begin{cases} \Delta\psi = 0 & \text{in } \mathcal{C}, \\ \psi = 0 & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial \psi}{\partial \nu} \equiv -\partial_y \psi(\cdot, 0) = v(\cdot, 0) = B_{1/2} g & \text{on } \Omega \times \{0\}, \end{cases}$$

we conclude that

$$(B_{1/2} \circ B_{1/2}) g = B_{1/2} v(\cdot, 0) = \psi(\cdot, 0) = \varphi = (-\Delta)^{-1} g.$$

Summarizing the above argument, we have:

Proposition 2.12. $B_{1/2} \circ B_{1/2}|_{L^2(\Omega)} = (-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$, *where* $(-\Delta)^{-1}$ *is the inverse Laplacian in* Ω *with zero Dirichlet boundary conditions.*

Note that $B_{1/2} : L^2(\Omega) \to L^2(\Omega)$ is a self-adjoint operator. In fact, since for $v_1, v_2 \in H^1_{0,L}(\mathcal{C})$,

$$\int_{\mathcal{C}} (v_2 \Delta v_1 - v_1 \Delta v_2)\, dx\, dy = \int_{\Omega} \left(v_2 \frac{\partial v_1}{\partial \nu} - v_1 \frac{\partial v_2}{\partial \nu} \right) dx,$$

we see

$$\int_{\Omega} B_{1/2} g_2 \cdot g_1\, dx = \int_{\Omega} B_{1/2} g_1 \cdot g_2\, dx$$

and

$$\int_{\Omega} v_2(x, 0) A_{1/2} v_1(x, 0)\, dx = \int_{\Omega} v_1(x, 0) A_{1/2} v_2(x, 0)\, dx.$$

On the other hand, by using (2.21) with $\xi = v$ and Lemma 2.5, we obtain that $B_{1/2}$ is a positive compact operator in $L^2(\Omega)$. Hence by the spectral theory for self-adjoint compact operators, we have that all the eigenvalues of $B_{1/2}$ are real, positive, and that there are corresponding eigenfunctions which make up an orthonormal basis of $L^2(\Omega)$. Furthermore, such basis and eigenvalues are explicit in terms of those of the Laplacian with Dirichlet boundary conditions, since $(-\Delta)^{-1}$ has $B_{1/2}$ as unique, positive and self-adjoint square root, by Proposition 2.12. Summarizing:

Proposition 2.13. *Let* $\{\varphi_k\}$ *be an orthonormal basis of* $L^2(\Omega)$ *forming a spectral decomposition of* $-\Delta$ *in* Ω *with Dirichlet boundary conditions, as in* (2.3)*, with* $\{\lambda_k\}$ *the corresponding Dirichlet eigenvalues of* $-\Delta$ *in* Ω*. Then, for all* $k \geqslant 1$,

$$\begin{cases} A_{1/2}\varphi_k = \lambda_k^{1/2}\varphi_k & \textit{in } \Omega, \\ \varphi_k = 0 & \textit{on } \partial\Omega. \end{cases} \tag{2.22}$$

In particular, $\{\varphi_k\}$ *is also a basis formed by the eigenfunctions of* $A_{1/2}$*, with eigenvalues* $\{\lambda_k^{1/2}\}$.

Proof of Proposition 2.1. It follows from Lemma 2.7 and Lemma 2.10. □

Proof of Proposition 2.2. It follows from Lemma 2.8, Lemma 2.10 and its proof, and Propositions 2.12 and 2.13. □

3. Regularity of solutions

In this section we study the regularity of weak solutions for linear and nonlinear problems involving $A_{1/2}$. First we consider the linear problem

$$\begin{cases} A_{1/2}u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $g \in \mathcal{V}_0^*(\Omega)$ and Ω is a smooth bounded domain in $\mathbb{R}^n$. By the construction of the previous section, the precise meaning of (3.1) is that $u = \mathrm{tr}_\Omega\, v$, where the function $v \in H^1_{0,L}(\mathcal{C})$ with $v(\cdot, 0) = u \in \mathcal{V}_0(\Omega)$ satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C}, \\ v = 0 & \text{on } \partial_L\mathcal{C}, \\ \dfrac{\partial v}{\partial \nu} = g(x) & \text{on } \Omega \times \{0\}. \end{cases} \tag{3.2}$$

We will say then that v is a *weak solution* of (3.2) and that u is a *weak solution* of (3.1).

Most of this section contains the proof of the following analogues of the $W^{2,p}$-estimates of Calderón–Zygmund and of the Schauder estimates.

Proposition 3.1. *Let* $\alpha \in (0, 1)$*,* Ω *be a* $C^{2,\alpha}$ *bounded domain of* $\mathbb{R}^n$*,* $g \in \mathcal{V}_0^*(\Omega)$*,* $v \in H^1_{0,L}(\mathcal{C})$ *be the weak solution of* (3.2)*, and* $u = \mathrm{tr}_\Omega\, v$ *be the weak solution of* (3.1). *Then,*

(i) *If* $g \in L^2(\Omega)$*, then* $u \in H_0^1(\Omega)$.
(ii) *If* $g \in H_0^1(\Omega)$*, then* $u \in H^2(\Omega) \cap H_0^1(\Omega)$.
(iii) *If* $g \in L^\infty(\Omega)$*, then* $v \in W^{1,q}(\Omega \times (0, R))$ *for all* $R > 0$ *and* $1 < q < \infty$. *In particular,* $v \in C^\alpha(\bar{\mathcal{C}})$ *and* $u \in C^\alpha(\overline{\Omega})$.
(iv) *If* $g \in C^\alpha(\overline{\Omega})$ *and* $g|_{\partial\Omega} \equiv 0$*, then* $v \in C^{1,\alpha}(\bar{\mathcal{C}})$ *and* $u \in C^{1,\alpha}(\overline{\Omega})$.
(v) *If* $g \in C^{1,\alpha}(\overline{\Omega})$ *and* $g|_{\partial\Omega} \equiv 0$*, then* $v \in C^{2,\alpha}(\bar{\mathcal{C}})$ *and* $u \in C^{2,\alpha}(\overline{\Omega})$.

As a consequence, we deduce the regularity of bounded weak solutions to the nonlinear problem

$$\begin{cases} A_{1/2}u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

As before, the precise meaning for (3.3) is that $v \in H^1_{0,L}(\mathcal{C})$, $v(\cdot,0)=u$, and v is a weak solution of

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C}, \\ v = 0 & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial v}{\partial \nu} = f\big(v(\cdot,0)\big) & \text{on } \Omega \times \{0\}. \end{cases} \tag{3.4}$$

Here the weak solution u is assumed to be bounded. Regularity results for subcritical and critical problems and for weak solutions which are not assumed a priori to be bounded will be proved in Section 5.

By $C_0(\overline{\Omega})$ we denote the space of continuous functions in $\overline{\Omega}$ vanishing on the boundary $\partial\Omega$. In the following result, $f(0)=0$ is assumed to ensure the $C^1(\overline{\Omega})$ regularity of solutions of (3.3).

Proposition 3.2. *Let $\alpha \in (0,1)$, Ω be a $C^{2,\alpha}$ bounded domain of $\mathbb{R}^n$, and f be a $C^{1,\alpha}$ function such that $f(0)=0$. If $u \in L^\infty(\Omega)$ is a weak solution of* (3.3)*, and thus $v \in H^1_{0,L}(\mathcal{C}) \cap L^\infty(\mathcal{C})$ is a weak solution of* (3.4)*, then $u \in C^{2,\alpha}(\overline{\Omega}) \cap C_0(\overline{\Omega})$. In addition, $v \in C^{2,\alpha}(\bar{\mathcal{C}})$.*

Proof. By (iii) of Proposition 3.1 we have that $u \in C^\alpha(\overline{\Omega})$. Next, by (iv) of Proposition 3.1 and since on $\partial\Omega \times \{0\}$, $g := f(v(\cdot,0)) = f(0) = 0$, we have $u \in C^{1,\alpha}(\overline{\Omega})$. Finally, $v \in C^{2,\alpha}(\bar{\mathcal{C}})$ and $u \in C^{2,\alpha}(\overline{\Omega})$ from (v) of Proposition 3.1 since $g = f(u)$ vanishes on $\partial\Omega$ and it is of class $C^{1,\alpha}$, since both f and u are $C^{1,\alpha}$. □

Proof of Proposition 3.1. (i) and (ii). Both statements follow immediately from Propositions 2.1 and 2.2. Simply use that $\{\varphi_k\}$ is an orthonormal basis of $L^2(\Omega)$ and that $\{\varphi_k/\lambda_k^{1/2}\}$ is an orthonormal basis of $H^1_0(\Omega)$. For part (ii), note that if $A_{1/2}u = g \in H^1_0(\Omega)$, then we have $\Delta u \in L^2(\Omega)$.

(iii) Let v be a weak solution of (3.2). We proceed with a useful method, introduced by Cabré and Solà-Morales in [5], which consists of using the auxiliary function

$$w(x,y) = \int_0^y v(x,t)\,dt \quad \text{for } (x,y) \in \mathcal{C}. \tag{3.5}$$

Since $(\Delta w)_y = 0$ in $\mathcal{C}$, we have that Δw is independent of y. Hence we can compute it on $\{y=0\}$. On $\{y=0\}$, since $w \equiv 0$, we have $\Delta w = w_{yy} = v_y$. Thus w is a solution of the Dirichlet problem

$$\begin{cases} -\Delta w(x,y) = g(x) & \text{in } \mathcal{C}, \\ w = 0 & \text{on } \partial\mathcal{C}. \end{cases} \tag{3.6}$$

We extend w to the whole cylinder $\Omega \times \mathbb{R}$ by *odd* reflection:

$$w_{od}(x,y) = \begin{cases} w(x,y) & \text{for } y \geqslant 0, \\ -w(x,-y) & \text{for } y \leqslant 0. \end{cases}$$

Moreover, we put

$$g_{od}(x,y)=\begin{cases} g(x) & \text{for } y>0,\\ -g(x) & \text{for } y<0.\end{cases}$$

Then we obtain

$$\begin{cases} -\Delta w_{od}=g_{od} & \text{in } \Omega\times\mathbb{R},\\ w_{od}=0 & \text{on } \partial\Omega\times\mathbb{R}.\end{cases} \tag{3.7}$$

Since $g_{od}\in L^q(\Omega\times(-2R,2R))$ for all $R>0$ and $1<q<\infty$, regularity for the Dirichlet problem (3.7) gives $w_{od}\in W^{2,q}(\Omega\times(-R,R))$ for all $R>0$ and $1<q<\infty$. In particular, $w\in C^{1,\alpha}(\bar{\mathcal{C}})$. Therefore, $v=w_y\in C^{\alpha}(\bar{\mathcal{C}})$ and $u\in C^{\alpha}(\overline{\Omega})$.

(iv) Choose a smooth domain H such that $\overline{\Omega}\subset H$, and let

$$g_H=\begin{cases} g & \text{in } \overline{\Omega},\\ 0 & \text{in } H\setminus\overline{\Omega}.\end{cases}$$

We have that $g_H\in C^{\alpha}(\overline{H})$, since $g|_{\partial\Omega}=0$, by assumption. Consider the weak solution v_H of

$$\begin{cases} \Delta v_H=0 & \text{in } H\times(0,\infty),\\ v_H=0 & \text{on } \partial H\times[0,\infty),\\ \dfrac{\partial v_H}{\partial\nu}=g_H(x) & \text{on } H\times\{0\}.\end{cases}$$

Consider also the auxiliary function

$$w_H(x,y)=\int_0^y v_H(x,t)\,dt \quad \text{in } \overline{H}\times[0,\infty),$$

which solves problem (3.6) with Ω and g replaced by H and g_H.

Using boundary regularity theory (but away from the corners of $H\times[0,\infty)$) for this Dirichlet problem, we see that w_H is $C^{2,\alpha}(H\times(0,\infty))$ (again, here we do not claim regularity at the corners $\partial H\times\{0\}$). Thus, $w_H\in C^{2,\alpha}(\bar{\mathcal{C}})$ (here instead we include the corners $\partial\Omega\times\{0\}$ of $\mathcal{C}$).

Consider the difference $\varphi=w_H-w$ in $\mathcal{C}$, where w is defined by (3.5). It is clear that

$$\begin{cases} \Delta\varphi=0 & \text{in } \mathcal{C},\\ \varphi=w_H & \text{on } \partial_L\mathcal{C},\\ \varphi=0 & \text{on } \Omega\times\{0\}.\end{cases}$$

We extend φ to the whole cylinder $\Omega\times\mathbb{R}$ by *odd* reflection:

$$\varphi_{od}(x,y)=\begin{cases} \varphi(x,y) & \text{for } y\geqslant 0,\\ -\varphi(x,-y) & \text{for } y\leqslant 0.\end{cases}$$

Moreover, we put

$$w_{H,od}(x,y)=\begin{cases} w_H(x,y) & \text{for } y>0,\\ -w_H(x,-y) & \text{for } y\leqslant 0.\end{cases}$$

Then we have

$$\begin{cases} \Delta\varphi_{od} = 0 & \text{in } \Omega \times \mathbb{R}, \\ \varphi_{od} = w_{H,od} & \text{on } \partial\Omega \times \mathbb{R}. \end{cases} \tag{3.8}$$

Since $w_H \in C^{2,\alpha}(\bar{\mathcal{C}})$, $w_H \equiv 0$ on $\partial\Omega \times \{0\}$, and $\partial_{yy} w_H = \partial_y v_H = -g_H = -g = 0$ on $\partial\Omega \times \{0\}$, we deduce that $w_{H,od} \in C^{2,\alpha}(\partial\Omega \times \mathbb{R})$. It follows from elliptic regularity for (3.8) that $\varphi_{od} \in C^{2,\alpha}(\overline{\Omega} \times \mathbb{R})$. Thus, $\varphi \in C^{2,\alpha}(\bar{\mathcal{C}})$, $w \in C^{2,\alpha}(\bar{\mathcal{C}})$ and $v = \partial_y w \in C^{1,\alpha}(\bar{\mathcal{C}})$.

(v) Choose a smooth bounded domain B such that $\Omega \subset \overline{B}$. B could be the same as H in (ii), for instance a ball, but we change its name for notation clarity. Since $g \in C^{1,\alpha}(\overline{\Omega})$, there exists an extension $g_B \in C^{1,\alpha}(\overline{B})$; see [15]. Consider the solution v_B of

$$\begin{cases} \Delta v_B = 0 & \text{in } B \times (0,\infty), \\ v_B = 0 & \text{on } \partial B \times [0,\infty), \\ \dfrac{\partial v_B}{\partial \nu} = g_B & \text{on } B \times \{0\}. \end{cases}$$

Consider the auxiliary function

$$w_B(x,y) = \int_0^y v_B(x,t)\,dt \quad \text{in } \overline{B} \times [0,\infty).$$

As before, from interior boundary regularity for the Dirichlet problem of the type (3.6) satisfied by w_B, we obtain that $w_B \in C^{3,\alpha}(B \times [0,\infty))$ since $g_B \in C^{1,\alpha}(\overline{B})$ (away from the corners $\partial B \times \{0\}$). Thus, $v_B \in C^{2,\alpha}(B \times [0,\infty))$. Thus, $v_B \in C^{2,\alpha}(\bar{\mathcal{C}})$. Consider the difference $\psi = v_B - v$ in $\mathcal{C}$, where v is a weak solution of (3.2). We have that $\psi = v_B - v$ satisfies

$$\begin{cases} \Delta\psi = 0 & \text{in } \mathcal{C}, \\ \psi = v_B & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial \psi}{\partial \nu} = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

We extend ψ to the whole cylinder $\overline{\Omega} \times \mathbb{R}$ now by *even* reflection:

$$\psi_{ev}(x,y) = \begin{cases} \psi(x,y) & \text{for } y \geqslant 0, \\ \psi(x,-y) & \text{for } y \leqslant 0. \end{cases}$$

Moreover, we put

$$v_{B,ev}(x,y) = \begin{cases} v_B(x,y) & \text{for } y > 0, \\ v_B(x,-y) & \text{for } y \leqslant 0. \end{cases}$$

Then, since $\frac{\partial\psi}{\partial\nu} = 0$ on $\Omega \times \{0\}$, we have

$$\begin{cases} \Delta\psi_{ev} = 0 & \text{in } \Omega \times \mathbb{R}, \\ \psi_{ev} = v_{B,ev} & \text{on } \partial\Omega \times \mathbb{R}. \end{cases}$$

Since $v_B \in C^{2,\alpha}(\bar{\mathcal{C}})$, $-\partial_y v_B = g_B = g = 0$ on $\partial\Omega \times \{0\}$, we deduce that $v_{B,ev} \in C^{2,\alpha}(\partial\Omega \times \mathbb{R})$. Therefore, it follows from classical regularity that $\psi_{ev} \in C^{2,\alpha}(\overline{\Omega} \times \mathbb{R})$. Thus, $\psi \in C^{2,\alpha}(\bar{\mathcal{C}})$, and $v \in C^{2,\alpha}(\bar{\mathcal{C}})$. □

4. Maximum principles

In this section we establish several maximum principles for $A_{1/2}$. We denote by $C_0(\overline{\Omega})$ the space of continuous functions in $\overline{\Omega}$ vanishing on the boundary $\partial\Omega$. For convenience, we state the results for functions in $C_0(\overline{\Omega}) \cap C^2(\overline{\Omega})$ (a space contained in $H_0^1(\Omega) \subset \mathcal{V}_0(\Omega)$), but this can be weakened.

The first statement is the weak maximum principle.

Lemma 4.1. *Assume that $u \in C^2(\overline{\Omega})$ satisfies*

$$\begin{cases} A_{1/2}u + c(x)u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $c \geqslant 0$ in Ω. Then, $u \geqslant 0$ in Ω.

Proof. Consider the extension $v = \text{h-ext}(u)$. If we prove that $v \geqslant 0$ in $\mathcal{C}$, then $u \geqslant 0$ in Ω. Suppose by contradiction that v is negative somewhere in $\mathcal{C}$. Then, since $\Delta v = 0$ in $\mathcal{C}$ and $v = 0$ on $\partial_L\mathcal{C}$, we deduce that v is negative somewhere in $\Omega \times \{0\}$ and that $\inf_{\mathcal{C}} v < 0$ is achieved at some point $(x_0, 0) \in \Omega \times \{0\}$. Thus, we have

$$\inf_{\mathcal{C}} v = v(x_0, 0) < 0.$$

By Hopf's lemma,

$$v_y(x_0, 0) > 0.$$

It follows

$$\frac{\partial v}{\partial \nu} = -v_y(x_0, 0) = A_{1/2}v(x_0, 0) < 0.$$

Therefore, since $c \geqslant 0$,

$$A_{1/2}v(x_0, 0) + c(x_0)v(x_0, 0) < 0.$$

This is a contradiction with the hypothesis $A_{1/2}u + c(x)u \geqslant 0$. □

The next statement is the strong maximum principle for $A_{1/2}$.

Lemma 4.2. *Assume that $u \in C^2(\overline{\Omega})$ satisfies*

$$\begin{cases} A_{1/2}u + c(x)u \geqslant 0 & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $c \in L^\infty(\Omega)$. Then, either $u > 0$ in Ω, or $u \equiv 0$ in Ω.

Proof. The proof is similar to that of Lemma 4.1. Consider $v = \text{h-ext}(u)$. We observe that $v \geqslant 0$ in $\mathcal{C}$. Suppose that $v \not\equiv 0$ but $u = 0$ somewhere in Ω. Then there exists a minimum point $(x_0, 0) \in \Omega \times \{0\}$ of v where $v(x_0, 0) = 0$. Then by Hopf's lemma we see that $A_{1/2}u(x_0) = -v_y(x_0, 0) < 0$. This implies that $A_{1/2}u(x_0) + c(x_0)u(x_0) < 0$, because of $v(x_0, 0) = u(x_0) = 0$. □

Next we establish Hopf's lemma for $A_{1/2}$, following the proof from [9].

Lemma 4.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $c \in L^\infty(\Omega)$.*

(i) *Assume that Ω is smooth and that $0 \not\equiv u \in C^2(\overline{\Omega})$ satisfies*

$$\begin{cases} A_{1/2}u + c(x)u \geqslant 0 & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, $\frac{\partial u}{\partial \nu_0} < 0$ on $\partial\Omega$, where ν_0 is the unit outer normal to $\partial\Omega$.

(ii) *Assume that $P \in \partial\Omega$ and that $\partial\Omega$ is smooth in a neighborhood of P. Let $0 \not\equiv v \in C^2(\bar{\mathcal{C}}) \cap L^\infty(\mathcal{C})$, where $\mathcal{C} = \Omega \times (0, \infty)$, satisfy*

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C}, \\ v \geqslant 0 & \text{on } \partial_L\mathcal{C}, \\ \dfrac{\partial v}{\partial \nu} + c(x)v \geqslant 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

If $v(P, 0) = 0$, then $\frac{\partial v(P,0)}{\partial \nu_0} < 0$, where ν_0 is the unit outer normal in $\mathbb{R}^n$ to $\partial\Omega$.

Proof. We follow the proof given in [9]. Note that statement (i) is a particular case of (ii). Thus, we only need to prove (ii).

Step 1. We shall first prove the lemma in the case $c \equiv 0$. Without loss of generality we may assume that $(P, 0) = P_1 = (b_1, 0, \ldots, 0) \in \partial\Omega \times \{0\}$, $b_1 > 0$ and $\nu_0 = (1, 0, \ldots, 0)$. Hence we need to prove

$$\frac{\partial v(P_1)}{\partial x_1} < 0.$$

Since Ω is smooth in a neighborhood of P, there is a half-ball in $\mathbb{R}^{n+1}_+$ included in the domain $\mathcal{C}$, such that P_1 is the only point in the closed half-ball belonging also to $\partial_L\mathcal{C}$. Let $P_2 \in \Omega \times \{0\}$ and $r > 0$ be the center and radius of such ball. Then we have $P_2 = (b_2, 0, \ldots, 0) \in \Omega \times \{0\}$. Denote

$$\begin{aligned} &B_r^+(P_2) := \big\{ z = (x, y) \bigm| |z - P_2| < |P_1 - P_2| =: r,\ y > 0 \big\} \subset \mathcal{C}, \\ &B_{r/2}^+(P_2) := \big\{ z = (x, y) \bigm| |z - P_2| < |P_1 - P_2|/2,\ y > 0 \big\}, \\ &A = B_r^+(P_2) \setminus \overline{B_{r/2}^+(P_2)}. \end{aligned}$$

Recall that $P_1 \in \partial B_r^+(P_2) \cap (\partial\Omega \times \{0\})$.

Consider the function on A:

$$\varphi(z) = \exp\bigl(-\lambda|z - P_2|^2\bigr) - \exp\bigl(-\lambda|P_1 - P_2|^2\bigr),$$

with $\lambda > 0$ to be determined later. Note that

$$\Delta\varphi = \exp\bigl(-\lambda|z - P_2|^2\bigr)\bigl\{4\lambda^2|z - P_2|^2 - 2(n+1)\lambda\bigr\}.$$

We can choose $\lambda > 0$ large enough such that $\Delta\varphi \geqslant 0$ in A.

On the other hand, by Lemma 4.2, we see that $v > 0$ in $\overline{A} \setminus \{P_1\}$. Hence, since $\varphi \equiv 0$ on $\partial B_r^+(P_2) \cap \{y > 0\}$, we can take $\varepsilon > 0$ such that

$$v - \varepsilon\varphi \geqslant 0 \quad \text{on } \partial A \cap \{y > 0\}.$$

Since $-\Delta(v - \varepsilon\varphi) \geqslant 0$ in A, and

$$-\partial_y(v - \varepsilon\varphi) = \frac{\partial v}{\partial \nu} \geqslant 0 \quad \text{on } \partial A \cap \{y = 0\}$$

(recall that $c \equiv 0$), by the maximum principle as in Lemma 4.1 we obtain

$$v - \varepsilon\varphi \geqslant 0 \quad \text{in } A.$$

Thus, from $v - \varepsilon\varphi = 0$ at P_1 we see that $\partial_{x_1}(v - \varepsilon\varphi)(P_1) \leqslant 0$. Therefore, $\partial_{x_1} v(P_1) \leqslant \varepsilon\partial_{x_1}\varphi(P_1) = -2\lambda(b_1 - b_2)e^{-\lambda|P_1 - P_2|^2} < 0$. Thus we have the desired result.

Step 2. In the case $c \not\equiv 0$, we define the function $w = v\exp(-\beta y)$ for some $\beta > 0$ to be determined. From a direct calculation, we see that

$$-\Delta w - 2\beta\partial_y w = \beta^2 w \geqslant 0 \quad \text{in } \mathcal{C}$$

and, choosing $\beta \geqslant \|c\|_{L^\infty(\Omega)}$,

$$-\partial_y w \geqslant \bigl[\beta - c(x)\bigr]w \geqslant 0 \quad \text{on } \Omega \times \{0\}.$$

Now we can apply to w the same approach as in Step 1, with Δ replaced by $\Delta + 2\beta\partial_y$, and obtain the assertion. □

Finally, we establish a maximum principle for $A_{1/2}$ in domains of small measure. Note that in part (ii) of its statement, the hypothesis on small measure is made only on the base Ω of the cylinder $\mathcal{C}$.

Proposition 4.4.

(i) *Assume that $u \in C^2(\overline{\Omega})$ satisfies*

$$\begin{cases} A_{1/2}u + c(x)u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $c \in L^\infty(\Omega)$. Then, there exists $\delta > 0$ depending only on n and $\|c^-\|_{L^\infty(\Omega)}$, such that if $|\Omega \cap \{u < 0\}| \leqslant \delta$, then $u \geqslant 0$ in Ω.

(ii) *Assume that Ω is a bounded (not necessary smooth) domain of $\mathbb{R}^n$ and $c \in L^\infty(\Omega)$. Let $v \in C^2(\overline{\mathcal{C}}) \cap L^\infty(\mathcal{C})$, where $\mathcal{C} = \Omega \times (0, \infty)$, satisfy*

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C}, \\ v \geqslant 0 & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial v}{\partial \nu} + c(x)v \geqslant 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Then, there exists $\delta > 0$ depending only on n and $\|c^-\|_{L^\infty(\Omega)}$, such that if

$$\left|\Omega \cap \left\{v(\cdot, 0) < 0\right\}\right| \leqslant \delta$$

then $v \geqslant 0$ in $\mathcal{C}$.

Proof. For part (i) of the theorem, consider $v = \text{h-ext}(u)$. We see that v satisfies the assumptions on part (ii) of the theorem. Hence, it is enough to prove part (ii). For this, let $v^- = \max\{0, -v\} \geqslant 0$. Since $v^- = 0$ on $\partial\Omega \times [0, \infty)$, we see

$$0 = \int_{\mathcal{C}} v^- \Delta v \, dx \, dy = \int_{\Omega \times \{0\}} v^- \frac{\partial v}{\partial \nu} \, dx + \int_{\mathcal{C}} |\nabla v^-|^2 \, dx \, dy.$$

Then,

$$\begin{aligned} \int_{\mathcal{C}} |\nabla v^-|^2 \, dx \, dy &= - \int_{\Omega \times \{0\}} v^- \frac{\partial v}{\partial \nu} \, dx \\ &\leqslant \int_{\Omega \times \{0\}} v^- c v \, dx = \int_{\Omega} -c\left(v^-\right)^2 dx \\ &\leqslant \int_{\Omega \cap \{v^-(\cdot,0) > 0\}} c^- \left(v^-(\cdot, 0)\right)^2 dx \\ &\leqslant \left|\Omega \cap \left\{v^-(\cdot, 0) > 0\right\}\right|^{1/n} \|c^-\|_{L^\infty(\Omega)} \left\|v^-(\cdot, 0)\right\|^2_{L^{2n/(n-1)}(\Omega)}. \end{aligned}$$

Thus, extending v^- by 0 outside $\overline{\mathcal{C}}$ we obtain an $H^1(\mathbb{R}^{n+1}_+)$ function and thus we have, if $v^- \not\equiv 0$,

$$0 < S_0 \leqslant \frac{\int_{\mathbb{R}^{n+1}_+} |\nabla v^-|^2 \, dx \, dy}{\|v^-(\cdot, 0)\|^2_{L^{2n/(n-1)}(\mathbb{R}^n)}} = \frac{\int_{\mathcal{C}} |\nabla v^-|^2 \, dx \, dy}{\|v^-(\cdot, 0)\|^2_{L^{2n/(n-1)}(\Omega)}}$$

$$\leqslant \left|\Omega \cap \left\{v^-(\cdot,0) > 0\right\}\right|^{1/n} \|c^-\|_{L^\infty(\Omega)},$$

where S_0 is the best constant of the Sobolev trace inequality in $\mathbb{R}^{n+1}_+$. If $|\Omega \cap \{v^-(\cdot,0) > 0\}|$ is small enough, we arrive at a contradiction. □

5. Subcritical case and L^∞ estimate of Brezis–Kato type

In this section, we study the nonlinear problem (1.1) with $f(u) = u^p$ in the subcritical and critical cases. In the subcritical case we look for a function $v(x, y)$ satisfying for $x \in \Omega$ and $y \in \mathbb{R}^+$,

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C} = \Omega \times (0,\infty), \\ v = 0 & \text{on } \partial_L \mathcal{C} = \partial\Omega \times [0,\infty), \\ \dfrac{\partial v}{\partial \nu} = v^p & \text{on } \Omega \times \{0\}, \\ v > 0 & \text{in } \mathcal{C}, \end{cases} \tag{5.1}$$

where ν is the unit outer normal to $\mathcal{C}$ at $\Omega \times \{0\}$ and $1 < p < 2^\sharp - 1$ if $n \geqslant 2$, or $1 < p < \infty$ if $n = 1$. If v is a solution of (5.1), then $v(x, 0) = u(x)$ is a solution of (1.1) with the nonlinearity $f(u) = u^p$.

In order to find a solution of (5.1) as stated in Theorem 1.1, we consider the following minimization problem:

$$I_0 = \inf\left\{ \int_{\mathcal{C}} \left|\nabla v(x,y)\right|^2 dx\,dy \;\middle|\; v \in H^1_{0,L}(\mathcal{C}),\ \int_{\Omega} \left|v(x,0)\right|^{p+1} dx = 1 \right\}.$$

We show that I_0 is achieved.

Proposition 5.1. *Assume that $1 < p < 2^\sharp - 1$ if $n \geqslant 2$ or $1 < p < \infty$ if $n = 1$. Then I_0 is achieved in $H^1_{0,L}(\mathcal{C})$ by a nonnegative function v.*

Proof. First, there is a function $v \in H^1_{0,L}(\mathcal{C})$ such that

$$\int_{\mathcal{C}} \left|\nabla v(x,y)\right|^2 dx\,dy < \infty \quad \text{and} \quad \int_{\Omega} \left|v(x,0)\right|^{p+1} dx = 1.$$

In fact, it suffices to take any C^∞ function with compact support in $\Omega \times [0,\infty)$ and not identically zero on $\Omega \times \{0\}$, and multiply it by an appropriate constant. Next we complete the proof by weak lower semi-continuity of the Dirichlet integral and by the compact embedding property in Lemma 2.5. Finally, note that $|v| \geqslant 0$ is a nonnegative minimizer if v is a minimizer. □

To establish the regularity of the minimizer just obtained, we prove an L^∞-estimate of Brezis–Kato type by the technique of bootstrap for subcritical or critical nonlinear problems. Let g_0 be a Carathéodory function in $\Omega \times \mathbb{R}$ satisfying the growth condition

$$\left|g_0(x,s)\right| \leqslant C\left(1 + |s|^p\right) \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}, \tag{5.2}$$

where Ω is a smooth domain in $\mathbb{R}^n$, $1 \leqslant p \leqslant \frac{n+1}{n-1}$ if $n \geqslant 2$, or $1 \leqslant p < \infty$ if $n = 1$. We consider the problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C} = \Omega \times (0, \infty), \\ v = 0 & \text{on } \partial_L \mathcal{C} = \partial\Omega \times [0, \infty), \\ \dfrac{\partial v}{\partial \nu} = g_0(\cdot, v) & \text{on } \Omega \times \{0\}. \end{cases} \tag{5.3}$$

Theorem 5.2. *Let $v \in H^1_{0,L}(\mathcal{C})$ be a weak solution of* (5.3) *and assume the growth condition* (5.2) *for g_0, with $1 \leqslant p \leqslant \frac{n+1}{n-1}$ if $n \geqslant 2$, or $1 \leqslant p < \infty$ if $n = 1$. Then, $v(\cdot, 0) \in L^\infty(\Omega)$.*

Proof. The proof follows the one of Brezis–Kato for the Laplacian. First of all, let us rewrite the condition on g_0 as

$$\left|g_0(x, v)\right| \leqslant a(x)\left(1 + \left|v(x, 0)\right|\right)$$

with the function

$$a(x) := \frac{|g_0(x, v(x, 0))|}{1 + |v(x, 0)|}$$

which satisfies

$$0 \leqslant a \leqslant C\left(1 + \left|v(\cdot, 0)\right|^{p-1}\right) \in L^n(\Omega),$$

since $v \in H^1_{0,L}(\mathcal{C})$, $v(\cdot, 0) \in L^{\frac{2n}{n-1}}(\Omega)$ and $p - 1 \leqslant \frac{2}{n-1}$.

Denote

$$B_r^+ = \left\{(x, y) \mid \left|(x, y)\right| < r \text{ and } y > 0\right\}.$$

For $\beta \geqslant 0$ and $T > 1$, let $\varphi_{\beta,T} = v v_T^{2\beta} \in H^1_{0,L}(\mathcal{C})$ with $v_T = \min\{|v|, T\}$. Denote

$$D_T = \left\{(x, y) \in \mathcal{C} \mid \left|v(x, y)\right| < T\right\}.$$

By direct computation, we see

$$\int_{\mathcal{C}} \left|\nabla\left(v v_T^\beta\right)\right|^2 dx\, dy = \int_{\mathcal{C}} v_T^{2\beta} |\nabla v|^2 \, dx\, dy + \int_{D_T} \left(2\beta + \beta^2\right) |v|^{2\beta} |\nabla v|^2 \, dx\, dy.$$

Multiplying (5.3) by $\varphi_{\beta,T}$ and integrating by parts, we obtain

$$\int_{\mathcal{C}} v_T^{2\beta}|\nabla v|^2\,dx\,dy+2\beta\int_{D_T}|v|^{2\beta}|\nabla v|^2\,dx\,dy$$
$$=\int_{\mathcal{C}}\nabla v\nabla\big(vv_T^{2\beta}\big)\,dx\,dy=\int_{\Omega\times\{0\}}g_0(x,v)vv_T^{2\beta}\,dx$$
$$\leqslant\int_{\Omega\times\{0\}}a(x)\big(1+|v|\big)^2v_T^{2\beta}\,dx.$$

Combining these facts, we have

$$\int_{\mathcal{C}}\big|\nabla\big(vv_T^{\beta}\big)\big|^2\,dx\,dy\leqslant C(\beta+1)\int_{\Omega\times\{0\}}a(x)\big(1+|v|^2\big)v_T^{2\beta}\,dx,$$

where C denotes different constants independent of T and of β. By Lemma 2.4, we deduce

$$\bigg(\int_{\Omega\times\{0\}}\big|vv_T^{\beta}\big|^{2^\sharp}\,dx\bigg)^{2/2^\sharp}\leqslant C(\beta+1)\int_{\Omega\times\{0\}}a(x)\big(1+|v|^2\big)v_T^{2\beta}\,dx. \tag{5.4}$$

Assume that $|v(\cdot,0)|^{\beta+1}\in L^2(\Omega)$ for some $\beta\geqslant 0$. Then we obtain that $\int_{\Omega\times\{0\}}|v|^2v_T^{2\beta}\,dx$ and $\int_{\Omega\times\{0\}}v_T^{2\beta}\,dx$ are bounded uniformly in T. In what follows, let C denote constants independent of T—but that may depend on β and $\|v(\cdot,0)^{\beta+1}\|_{L^2(\Omega)}$. Given $M_0>0$, we have

$$\int_{\Omega\times\{0\}}a|v|^2v_T^{2\beta}\,dx\leqslant M_0\int_{\Omega\times\{0\}}|v|^2v_T^{2\beta}\,dx+\int_{\{a\geqslant M_0\}}a|v|^2v_T^{2\beta}\,dx$$
$$\leqslant CM_0+\bigg(\int_{\{a\geqslant M_0\}}a^n\,dx\bigg)^{1/n}\bigg(\int_{\Omega\times\{0\}}\big|vv_T^{\beta}\big|^{2^\sharp}\,dx\bigg)^{2/2^\sharp}$$
$$\leqslant CM_0+\varepsilon(M_0)\bigg(\int_{\Omega\times\{0\}}\big|vv_T^{\beta}\big|^{2^\sharp}\,dx\bigg)^{2/2^\sharp},$$

where $\varepsilon(M_0)=(\int_{\{a\geqslant M_0\}}a^n\,dx)^{1/n}\to 0$ as $M_0\to\infty$. Note that we can deal with $\int_{\Omega\times\{0\}}av_T^{2\beta}\,dx$ in the analogue procedure. Therefore, we deduce from the last inequalities and (5.4), taking M_0 large enough so that $C(\beta+1)\varepsilon(M_0)=\frac{1}{2}$, that

$$\bigg(\int_{\Omega\times\{0\}}\big|vv_T^{\beta}\big|^{2^\sharp}\,dx\bigg)^{2/2^\sharp}\leqslant C(1+M_0). \tag{5.5}$$

Thus letting $T\to\infty$, since C is independent of T, we obtain that $|v(\cdot,0)|^{\beta+1}\in L^{2^\sharp}(\Omega)$. This conclusion followed simply from assuming $|v(\cdot,0)|^{\beta+1}\in L^2(\Omega)$.

Hence, by iterating $\beta_0 = 0$, $\beta_i + 1 = (\beta_{i-1} + 1)\frac{n}{n-1}$ if $i \geqslant 1$ in (5.5), we conclude that $v(\cdot, 0) \in L^q(\Omega)$ for all $q < \infty$. Finally, the proof of part (iii) in Proposition 3.1—which only uses $g \in L^q(\Omega)$ for all $q < \infty$ and not $g \in L^\infty(\Omega)$—applied with $g(x) = g_0(x, v(x, 0))$, which satisfies $|g| \leqslant C(1 + |v(\cdot, 0)|^p) \in L^q(\Omega)$ for all $q < \infty$, leads to $v(\cdot, 0) \in C^\alpha(\overline{\Omega}) \subset L^\infty(\Omega)$. $\Box$

Proof of Theorem 1.1. Proposition 5.1 gives the existence of a weak nonnegative solution v to (5.1) after multiplying the nonnegative minimizer of I_0 by a constant to take care of the Lagrange multiplier. Then, Theorem 5.2 gives that $v(\cdot, 0) \in L^\infty(\Omega)$. Next, Proposition 3.2 gives that $u \in C^{2,\alpha}(\overline{\Omega})$, since $f(s) = |s|^p$ is a $C^{1,\alpha}$ function for some $\alpha \in (0, 1)$. Finally, the strong maximum principle, Lemma 4.2, leads to $u > 0$ in Ω. $\Box$

6. A priori estimates for positive solutions

In this section we prove Theorem 1.3. Namely, we establish a priori estimates of Gidas–Spruck type for weak solutions of

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C} = \Omega \times (0, \infty) \subset \mathbb{R}^{n+1}_+, \\ v = 0 & \text{on } \partial_L \mathcal{C} = \partial\Omega \times [0, \infty), \\ \dfrac{\partial v}{\partial \nu} = v^p & \text{on } \Omega \times \{0\}, \\ v > 0 & \text{in } \mathcal{C}, \end{cases} \tag{6.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $n \geqslant 2$, and $1 < p < \frac{n+1}{n-1}$.

For this, we need two nonlinear Liouville theorems for problems involving the square root of the Laplacian in unbounded domains—one in the whole space, another in the half-space. The first one was proved by Y.Y. Li, Zhang and Zhu in [18,17] and Ou in [22]. Its statement is the following—and it is equivalent to Theorem 1.4 in the Introduction.

Theorem 6.1. *(See [18,17,22].) For $n \geqslant 2$ and $1 < p < 2^\sharp - 1 = \frac{n+1}{n-1}$, there exists no weak solution of the problem*

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial v}{\partial \nu} = v^p & \text{on } \partial\mathbb{R}^{n+1}_+, \\ v > 0 & \text{in } \mathbb{R}^{n+1}_+. \end{cases} \tag{6.2}$$

We need to prove an analogue nonlinear Liouville type result involving the square root of $-\Delta$ with Dirichlet boundary value in the half-space. This is Theorem 1.5 of the Introduction and Proposition 6.3 in this section. As we will see, this nonlinear Liouville theorem in $\mathbb{R}^n_+$ will be first reduced to the one-dimensional case $\mathbb{R}_+$, by using the moving planes method. After this, we prove that there exists no positive bounded solution for the nonlinear Neumann boundary problem in the quarter $\mathbb{R}^2_{++}$, which corresponds to the nonlinear Liouville theorem involving the square root of $-\Delta$ with Dirichlet boundary value in the half-line; see Proposition 6.4. To complete the proof of Theorem 1.5 we will use the following Liouville theorem in dimension $n + 1 = 2$.

Proposition 6.2. *(See [9].) Suppose that v weakly solves*

$$\begin{cases} -\Delta v \geqslant 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial v}{\partial \nu} \geqslant 0 & \text{on } \partial\mathbb{R}^2_+, \\ v \geqslant 0 & \text{in } \mathbb{R}^2_+. \end{cases} \tag{6.3}$$

Then, v is a constant.

As usual, very strong Liouville theorems (but quite simple to prove) hold in low dimensions, but not in higher ones. Compare (6.3) in low dimensions for supersolutions of the homogeneous linear problem with (6.2) for solutions of a precise nonlinear problem. The proof of Proposition 6.2 in [9] compared in an appropriate way the solution v with $\log(|\cdot|)$. For completeness, we give here an alternative proof.

Proof of Proposition 6.2. Replacing v by $v - \inf_{\mathbb{R}^2_+} v \geqslant 0$, we may assume $\inf_{\mathbb{R}^2_+} v = 0$. Letting $w = 1 - v$, we have

$$\begin{cases} -\Delta w \leqslant 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial w}{\partial \nu} \leqslant 0 & \text{on } \partial\mathbb{R}^2_+, \\ w \leqslant 1 & \text{in } \mathbb{R}^2_+. \end{cases} \tag{6.4}$$

In addition, $\sup_{\mathbb{R}^2_+} w = 1$. Let $\xi_R \in C^\infty(\mathbb{R}^2)$ be a function with compact support in $B_{2R}(0)$, equal to 1 in $B_R(0)$, and with $|\nabla \xi_R| \leqslant \frac{C}{R}$. Let

$$D^+_{R,2R} := \{(x,y) \in \mathbb{R}^2 \mid R \leqslant |(x,y)| \leqslant 2R,\ y > 0\}.$$

Multiplying the first equation in (6.4) by $w^+\xi_R^2$, integrating in $\mathbb{R}^2_+$ and using the Neumann condition and $w^+ \leqslant 1$, we see that

$$\int_{\mathbb{R}^2_+} \xi_R^2 |\nabla w^+|^2 \leqslant 2 \int_{D^+_{R,2R}} \xi_R \nabla \xi_R w^+ \nabla w^+ \tag{6.5}$$

$$\leqslant C\Bigg(\int_{D^+_{R,2R}} |\nabla \xi_R|^2\Bigg)^{1/2}\Bigg(\int_{D^+_{R,2R}} \xi_R^2 |\nabla w^+|^2\Bigg)^{1/2}$$

$$\leqslant C\Bigg(\int_{D^+_{R,2R}} \xi_R^2 |\nabla w^+|^2\Bigg)^{1/2}. \tag{6.6}$$

This leads, letting $R \uparrow \infty$, to $\int_{\mathbb{R}^2_+} |\nabla w^+|^2 < \infty$. As a consequence of this, the integral in (6.6) tends to zero as $R \to \infty$. Thus, by (6.5) and (6.6),

$$\int_{\mathbb{R}^2_+} \left|\nabla w^+\right|^2 = 0.$$

Thus, w^+ is constant, and since $\sup_{\mathbb{R}^2_+} w = 1$, we conclude $w \equiv 1$. □

Proposition 6.3. *Let $n \geqslant 2$ and*

$$\mathbb{R}^{n+1}_{++} = \left\{z = (x_1, x_2, \ldots, x_n, y) \;\middle|\; x_n > 0,\ y > 0\right\}.$$

Assume that v is a classical solution of

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_{++}, \\ v = 0 & \text{on } \{x_n = 0,\ y \geqslant 0\}, \\ \dfrac{\partial v}{\partial \nu} = v^p & \text{on } \{x_n > 0,\ y = 0\}, \\ v > 0 & \text{in } \mathbb{R}^{n+1}_{++}, \end{cases} \tag{6.7}$$

where $1 \leqslant p \leqslant \frac{n+1}{n-1}$. Then, v depends only on x_n and y.

Proof. We shall follow the steps of [14]. Let $e_n = (0, \ldots, 0, 1, 0)$ and $N = n + 1$. Consider the conformal transformation

$$\bar{z} = T(z) = \frac{z + e_n}{|z + e_n|^2}$$

and the Kelvin transformation w of v

$$w(\bar{z}) = |z + e_n|^{N-2} v(z) = |\bar{z}|^{2-N} v(z).$$

Denote $B^+_{1/2}(\frac{e_n}{2}) := \{\bar{z} = (\bar{x}, \bar{y}) \mid |\bar{z} - \frac{1}{2}e_n| < \frac{1}{2},\ \bar{y} > 0\}$, $S^+_{1/2}(\frac{e_n}{2}) := \partial B^+_{1/2}(\frac{e_n}{2}) \cap \{\bar{y} > 0\}$, $\Gamma_{0,1/2} := \partial B^+_{1/2}(\frac{e_n}{2}) \cap \{\bar{y} = 0\}$.

Note that, through T, $\mathbb{R}^{n+1}_{++} = \{x_n > 0,\ y > 0\}$ gets mapped into the half-ball $B^+_{1/2}(\frac{e_n}{2})$, the boundary $\{x_n > 0,\ y = 0\}$ becomes the ball $\Gamma_{0,1/2}$, $\{x_n = 0,\ y \geqslant 0\}$ goes to the half-sphere $S^+_{1/2}(\frac{e_n}{2})$, and the infinity goes to $\bar{z} = 0$.

We see that w satisfies

$$\begin{cases} \Delta w = 0 & \text{in } B^+_{1/2}\left(\dfrac{e_n}{2}\right), \\ w = 0 & \text{on } S^+_{1/2}\left(\dfrac{e_n}{2}\right), \\ \dfrac{\partial w(\bar{z})}{\partial \nu} = |\bar{z}|^{p(N-2)-N} w^p(\bar{z}) & \text{on } \Gamma_{0,1/2}, \\ w > 0 & \text{in } B^+_{1/2}\left(\dfrac{e_n}{2}\right). \end{cases}$$

Since $|\bar{z}|^{p(n-1)-(n+1)}$ is nonincreasing in the $\bar{z}_i$ direction for all $i = 1, \dots, n-1$ (in fact, in any direction orthogonal to the $\bar{z}_n$-axis), the moving planes method used as in [9] gives that w is symmetric about all the $\bar{z}_i$-axis for $i = 1, \dots, n-1$. This leads to $w = w(|\bar{z}'|, \bar{z}_n, \bar{y})$, where $\bar{z}' = (\bar{z}_1, \dots, \bar{z}_{n-1})$ and hence $v = v(|x'|, x_n, y)$. Now, since we may perform the Kelvin's transform with respect to any point $(-x_0', -1, 0)$—and not only with respect to $x_0' = 0$ as before—we conclude that $v = v(x_n, y)$ as claimed. □

Proposition 6.4. *Assume that f is a $C^{1,\alpha}$ function for some $\alpha \in (0,1)$, such that $f > 0$ in $(0, \infty)$ and $f(0) = 0$. Let C be a positive constant. Then there is no bounded solution of the problem*

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_{++} = \{x > 0,\ y > 0\}, \\ v = 0 & \text{on } \{x = 0,\ y \geqslant 0\}, \\ \dfrac{\partial v}{\partial \nu} = f(v) & \text{on } \{x > 0,\ y = 0\}, \\ 0 < v \leqslant C & \text{in } \mathbb{R}^2_{++}. \end{cases} \tag{6.8}$$

Proof. We use some tools developed in [5].

Suppose by contradiction that there is such solution v. First, we claim that $v(x,0) \to 0$ as $x \to \infty$. Suppose by contradiction that there exists a sequence $a_m \to \infty$ ($m \to \infty$) such that $v(a_m, 0) \to \alpha > 0$. Let $v_m(x,y) := v(x + a_m, y)$. It is clear that v_m is a solution of (6.8) in $U_m := \{(x,y) \mid x > -a_m,\ y > 0\}$. Moreover, $v_m(0,0) = v(a_m, 0) \to \alpha$. Therefore there exists a subsequence, still denoted by v_m, such that $v_m \to w$ in $C^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ as $m \to \infty$, and w is a solution of

$$\begin{cases} \partial_{xx} w + \partial_{yy} w = 0 & \text{in } \mathbb{R}^2_+ = \{(x,y) \mid y > 0\}, \\ \dfrac{\partial w}{\partial \nu} = f(w) \geqslant 0 & \text{on } \{y = 0\}, \\ 0 \leqslant w \leqslant C & \text{in } \{y > 0\}. \end{cases} \tag{6.9}$$

Notice that

$$w(0,0) = \alpha > 0.$$

On the other hand, by Proposition 6.2 we know that w is identically constant. This is impossible due to the nonlinear Neumann condition, since $f > 0$ in $(0, \infty)$ and $f(w(0,0)) = f(\alpha) > 0$. We conclude the claim, that is, $v(x,0) \to 0$ as $x \to +\infty$.

Note that we can reflect the function v with respect to $\{x = 0,\ y > 0\}$, $\tilde{v}(x,y) = -v(-x,y)$ for $x < 0$, and obtain a bounded harmonic function $\tilde{v}$ in all $\mathbb{R}^2_+ = \{y > 0\}$, since $v \equiv 0$ on $\{x = 0,\ y > 0\}$. Applying interior gradient estimates to the bounded harmonic function $\tilde{v}$ in the ball $B_t(x,t) \subset \mathbb{R}^2_+$, we obtain

$$\left|\nabla v(x,t)\right| \leqslant \frac{C\|v\|_\infty}{t} \leqslant \frac{C}{t}, \quad \text{for all } t > \frac{1}{2},\ x > 0.$$

On the other hand, by the results of [5] applied to the solution $\tilde{v}$ in $\mathbb{R}^2_+$ (or equivalently by the proof of Proposition 3.2 of this paper; note that $f(0) = 0$), we have that $|\nabla v|$ and $|D^2 v|$ are bounded in $\overline{\mathbb{R}^2_{++}} \cap \{0 \leqslant y \leqslant 1\}$. We conclude that $|\nabla v|$ and $|D^2 v|$ are bounded in $\overline{\mathbb{R}^2_{++}}$ and

$$\left|\nabla v(x,t)\right| \leqslant \frac{C}{t+1}, \quad \text{for all } t>0,\ x>0.$$

Using interior estimates for harmonic functions as before, but now with the partial derivatives of v instead of v, it follows that

$$\left|D^2 v(x,t)\right| \leqslant \frac{C}{t^2+1}, \quad \text{for all } t>0,\ x>0.$$

Moreover, we have

$$\left|\frac{\partial}{\partial x}\left\{\frac{|\partial_x v(x,t)|^2 - |\partial_y v(x,t)|^2}{2}\right\}\right| \leqslant \frac{C}{t^3+1}.$$

By these facts, we see that the function

$$\Phi(x) := \int_0^{+\infty} \frac{|\partial_x v(x,t)|^2 - |\partial_y v(x,t)|^2}{2}\, dt$$

is well defined and $\frac{d\Phi}{dx}$ is also.

Using that $\lim_{t\to\infty} |\nabla v(x,t)| = 0$, we obtain, for $F(v) = \int_0^v f(s)\, ds$,

$$\begin{aligned}
&\frac{d}{dx}\big[\Phi(x) + F\big(v(x,0)\big)\big] \\
&\quad = \int_0^{+\infty} [\partial_{xx} v \partial_x v - \partial_y v \partial_{xy} v](x,t)\, dt + \big[f(v)\partial_x v\big](x,0) \\
&\quad = \big[\partial_y v \partial_x v + f(v)\partial_x v\big](x,0) = 0,
\end{aligned}$$

thanks to the harmonicity of v and the Neumann boundary condition. This leads to the Hamiltonian-type identity

$$\Phi(\cdot) + F\big(v(\cdot,0)\big) \text{ is identically constant in } (0,+\infty).$$

Furthermore, using that $\lim_{x\to+\infty} v(x,0) = 0$, and that $\lim_{x\to+\infty} v(x,y) = 0$ uniformly in compact sets in y (we can prove this by the same previous argument leading to $\lim_{x\to+\infty} v(x,0) = 0$), together with the above bounds for $|\nabla v(x,y)|$ for y large, we deduce

$$\lim_{x\to+\infty} \Phi(x) = 0.$$

From all these we obtain

$$\Phi(x) + F\big(v(x,0)\big) \equiv 0, \quad \text{for } x>0.$$

Since $v=0$ and thus $\partial_y v = 0$ along the y-axis, we see by the definition of $\Phi(0)$ that

$$0 = \Phi(0) + F\big(v(0,0)\big) = \Phi(0) = \frac{1}{2}\int_0^{+\infty} |\partial_x v|^2(0,t)\,dt.$$

This implies that $\partial_x v = 0$ on $\{x = 0,\ y > 0\}$, which contradicts Hopf's lemma. Thus, the contradiction means that there is no positive bounded solution of the problem. $\square$

Before proving Theorems 1.5 and 1.3, let us make some comments.

Remark 6.5. Theorem 1.5 is still open without the boundedness assumption on v.

In this respect, let us give some examples of problems in the quarter plane $\mathbb{R}^2_{++}$. The function $v(x, y) = x$ is an unbounded solution of the problem

$$\begin{cases} -\Delta v = 0,\ v \geqslant 0 & \text{in } \mathbb{R}^2_{++},\\ v = 0 & \text{on } \{x = 0,\ y > 0\},\\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \{x > 0,\ y = 0\}. \end{cases}$$

This tells us that the result of Proposition 6.2 (which did not require boundedness of the solution in the half-plane) does not hold in the quarter plane.

On the other hand, it is clear that $v(x, y) = \frac{\pi}{2}\arctan\frac{x}{y+1}$ satisfies $\Delta v = 0$ and $-\partial_y v|_{y=0} = \frac{\pi x}{2(1+x^2)} \geqslant 0$ for $x > 0$. Hence, there exists a bounded harmonic function in the quarter plane $\mathbb{R}^2_{++}$ such that

$$\begin{cases} -\Delta v = 0,\ v \geqslant 0 & \text{in } \mathbb{R}^2_{++},\\ v = 0 & \text{on } \{x = 0,\ y > 0\},\\ \dfrac{\partial v}{\partial \nu} \geqslant 0 & \text{on } \{x > 0,\ y = 0\}. \end{cases}$$

Thus the nonlinear condition $\frac{\partial v}{\partial \nu} = v^p$ on $\{y = 0\}$ is important in Theorem 1.5.

Proof of Theorem 1.5. It follows from Propositions 6.3 and 6.4. $\square$

Proof of Theorem 1.3. We know by Theorem 5.2 and Proposition 3.2 that all weak solutions u of (1.1), with f as in Theorem 1.3, belong to $C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$. Assume by contradiction that the theorem is not true and hence that there is a sequence u_m of solutions of (1.1) with

$$K_m = \|u_m\|_{L^\infty(\Omega)} \to \infty.$$

Since $v_m = \text{h-ext}(u_m)$ is a positive harmonic function in $\mathcal{C}$ vanishing on $\partial_L\mathcal{C}$, we have that v_m has also K_m as maximum in $\mathcal{C}$ and that it is attained at a point $(x_m, 0) \in \Omega \times \{0\}$. Let

$$\Omega_m = K_m^{p-1}(\Omega - x_m)$$

and define

$$\tilde{v}_m(x, y) = K_m^{-1} v\big(x_m + K_m^{1-p}x, K_m^{1-p}y\big), \quad x \in \Omega_m,\ y > 0.$$

We have that $\|\tilde{v}_m\|_{L^\infty(\Omega_m\times(0,\infty))} \leqslant 1$ and

$$\begin{cases} \Delta\tilde{v}_m = 0 & \text{in } \mathcal{C}_m := \Omega_m \times (0,\infty), \\ \tilde{v}_m = 0 & \text{on } \partial\Omega_m \times (0,\infty), \\ \dfrac{\partial\tilde{v}_m}{\partial\nu} = \tilde{v}_m^p & \text{on } \Omega_m \times \{0\}, \\ \tilde{v}_m > 0 & \text{in } \mathcal{C}_m. \end{cases} \tag{6.10}$$

Notice that

$$\tilde{v}_m(0,0) = 1.$$

Let

$$d_m = \operatorname{dist}(x_m, \partial\Omega).$$

Two cases may occur as $m \to \infty$; either case (a):

$$K_m^{p-1} d_m \to \infty$$

for a subsequence still denoted as before, or case (b):

$$K_m^{p-1} d_m \text{ is bounded.}$$

If case (a) occurs, we have that $B_{K_m^{p-1}d_m}(0) = K_m^{p-1}B_{d_m}(0) \subset \Omega_m$ and that $K_m^{p-1}d_m \to \infty$. By local compactness (Arzelà–Ascoli) of bounded solutions to (6.10) (recall $\|\tilde{v}_m\|_{L^\infty(\Omega_m)} \leqslant 1$), through a subsequence, we obtain a solution $\tilde{v}$ of problem (6.2) in all of $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$—note that $\tilde{v}_m(0,0) = 1$ leads to $\tilde{v}(0,0) = 1$ and hence $\tilde{v} \not\equiv 0$ and $\tilde{v} > 0$. This is a contradiction to Theorem 6.1.

Assume now that case (b), $K_m^{p-1}d_m$ is bounded, occurs. Note first that since the right-hand side of problem (6.1) for v_m satisfies $|v_m|^p = v_m^p \leqslant K_m^p$, we deduce from the proofs of Proposition 3.1 (iii) and (iv) that $\|\nabla u_m\|_{L^\infty(\Omega)} \leqslant CK_m^p$ for a constant C independent of m. Now, since $u_m|_{\partial\Omega} \equiv 0$ (where $u_m = v_m(\cdot,0)$), we get

$$K_m = v_m(x_m,0) \leqslant \|\nabla u_m\|_{L^\infty(\Omega)} \operatorname{dist}(x_m,\partial\Omega) \leqslant CK_m^p d_m.$$

We deduce that

$$0 < c \leqslant K_m^{p-1} d_m$$

for some positive constant c. Thus, in this case (b), we may assume that, up to a subsequence,

$$K_m^{p-1} d_m \to a \in (0,\infty) \tag{6.11}$$

for some constant $a > 0$.

We deduce that, up to a certain rotation of $\mathbb{R}^n$ for each index m, since we have (6.11), $K_m^{p-1} \to \infty$, $d_m \to 0$, and that $B_{K_m^{p-1}d_m}(0)$ is tangent to $\partial\Omega_m$, the domains Ω_m converge to the half-space

$\mathbb{R}^n_+ = \{x_n > -a\}$. Thus, through a subsequence of $\tilde{v}_m$, we obtain a solution $\tilde{v}$ of problem (6.7) in $\mathbb{R}^{n+1}_{++} = \{x_n > -a,\ y > 0\}$ with $\tilde{v}$ bounded by 1 and $\tilde{v} > 0$ (since $\tilde{v}_m(0,0) = 1$ for all m). This is a contradiction with Theorem 1.5. $\square$

Remark 6.6. From Theorem 1.3 we have a priori bounds for solutions of problem (1.1) with $f(u) = u^p$, $1 < p < \frac{n+1}{n-1}$. As a consequence, by using blow-up techniques and topological degree theory, one can obtain existence of positive solutions for related problems—for instance, for nonlinearities $f(x,u)$ of power type, as well as other boundary conditions. See Gidas and Spruck [14] for some of these applications when the operator is the classical Laplacian.

7. Symmetry of solutions

The goal of this section is to prove a symmetry result of Gidas–Ni–Nirenberg type for positive solutions of nonlinear problems involving the operator $A_{1/2}$, as stated in Theorem 1.6, by using the moving planes method. For this, we work with the equivalent local problem (1.8) and derive the following.

Theorem 7.1. *Assume that Ω is a bounded smooth domain of $\mathbb{R}^n$ which is convex in the x_1 direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let f be Lipschitz continuous and let $v \in C^2(\overline{\mathcal{C}})$ be a solution of* (1.8)*, where $\mathcal{C} = \Omega \times (0, +\infty)$. Then, v is symmetric with respect to x_1, i.e., $v(-x_1, x', y) = v(x_1, x', y)$ for all $(-x_1, x', y) \in \mathcal{C}$. In addition, $\frac{\partial v}{\partial x_1} < 0$ for $x_1 > 0$.*

Proof of Theorems 1.6 and 7.1. It suffices to prove Theorem 7.1. From it, Theorem 1.6 follows immediately.

Let $x = (x_1, x') \in \Omega$ and $\lambda > 0$. Consider the sets

$$\Sigma_\lambda = \big\{(x_1, x') \in \Omega \bigm| x_1 > \lambda\big\} \quad \text{and} \quad T_\lambda = \big\{(x_1, x') \in \Omega \bigm| x_1 = \lambda\big\}.$$

For $x \in \Sigma_\lambda$, define $x_\lambda = (2\lambda - x_1, x')$. By the hypotheses on the domain Ω we see that

$$\{x_\lambda \mid x \in \Sigma_\lambda\} \subset \Omega.$$

Recall that $v \in C^2(\overline{\mathcal{C}})$ is a solution of

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{C} = \Omega \times (0, \infty), \\ v = 0 & \text{on } \partial_L \mathcal{C} = \partial\Omega \times [0, \infty), \\ \dfrac{\partial v}{\partial \nu} = f(v) & \text{on } \Omega \times \{0\}, \\ v > 0 & \text{in } \mathcal{C}. \end{cases}$$

For $(x, y) \in \Sigma_\lambda \times [0, \infty)$, let us define

$$v_\lambda(x, y) = v(x_\lambda, y) = v\big(2\lambda - x_1, x', y\big)$$

and

$$w_\lambda(x,y) = (v_\lambda - v)(x,y).$$

Note that v_λ satisfies

$$\begin{cases} \Delta v_\lambda = 0 & \text{in } \Sigma_\lambda \times (0,\infty), \\ v_\lambda \geqslant 0 & \text{on } (\partial\Omega \cap \overline{\Sigma}_\lambda) \times (0,\infty), \\ \dfrac{\partial v_\lambda}{\partial \nu} = f(v_\lambda) & \text{on } \Sigma_\lambda \times \{0\}. \end{cases}$$

Thus, since $\partial\Sigma_\lambda = (\partial\Omega \cap \overline{\Sigma}_\lambda) \cup T_\lambda$ and $w_\lambda \equiv 0$ on T_λ, we have that w_λ satisfies

$$\begin{cases} \Delta w_\lambda = 0 & \text{in } \Sigma_\lambda \times (0,\infty), \\ w_\lambda \geqslant 0 & \text{on } (\partial\Sigma_\lambda) \times (0,\infty), \\ \dfrac{\partial w_\lambda}{\partial \nu} + c_\lambda(x) w_\lambda = 0 & \text{on } \Sigma_\lambda \times \{0\}, \end{cases} \tag{7.1}$$

where

$$c_\lambda(x,0) = -\frac{f(v_\lambda) - f(v)}{v_\lambda - v}.$$

Note that $c_\lambda(x,0) \in L^\infty(\Sigma_\lambda)$.

Let $\lambda^* = \sup\{\lambda \mid \Sigma_\lambda \neq \emptyset\}$ and let $\varepsilon > 0$ be a small number. If $\lambda \in (\lambda^* - \varepsilon, \lambda^*)$, then Σ_λ has small measure and we have, by part (ii) of Proposition 4.4 (applied with Ω replaced with Σ_λ), that

$$w_\lambda \geqslant 0 \quad \text{in } \Sigma_\lambda \times (0,\infty).$$

Note here that Σ_λ is not a smooth domain but that part (ii) of Proposition 4.4 does not require smoothness of the domain. By the strong maximum principle, Lemma 4.2, for problem (7.1) we see that w_λ is identically equal to zero or strictly positive in $\Sigma_\lambda \times (0,\infty)$. Since $\lambda > 0$, we have $w_\lambda > 0$ in $(\partial\Omega \cap \partial\Sigma_\lambda) \times (0,\infty)$, and hence we conclude that $w_\lambda > 0$ in $\Sigma_\lambda \times (0,\infty)$.

Let $\lambda_0 = \inf\{\lambda > 0 \mid w_\lambda \geqslant 0 \text{ in } \Sigma_\lambda \times (0,\infty)\}$. We are going to prove that $\lambda_0 = 0$. Suppose that $\lambda_0 > 0$ by contradiction. First, by continuity, we have $w_{\lambda_0} \geqslant 0$ in $\Sigma_{\lambda_0} \times (0,\infty)$. Then, as before, we deduce $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0} \times (0,\infty)$. Next, let $\delta > 0$ be a constant and $K \subset \Sigma_{\lambda_0}$ be a compact set such that $|\Sigma_{\lambda_0} \setminus K| \leqslant \delta/2$. We have $w_{\lambda_0}(\cdot,0) \geqslant \eta > 0$ in K for some constant η, since K is compact. Thus, we obtain that $w_{\lambda_0-\varepsilon}(\cdot,0) > 0$ in K and that $|\Sigma_{\lambda_0-\varepsilon} \setminus K| \leqslant \delta$ for ε small enough.

Now we apply again part (ii) of Proposition 4.4 in $\Sigma_{\lambda_0-\varepsilon} \times (0,\infty)$ to the function $w_{\lambda_0-\varepsilon}$. We know that $w_{\lambda_0-\varepsilon}(\cdot,0) \geqslant 0$ in K, and hence $\{w_{\lambda_0-\varepsilon} < 0\} \subset \Sigma_{\lambda_0-\varepsilon} \setminus K$, which has measure at most δ. We take δ to be the constant of part (ii) of Proposition 4.4. We deduce that

$$w_{\lambda_0-\varepsilon} \geqslant 0 \quad \text{in } \Sigma_{\lambda_0-\varepsilon} \times (0,\infty).$$

This is a contradiction to the definition of λ_0. Thus, $\lambda_0 = 0$.

We have proved, letting $\lambda \downarrow \lambda_0 = 0$ that

$$v\big(-x_1, x', y\big) \geqslant v\big(x_1, x', y\big) \quad \text{in } \big(\Omega \cap \{x_1 > 0\}\big) \times (0,\infty)$$

and, since $w_\lambda = 0$ on T_λ,

$$\partial_{x_1} v = -\frac{1}{2}\frac{\partial w_\lambda}{\partial x_1} < 0 \quad \text{for } x_1 > 0,$$

by Hopf's lemma. Finally replacing x_1 by $-x_1$, we deduce the desired symmetry $v(-x_1, x', y) = v(x_1, x', y)$. $\square$

Acknowledgments

Both authors were supported by the Spain Government grants MTM2005-07660-C02-01 and MTM2008-06349-C03-01. The first author was supported by the Catalan Government grant SGR2009-345. The second author was supported by CONICYT Becas de Postgrado of Chile and the Programa de Recerca del Centre de Recerca Matemàtica, Barcelona, Spain.

References

[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical points theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[2] D. Applebaum, Lévy processes—from probability to finance and quantum groups, Notices Amer. Math. Soc. 51 (2004) 1336–1347.
[3] H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, Bol. Soc. Brasil. Mat. 22 (1991) 1–37.
[4] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983) 437–477.
[5] X. Cabré, J. Solà-Morales, Layer solutions in a halfspace for boundary reactions, Comm. Pure Appl. Math. 58 (2005) 1678–1732.
[6] L. Caffarelli, S. Salsa, L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, Invent. Math. 171 (2008) 425–461.
[7] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007) 1245–1260.
[8] L. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math., in press.
[9] M. Chipot, M. Chlebík, M. Fila, I. Shafrir, Existence of positive solutions of a semilinear elliptic equation in $\mathbb{R}^n_+$ with a nonlinear boundary condition, J. Math. Anal. Appl. 223 (1998) 429–471.
[10] J. Davila, Singular solutions of semi-linear elliptic problems, in: Handbook of Differential Equations, vol. 2, Stationary Partial Differential Equations, Elsevier Science, 2009, Chapter 2.
[11] J. Davila, L. Dupaigne, M. Montenegro, The extremal solution of a boundary reaction problem, Commun. Pure Appl. Anal. 7 (2008) 795–817.
[12] J. Escobar, Sharp constant in a Sobolev trace inequality, Indiana Univ. Math. J. 37 (1988) 687–698.
[13] B. Gidas, W.-M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209–243.
[14] B. Gidas, J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, Comm. Partial Differential Equations 6 (1981) 883–901.
[15] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[16] N.S. Landkof, Foundations of Modern Potential Theory, Springer-Verlag, 1972.
[17] Y.Y. Li, L. Zhang, Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations, J. Anal. Math. 90 (2003) 27–87.
[18] Y.Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. 80 (1995) 383–417.
[19] P.L. Lions, The concentration-compactness principle in the calculus of variations, The limit case II, Rev. Mat. Iberoamericana 1 (1985) 45–121.
[20] J.L. Lions, E. Magenes, Non-homogeneous Boundary Value Problems and Applications, vol. I, Die Grundlehren der Math. Wissenschaften, vol. 181, Springer-Verlag, 1972.
[21] A. Nekvinda, Characterization of traces of the weighted Sobolev space $W^{1,p}(\Omega, d^{\epsilon}_M)$ on M, Czechoslovak Math. J. 43 (118) (1993) 695–711.

[22] B. Ou, Positive harmonic functions on the upper half space satisfying a nonlinear boundary condition, Differential Integral Equations 9 (1996) 1157–1164.
[23] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Comm. Pure Appl. Math. 60 (2006) 67–112.
[24] M. Struwe, Variational Methods, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer-Verlag, 1996.
[25] S. Sugitani, On nonexistence of global solutions for some nonlinear integral equations, Osaka J. Math. 12 (1975) 45–51.
[26] J. Tan, The Brezis–Nirenberg type problem involving the square root of the Laplacian, preprint.